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$F-$geodesics on the second order tangent bundle over a Riemannian manifold

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Abstract. Let (M, g) be a Riemannian manifold and T^2M be its second order tangent bundle. In this paper, we deal with certain characterizations of $F-$geodesics (which generalize both classical geodesics and magnetic curves) on the second order tangent bundle T^2M and the hypersurface $T^2_{1,1}M$ with respect to some natural metrics.

1. Introduction

Magnetic curves represent, in physics, the trajectories of the charged particles moving on a Riemannian manifold under the action of the magnetic fields. A magnetic field F on a Riemannian manifold (M, g) is a closed $2-$form and the Lorentz force associated to F is a $(1, 1)-$tensor field ρ such that

$$F(X, Y) = g(\rho X, Y)$$

for all vector fields X, Y on M. A magnetic trajectory in such a magnetic field is thus modeled by a second order differential equation, that is,

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \rho\dot{\gamma},$$

usually known as the Lorentz equation. Such curves are sometimes called also magnetic geodesics since the Lorentz equation generalizes the equation of geodesics under arc-length parametrization, namely, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Here, ∇ denotes the Levi-Civita connection of the Riemannian metric g.

A smooth curve γ on a Riemannian manifold (M, g) endowed with a $(1, 1)-$tensor field F and with Levi-Civita connection ∇ is called an $F-$geodesic if γ satisfies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = F\dot{\gamma}.$$

$F-$geodesics are strictly related to $F-$planar curves and extended magnetic curves and hence, geodesics. Note that the notion of F-geodesic is slightly different from $F-$planar curve (see [12]). Inspired by the

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Lorentz force, the electro-magnetic tensor field, as well as some special forces involved in the Euler-Lagrange equations from Lagrangian mechanics, Bejan and Druţă-Romaniuc [2] defined $F-$geodesics on a manifold with a linear connection. They presented several examples of $F-$geodesics; for instance, they constructed $F-$geodesics on the tangent bundle of a manifold by using lifts. Also, they characterized $F-$geodesics according to some special connections such as Vranceanu connection on foliated manifolds and adapted connections on almost contact manifolds. Finally, they found conditions for a pair of symmetric connections to have the same system of $F-$geodesics. In this paper, we deal with certain characterizations of $F-$geodesics on the second order tangent bundle T^2M and the hypersurface $T^2_{1,1}M$.

1.1. Whitney tangent fiber bundle $TM \oplus TM$

Let M be an $n-$dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \to M$. We refer to [6, 16] for all the necessary background for the tangent bundle. The Whitney tangent fiber bundle $TM \oplus TM$ is defined by

$$TM \oplus TM = \Big\{ (u, \omega) \in TM \times TM; \quad \pi(u) = \pi(\omega) \Big\} = \bigcup_{x \in M} T_xM \times T_xM,$$

where $\pi_\oplus$ is denoted by

$$\begin{array}{rcl} \pi_\oplus : TM \oplus TM & \to & M \\ (u, \omega) & \mapsto & \pi_\oplus(u, \omega) = \pi(u) = \pi(\omega). \end{array}$$

A local chart $(U, \varphi) = \left(U, x^i\right)$ on M induces a chart $\left(\pi^{-1}(U), \widetilde{\varphi}\right) = \left(\pi^{-1}(U), x^i, y^i\right)$ on TM and $\left(\pi_\oplus^{-1}(U), \overline{\varphi}\right) = \left(\pi_\oplus^{-1}(U), x^i, y^i, z^i\right)$ on $TM \oplus TM$ such

$$\overline{\varphi}(x, u, \omega) = (\varphi(x), \widetilde{\varphi}_x(u), \widetilde{\varphi}_x(\omega)) = (\varphi(x), y, z).$$

Let $\widetilde{X}, \widetilde{Y}$ be vector fields on TM. Then $\left(\widetilde{X}, \widetilde{Y}\right)$ is a vector field on $TM \oplus TM$ if and only if

$$d\pi\left(\widetilde{X}\right) = d\pi\left(\widetilde{Y}\right).$$

Relative to the chart $\left(\pi_\oplus^{-1}(U), \overline{\varphi}\right) = \left(\pi_\oplus^{-1}(U), x^i, y^i, z^i\right)$, the local frame vector fields given in [5] are

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right), \\ \frac{\partial}{\partial y^i} &= \left(\frac{\partial}{\partial y^i}, 0\right), \\ \frac{\partial}{\partial z^i} &= \left(0, \frac{\partial}{\partial z^i}\right). \end{aligned}$$

For any vector field X on M and $f \in C^\infty(M)$, we have

$$\begin{aligned} (X^V, 0) &= X^i \frac{\partial}{\partial y^i}, \ (0, X^V) = X^i \frac{\partial}{\partial z^i}, \\ (X^H, X^H) &= X^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} - \Gamma^k_{ij} X^i z^j \frac{\partial}{\partial z^k}, \\ (X^V, 0)(f \circ \pi) &= (0, X^V)(f \circ \pi) = 0, \\ (X^H, X^H)(f \circ \pi) &= X(f) \circ \pi. \end{aligned}$$

If (M, g) is a Riemannian manifold, ∇ its Levi-Civita connection and $\gamma_1, \gamma_2 : 0 \in I \subset \mathbb{R} \to M$ are two smooth curves, then we have

$$[\gamma_1 \sim \gamma_2] \quad \Leftrightarrow \quad \left[\gamma_1(0) = \gamma_2(0), \quad \frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0) \quad and \quad \frac{d^2\gamma_1}{dt^2}(0) = \frac{d^2\gamma_2}{dt^2}(0)\right]$$

$j_0^2\gamma = \left\{\overline{\gamma}; \quad \overline{\gamma} \sim \gamma\right\}.$

The second order tangent bundle is the natural bundle of 2−jets of differentiable curves defined by

$$T^2M = \left\{j_0^2\gamma;\ \gamma : I \to M, \text{ is a smooth curve at } 0 \in \mathbb{R}\right\}.$$

The canonical projection P on T^2M is given by

$$\begin{array}{rcl} P : T^2M & \to & M \\ j_0^2\gamma & \mapsto & \gamma(0). \end{array}$$

A local chart (U, φ) induces a chart $(P^{-1}(U), \phi)$ on T^2M given by

$$\phi(j_0^2\gamma) = (\varphi(\gamma(0)), \frac{d\varphi\circ\gamma}{dt}(0), \frac{d^2\varphi\circ\gamma}{dt^2}(0)).$$

Proposition 1.1. *[5] If $TM \oplus TM$ denotes the Whitney sum, then*

$$S : T^2M \to TM \oplus TM, \quad j_0^2\gamma \mapsto (\dot\gamma(0), (\nabla_{\dot\gamma(0)}\dot\gamma)(0))$$

is a diffeomorphism of natural bundles.

In the induced coordinates, we have

$$S : (x^i, y^i, z^i) \mapsto (x^i, y^i, z^i + y^j y^k \Gamma^i_{jk}).$$

Proposition 1.2. *[5] Let T^2M be a second order tangent bundle endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$ ($\Gamma(T^2M)$ is the set of all sections from M onto T^2M), if we define two vector fields on M by*

$$X_\sigma = P_1 \circ S \circ \sigma, \quad Y_\sigma = P_2 \circ S \circ \sigma,$$

then $\sigma = S^{-1}(X_\sigma, Y_\sigma)$, where P_1 and P_2 denote the first and the second projection from $TM \oplus TM$ onto TM.

1.2. Lifts to T^2M

If (U, φ) is a local chart on M, then the diffeomorphism S induces a local chart $((\pi_\oplus \circ S)^{-1}(U), \overline{\varphi} \circ S)$ on T^2M such that

$$\frac{\partial}{\partial x^i} = S_*^{-1}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right), \quad \frac{\partial}{\partial y^i} = S_*^{-1}\left(\frac{\partial}{\partial y^i}, 0\right), \quad \frac{\partial}{\partial z^i} = S_*^{-1}\left(0, \frac{\partial}{\partial z^i}\right), \tag{1}$$

where $\pi_\oplus : (u, \omega) \in TM \oplus TM \mapsto \pi(u) = \pi(\omega) = x$.

Definition 1.3. *[3, 4] Let (M, g) be a Riemannian manifold, X and F respectively be a vector field and a $(1,1)$−tensor field on M. For $\lambda = 0, 1, 2$, the λ-lift of X to T^2M is defined by*

$$\begin{array}{rcl} X^{(0)} & = & S_*^{-1}(X^H, X^H), \\ X^{(1)} & = & S_*^{-1}(X^V, 0), \\ X^{(2)} & = & S_*^{-1}(0, X^V), \\ F^{(0)}(X^{(\lambda)}) & = & (FX)^{(\lambda)}, \qquad (\lambda = 0, 1, 2) \\ F^{(\lambda)}(X^{(0)}) & = & (FX)^{(\lambda)}, \qquad (\lambda = 1, 2) \\ F^{(1)}(X^{(\lambda)}) & = & 0 = F^{(2)}(X^{(\lambda)}), \qquad (\lambda = 1, 2). \end{array}$$

From the formulae (1) and Definition 1.3, we obtain the following lemma.

Lemma 1.4. *For any vector field X on M and any smooth function $f \in C^\infty(M)$, we have*

$$\begin{aligned} X^{(1)} &= X^i \frac{\partial}{\partial y^i}, \\ X^{(2)} &= X^i \frac{\partial}{\partial z^i}, \\ X^{(0)} &= X^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} - \Gamma^k_{ij} X^i z^j \frac{\partial}{\partial z^k}, \\ X^{(1)}(f \circ \pi) &= X^{(2)}(f \circ \pi) = 0, \\ X^{(0)}(f \circ \pi) &= X(f) \circ \pi. \end{aligned}$$

From Definition 1.3 and the Lie bracket operations of the horizontal and vertical lifts of any vector field X to the tangent bundle (see [6, 16]), we obtain the following proposition.

Proposition 1.5. *[5] Let (M, g) be a Riemannian manifold. If R denotes the Riemannian curvature tensor of (M, g), then for all vector fields X, Y on M and $p \in T^2M$ we have*

1. $[X^{(0)}, Y^{(0)}]_p = [X, Y]^{(0)}_p - (R_x(X, Y)u)^{(1)}_p - (R_x(X, Y)\omega)^{(2)}_p$,
2. $[X^{(0)}, Y^{(i)}]_p = (\nabla_X Y)^{(i)}_p$,
3. $[X^{(i)}, Y^{(j)}]_p = 0$,

where $(x, u, \omega) = S(p)$ and $i, j = 1, 2$.

Lemma 1.6. *Let (M, g) be a Riemannian manifold. For all $x \in M$, $u = u^i \frac{\partial}{\partial x^i}$, $\omega = \omega^i \frac{\partial}{\partial x^i} \in T_xM$ and any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have the following*

1. $X^{(0)}(g(Y, u))_p = g(\nabla_X Y, u)_x$,
2. $X^{(0)}(g(Y, \omega))_p = g(\nabla_X Y, \omega)_x$,
3. $X^{(0)}(f(r_1^2))_p = X^{(0)}(f(r_2^2))_p = 0 = X^{(0)}(g(u, u))_p = X^{(0)}(g(\omega, \omega))_p$,
4. $X^{(1)}(g(u, u))_p = 2g(X, u)_x$,
5. $X^{(1)}(g(\omega, \omega))_p = 0 = X^{(2)}(g(u, u))_p$,
6. $X^{(2)}(g(\omega, \omega))_p = 2g(X, \omega)_x$,
7. $X^{(1)}(g(Y, u))_p = g(X, Y)_x = X^{(2)}(g(Y, \omega))_p$,
8. $X^{(1)}(g(Y, \omega))_p = 0 = X^{(2)}(g(Y, u))_p$,
9. $X^{(1)}(f(r_1^2))_p = 2f'(r_1^2)g(X, u)$,
10. $X^{(1)}(f(r_2^2))_p = 0 = X^{(2)}(f(r_1^2))$,
11. $X^{(2)}(f(r_2^2))_p = 2f'(r_2^2)g(X, \omega)$,

where $p = S^{-1}(x, u, \omega)$, $r_1^2 = g(u, u) = |u|^2$, $r_2^2 = g(\omega, \omega) = |\omega|^2$.

2. F−geodesics on T^2M

Definition 2.1. *Let (M, g) be a Riemannian manifold. We define the Sasaki metric G_S on the second order tangent bundle T^2M by*

$$G_S = S_*^{-1}(g_S \oplus g_S),$$

where g_S is the Sasaki metric on the tangent bundle of (M, g) (for Sasaki metric, see [15, 16]).

Thus, we obtain the following definition.

Definition 2.2. *Let (M, g) be a Riemannian manifold. If $p \in T^2M$, then for all vector fields X, Y on M and $i, j \in \{0, 1, 2\}$ $(i \neq j)$, we obtain*

1 $G_S\left(X^{(0)}, Y^{(0)}\right)_p = g(X,Y)_x$,

2 $G_S(X^{(i)}, Y^{(j)})_p = 0, \quad for \quad i \neq j$

3 $G_S(X^{(1)}, Y^{(1)})_p = g(X,Y)_x$,

4 $G_S(X^{(2)}, Y^{(2)})_p = g(X,Y)_x$, *where* $S(p) = (x,u,\omega) \in T_xM \oplus T_xM$ *(also see [13]).*

From Lemma 1.6 and Definition 2.2, standard calculations give the following lemma.

Lemma 2.3. *Let* (M,g) *be a Riemannian manifold and* T^2M *its second order tangent bundle with the Sasaki metric* G_S*. Then*

$$
\begin{aligned}
X^{(0)}(G_S(Y^{(0)}, Z^{(0)}))_p &= X(g(Y,Z))_x, \\
X^{(0)}(G_S(Y^{(1)}, Z^{(1)}))_p &= G_S((\nabla_X Y)^{(1)}, Z^{(1)})_p + G_S(Y^{(1)}, (\nabla_X Z)^{(1)})_p, \\
X^{(0)}(G_S(Y^{(2)}, Z^{(2)}))_p &= G_S((\nabla_X Y)^{(2)}, Z^{(2)})_p + G_S(Y^{(2)}, (\nabla_X Z)^{(2)})_p, \\
X^{(1)}(G_S(Y^{(0)}, Z^{(0)}))_p &= 0 = X^{(2)}(G_S(Y^{(0)}, Z^{(0)})_p, \\
X^{(1)}(G_S(Y^{(1)}, Z^{(1)}))_p &= 0, \\
X^{(2)}(G_S(Y^{(2)}, Z^{(2)}))_p &= 0, \\
X^{(1)}(G_S(Y^{(2)}, Z^{(2)}))_p &= 0 = X^{(2)}(G_S(Y^{(1)}, Z^{(1)})_p
\end{aligned}
$$

for all vector fields X,Y,Z *on* M *and* $p \in T^2M$.

Proposition 2.4. *[5] Let* (M,g) *be a Riemannian manifold and* T^2M *be its second order tangent bundle equipped with the Sasaki metric* G_S*. If* $\widetilde{\nabla}$ *denotes the Levi-Civita connection of* T^2M*, then for* $p \in T^2M$ *and vector fields* X,Y *on* M *we have*

$$
\begin{aligned}
1.\quad (\widetilde{\nabla}_{X^{(0)}} Y^{(0)})_p &= (\nabla_X Y)^{(0)} - \frac{1}{2}\Big(R(X,Y)u\Big)^{(1)} - \frac{1}{2}\Big(R(X,Y)\omega\Big)^{(2)}, \\
2.\quad (\widetilde{\nabla}_{X^{(0)}} Y^{(1)})_p &= (\nabla_X Y)^{(1)} + \frac{1}{2}\Big(R(u,Y)X\Big)^{(0)}, \\
3.\quad (\widetilde{\nabla}_{X^{(0)}} Y^{(2)})_p &= (\nabla_X Y)^{(2)} + \frac{1}{2}\Big(R(\omega,Y)X\Big)^{(0)}, \\
4.\quad (\widetilde{\nabla}_{X^{(1)}} Y^{(0)})_p &= \frac{1}{2}\Big(R(u,X)Y\Big)^{(0)}, \\
5.\quad (\widetilde{\nabla}_{X^{(2)}} Y^{(0)})_p &= \frac{1}{2}\Big(R(\omega,X)Y\Big)^{(0)}, \\
6.\quad (\widetilde{\nabla}_{X^{(i)}} Y^{(j)})_p &= 0 \qquad i,j = 1,2,
\end{aligned}
$$

where $S(p) = (x,u,\omega)$*,* ∇ *and* R *denote the Levi-Civita connection and the Riemannian curvature tensor of* (M,g)*, respectively.*

Definition 2.5. *Let* M *be a smooth manifold,* F *be a (1,1)-tensor field on* M*,* $\overline{\nabla}$ *be a linear connection on* M *and* $\gamma : I \to M$ *be a smooth curve. Then*

1. γ *is said to be a magnetic curve with respect to* $(F, \overline{\nabla})$*, if* γ *satisfies :* $\overline{\nabla}_{\dot\gamma}\dot\gamma(t) = F\,\dot\gamma(t)$ *([1, 7]),*

2. γ *is said to be an* $F-$*planar curve with respect to* $\overline{\nabla}$ *if* γ *satisfies :* $\overline{\nabla}_{\dot\gamma}\dot\gamma(t) = \varrho_1(t)\dot\gamma(t) + \varrho_2(t)F\,\dot\gamma(t)$ *([11, 12]),*

where ϱ_1, ϱ_2 *are some smooth real functions.*

Definition 2.6. *[2] Let M be a smooth manifold, F be a (1,1)-tensor field on M, $\overline{\nabla}$ be a linear connection, and $\gamma : I \to M$ be a smooth curve. We say that γ is an $F-$geodesic with respect to $\overline{\nabla}$ if $\gamma(u)$ satifies*

$$\overline{\nabla}_{\dot{\gamma}(u)}\dot{\gamma}(u) = F(\dot{\gamma}(u)). \tag{2}$$

If t is another parameter for the same curve $\gamma(u)$ then the relation (2) becomes

$$\overline{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = \alpha(t)\dot{\gamma}(t) + \beta(t)F(\dot{\gamma}(t)), \tag{3}$$

where α and β are some functions on the curve $\gamma(t)$.
A curve $\gamma(t)$ satisfying the relation (3) describes an $F-$geodesic up to a reparameterization.

One can easily see that an $F-$geodesic is an $F-$planar curve, but in general an $F-$planar curve is not always an F-geodesic.

Definition 2.7. *Let (M, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. We define a curve $C : I \to T^2M$ by $C(t) = S^{-1}(x(t), y(t), z(t))$ for all $t \in I$, where $y(t) \in T_{x(t)}M$, i.e., $y(t), z(t)$ are vector fields along $x(t)$.*

(1) *The curve $C(t) = S^{-1}(x(t), \dot{x}(t), \dot{x}(t))$ is called a natural lift of the curve $x(t)$.*

(2) *The curve $C(t) = S^{-1}(x(t), y(t), z(t))$ is said to be a horizontal lift of the cure $x(t)$ if and only if $\nabla_{\dot{x}}y = 0$ and $\nabla_{\dot{x}}z = 0$.*

Lemma 2.8 ([14]). *Let (M, g) be a Riemannian manifold. If X, Y are vector fields on M and $(x, u) \in TM$ such that $X_x = u$, then we have*

$$d_xX(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Lemma 2.9. *Let (M, g) be a Riemannian manifold. If Z is a vector field on M and $\sigma \in \Gamma(T^2M)$ then for all $x \in M$, we have*

$$d_x\sigma(Z_x) = Z^{(0)}_p + (\nabla_Z X_\sigma)^{(1)}_p + (\nabla_Z Y_\sigma)^{(2)}_p,$$

where $p = \sigma(x)$.

Proof. Using Lemma 2.8, it follows that

$$\begin{aligned} d_x\sigma(Z) &= dS^{-1}(dX_\sigma(Z), dY_\sigma(Z))_{S(p)} \\ &= dS^{-1}(Z^H, Z^H)_{S(p)} + dS^{-1}((\nabla_Z X_\sigma)^V, (\nabla_Z Y_\sigma)^V)_{S(p)} \\ &= Z^{(0)}_p + (\nabla_Z X_\sigma)^{(1)}_p + (\nabla_Z Y_\sigma)^{(2)}_p. \end{aligned}$$

□

Lemma 2.10. *Let (M, g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Sasaki metric and let $x : I \to M$ be a curve on M. If $C : t \in I \to C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$ (i.e., $y(t), z(t) \in T_{x(t)}M$), then*

$$\dot{C} = \dot{x}^{(0)} + (\nabla_{\dot{x}}y)^{(1)} + (\nabla_{\dot{x}}z)^{(2)},$$

where $\dot{x} = \frac{dx}{dt}$ and $\dot{C} = \frac{dC}{dt}$.

Proof. If Y, Z are vector fields such $Y(x(t)) = y(t)$ and $Z(x(t)) = z(t)$, then we have

$$\dot{C}(t) = dC(t) = d\sigma(\dot{x}(t)),$$

where $\sigma = S^{-1}(Y, Z)$. Using Lemma 2.9 we obtain

$$\dot{C}(t) = d\sigma(\dot{x}(t)) = \dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}. \tag{4}$$

□

Theorem 2.11. *Let (M, g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$ and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on T^2M such that $y(t)$, $z(t)$ are vector fields along $x(t)$. Then we have*

$$\widetilde{\nabla}_{\dot{C}}\dot{C} = \Big[\nabla_{\dot{x}}\dot{x} + R(y, \nabla_{\dot{x}} y)\dot{x} + R(z, \nabla_{\dot{x}} z)\dot{x}\Big]^{(0)} + \Big[\nabla_{\dot{x}}\nabla_{\dot{x}} y\Big]^{(1)} + \Big[\nabla_{\dot{x}}\nabla_{\dot{x}} z\Big]^{(2)}. \tag{5}$$

Proof. The proof follows immediately from Proposition 2.4 and the formula (4).
□

Theorem 2.12. *Let (M, g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. A curve $C(t) = S^{-1}(x(t), y(t), z(t))$ on T^2M is an $F^{(0)}$-planar curve if and only if*

$$\begin{aligned}
\nabla_{\dot{x}}\dot{x} &= -R(y, \nabla_{\dot{x}} y)\dot{x} - R(z, \nabla_{\dot{x}} z)\dot{x} + \varrho_1(t)\,\dot{x} + \varrho_2(t)\,F(\dot{x}),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} y &= \varrho_1(t)\nabla_{\dot{x}} y + \varrho_2(t)F(\nabla_{\dot{x}} y),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} z &= \varrho_1(t)\nabla_{\dot{x}} z + \varrho_2(t)F(\nabla_{\dot{x}} z).
\end{aligned}$$

Proof. From the formula (4), we have

$$\begin{aligned}
\widetilde{\nabla}_{\dot{C}}\dot{C} &= \varrho_1(t)\,\dot{C} + \varrho_2(t)\,F^{(0)}(\dot{C})\\
&= \varrho_1(t)\Big[\dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}\Big]\\
&\qquad + \varrho_2(t)\Big[F^{(0)}\dot{x}^{(0)} + F^{(0)}(\nabla_{\dot{x}} y)^{(1)} + F^{(0)}(\nabla_{\dot{x}} z)^{(2)}\Big]\\
&= \Big[\varrho_1(t)\dot{x} + \varrho_2(t)F\dot{x}\Big]^{(0)} + \Big[\varrho_1(t)\nabla_{\dot{x}} y + \varrho_2(t)F\nabla_{\dot{x}} y\Big]^{(1)}\\
&\quad + \Big[\varrho_1(t)\nabla_{\dot{x}} z + \varrho_2(t)F\nabla_{\dot{x}} z\Big]^{(2)}.
\end{aligned}$$

Using the formula (5), the result immediately follows. □

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 2.12 , we obtain the following result.

Theorem 2.13. *Let (M, g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. A curve $C(t) = S^{-1}(x(t), y(t), z(t))$ on T^2M is an $F^{(0)}$-geodesic if and only if*

$$\begin{aligned}
\nabla_{\dot{x}}\dot{x} &= -R(y, \nabla_{\dot{x}} y)\dot{x} - R(z, \nabla_{\dot{x}} z)\dot{x} + F(\dot{x}),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} y &= F(\nabla_{\dot{x}} y),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} z &= F(\nabla_{\dot{x}} z).
\end{aligned}$$

Using Theorem 2.12 and Theorem 2.13, we obtain the following corollaries.

Corollary 2.14. *Let (M, g) be a locally flat Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. Then a curve $C(t) = S^{-1}(x(t), y(t), z(t))$ on T^2M is an $F^{(0)}$-geodesic if and only if*

$$\begin{aligned}
\nabla_{\dot{x}}\dot{x} &= F(\dot{x}),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} y &= F(\nabla_{\dot{x}} y),\\
\nabla_{\dot{x}}\nabla_{\dot{x}} z &= F(\nabla_{\dot{x}} z).
\end{aligned}$$

Corollary 2.15. *Let (M,g) be a locally flat Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. Then a curve $C(t) = S^{-1}(x(t), y(t), z(t))$ on T^2M is an $F^{(0)}$-geodesic up to a reparameterization (resp., $F^{(0)}$-planar curve) if and only if*

$$\begin{aligned} \nabla_{\dot{x}}\dot{x} &= \varrho_1(t)\,\dot{x} + \varrho_2(t)\,F(\dot{x}),\\ \nabla_{\dot{x}}\nabla_{\dot{x}}y &= \varrho_1(t)\nabla_{\dot{x}}y + \varrho_2(t)F(\nabla_{\dot{x}}y),\\ \nabla_{\dot{x}}\nabla_{\dot{x}}z &= \varrho_1(t)\nabla_{\dot{x}}z + \varrho_2(t)F(\nabla_{\dot{x}}z). \end{aligned}$$

Proposition 2.16. *Let (M,g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a horizontal lift of a curve $x(t)$, then $C(t)$ is an $F^{(0)}$-planar curve (resp., $F^{(0)}$-geodesic) if and only if $x(t)$ is an F-planar curve (resp., F-geodesic).*

Proof. From the formulas (4) and (5), we have

$$\begin{aligned} \dot{C}(t) &= (\dot{x})^{(0)}(t)\\ \widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{(\dot{x})^0}\dot{x}^{(0)} = (\nabla_{\dot{x}}\dot{x})^{(0)}. \end{aligned}$$

Let $C(t)$ be an $F^{(0)}$-planar curve. Then

$$\begin{aligned} \widetilde{\nabla}_{\dot{C}}\dot{C} &= \varrho_1(t)\,\dot{C} + \varrho_2(t)\,F^{(0)}(\dot{C})\\ &= \varrho_1(t)\,\dot{x}^{(0)} + \varrho_2(t)\,F^{(0)}(\dot{x}^{(0)})\\ &= \Big[\varrho_1(t)\,\dot{x} + \varrho_2(t)\,F(\dot{x})\Big]^{(0)}\\ &= (\nabla_{\dot{x}}\dot{x})^{(0)}. \end{aligned}$$

Hence, $C(t)$ is an $F^{(0)}$-planar curve if and only $x(t)$ is an F-planar curve. In the case of $\rho_1 = 0$ and $\rho_2 = 1$, we get that $C(t)$ is an $F^{(0)}$-geodesic if and only $x(t)$ is an F-geodesic.
□

Remark 2.17. *If $C(t) = S^{-1}(x(t), y(t), z(t))$ is the horizontal lift of the curve $x(t)$, then we have*

$$\begin{aligned} \Big[\nabla_{\dot{x}}y = 0\Big] &\Leftrightarrow \Big[\frac{dy^k}{dt} + \Gamma^k_{ij}y^i\frac{dx^j}{dt} = 0\Big] \Leftrightarrow \Big[y(t) = e^{-(\int A(t)dt)}.K\Big],\\ \Big[\nabla_{\dot{x}}z = 0\Big] &\Leftrightarrow \Big[\frac{dz^k}{dt} + \Gamma^k_{ij}z^i\frac{dx^j}{dt} = 0\Big] \Leftrightarrow \Big[z(t) = e^{-(\int A(t)dt)}.\overline{K}\Big], \end{aligned}$$

where $K, \overline{K} \in \mathbb{R}^n$ and $A(t) = [a_{ki}]$, $a_{ki} = \sum_{j=1}^{n} \Gamma^k_{ij}\frac{dx^j}{dt}$. Therefore, $C(t)$ is an $F^{(0)}$-geodesic (resp. $F^{(0)}$-planar curve) if and only if $\nabla_{\dot{x}}\dot{x} = F(\dot{x})$ (resp. $\nabla_{\dot{x}}\dot{x} = \varrho_1(t)\,\dot{x} + \varrho_2(t)\,F(\dot{x})$).

Using Remark 2.17, we can construct an infinity of examples of F-geodesics (resp. F-planar curve) on (T^2M, G_S).

Example 2.18. *Let $\mathbb{R}^n$ be equipped with the Riemannian metric $g = ds^2$ and $B \in \mathcal{M}_{n\times n}(\mathbb{R})$. If $F = B$ is an invertible matrix, then $C(t) = S^{-1}(B^{-1}\ \exp(B\,t)\,K_1 + K_2,\ const.,\ const.), K_1, K_2 \in \mathbb{R}^n$, is an $F^{(0)}$-geodesic.*

Example 2.19. *Let $\mathbb{R}$ be equipped with the Riemannian metric $g = e^x dx^2$and $F = a \in \mathbb{R}^*$. Then the Christoffel symbol of the Levi-Civita connection is given by*

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}$$

and $C(t) = S^{-1}(x(t),\ y(t),\ z(t)) = S^{-1}\left(2\ln\left(\frac{K_1e^{at}+aK_2}{2a}\right), \frac{2aK_3}{K_1e^{at}+aK_2}, \frac{2aK_4}{K_1e^{at}+aK_2}\right)$, $K_1, ..., K_4 \in \mathbb{R}$, is an $F^{(0)}$-geodesic such that $\nabla_{\dot{x}}y = 0$ and $\nabla_{\dot{x}}z = 0$.

Example 2.20. *Let $\mathbb{R}$ be equipped with the Riemannian metric $g = e^x dx^2$, $F = a \in \mathbb{R}^*$, $\rho_1(t) = \frac{1}{t}$ $\rho_2(t) = 1$. Then we have $\Gamma_{11}^1 = \frac{1}{2}$ and $x(t)$ is an F-planar curve if and only if it satisfies the following differential equation*

$$x'' + \frac{1}{2}x'^2 = \frac{at+1}{t}x'.$$

A solution of the previous equation is given by

$$x(t) = 2\ln\frac{K_1 e^{at}(at-1) + K_2}{2a^2}.$$

So, from Remark 2.17 we obtain

$$\begin{aligned} y(t) &= \frac{2a^2K_3}{K_1e^{at}(at-1)+K_2}, \\ z(t) &= \frac{2a^2K_4}{K_1e^{at}(at-1)+K_2}, \end{aligned}$$

where $K_1, .., K_4 \in \mathbb{R}$. Then $C(t) = S^{-1}(x(t),\ y(t),\ z(t))$, is an $F^{(0)}$-planar curve such that $\nabla_{\dot{x}} y = 0$ and $\nabla_{\dot{x}} z = 0$.

Example 2.21. *Let $\left(\mathbb{R} \setminus \{0\}\right)^2$ be equipped with the Riemannian metric h defined by*

$$h_{11} = x^2\ ,\ h_{22} = y^2,\ h_{12} = 0$$

and $F = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma_{11}^1 = \frac{1}{x}\ ,\ \Gamma_{22}^2 = \frac{1}{y},\ \Gamma_{ij}^k = 0\ \forall (i,j,k) \in \{1,2\}^3 \setminus \{(1,1),(2,2)\}.$$

Let $C(t) = S^{-1}(x(t), y(t), z(t))$ be the horizontal lift of the curve $x(t) = (x_1(t), x_2(t))$. From Remark 2.17, we have

$$\begin{aligned} A(t) &= \begin{pmatrix} \frac{x_1'(t)}{x_1(t)} & 0 \\ 0 & \frac{x_2'(t)}{x_2(t)} \end{pmatrix}, \\ y(t) &= \left(\frac{k_1}{x_1(t)}, \frac{k_2}{x_2(t)}\right) \text{ and } z(t) = \left(\frac{k_3}{x_1(t)}, \frac{k_4}{x_2(t)}\right), \end{aligned}$$

where $k_1, k_2, k_3, k_4 \in \mathbb{R}$. $x(t) = (x_1(t),\ x_2(t))$ is an F-geodesic if and only if it satisfies the following differential equations

$$\begin{cases} x_1'' + \frac{1}{x_1}x_1'^2 = a\ x_1' \\ x_2'' + \frac{1}{x_2}x_2'^2 = 0 \end{cases}$$

whose solution is given by

$$x(t) = (x_1(t),\ x_2(t)) = \left(\exp\sqrt{\frac{a}{2}}\ t,\ \sqrt{2k_5\ t + k_6}\right),$$

where $k_5, k_6 \in \mathbb{R}$. Therefore, $C(t) = S^{-1}\left(x_1(t),\ x_2(t), \frac{k_1}{x_1(t)}, \frac{k_2}{x_2(t)}, \frac{k_3}{x_1(t)}, \frac{k_4}{x_2(t)}\right)$ is an F- geodesic such that $\nabla_{\dot{x}} y = 0$ and $\nabla_{\dot{x}} z = 0$.

Proposition 2.22. *Let (M, g) be a Riemannian manifold equipped with the Levi-Civita connection ∇ and let (T^2M, G_S) be its second order tangent bundle equipped with the Levi-Civita connection $\widetilde{\nabla}$. Let F be a (1,1)-tensor field on M. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is the horizontal lift of a curve $x(t)$, then we have*

1. *An integral curve of any vector field X on M is an $F-$geodesic with respect to ∇ if and only if the integral curve of $X^{(0)}$ is an $F^{(0)}-$geodesic with respect to $\widetilde{\nabla}$.*
2. *An integral curve of any vector field X on M is an $F-$geodesic up to a reparameterization, with respect to ∇ if and only if the integral curve of $X^{(0)}$ is an $F^{(0)}-$geodesic up to a reparameterization, with respect to $\widetilde{\nabla}$.*
3. *$C(t)$ is an $F^{(0)}-$geodesic with respect to $\widetilde{\nabla}$ if and only if the curve $x(t)$ is an $F-$geodesic with respect to ∇.*
4. *$C(t)$ is an $F^{(0)}-$geodesic up to a reparameterization with respect to $\widetilde{\nabla}$ if and only if the curve $x(t)$ is an $F-$geodesic up to a reparameterization with respect to ∇.*

Proof. Let γ be an $F-$geodesic up to a reparameterization with respect to Levi-Civita connection ∇ on M. Then the relation (3) is satisfied and we obtain

$$\widetilde{\nabla}_{\dot\gamma^{(0)}}\dot\gamma^{(0)} = \left(\nabla_{\dot\gamma}\dot\gamma\right)^0 = \alpha \circ P\ \dot\gamma(t)^{(0)} + \beta \circ P\ F^{(0)}\dot\gamma(t)^{(0)},$$

where P is the canonical projection on T^2M. In the case of $\alpha = 0$ and $\beta = 1$, one can easily obtain (1). □

Remark 2.23. *The Proposition 2.22 remains true, if we replace $\widetilde{\nabla}$ by $\nabla^{(0)}$, where $\nabla^{(0)}$ is defined by*

$$\begin{aligned} \nabla^{(0)}_{X^{(0)}}Y^{(\lambda)} &= (\nabla_X Y)^{(\lambda)}, \\ \nabla^{(0)}_{X^{(i)}}Y^{(\lambda)} &= 0 \end{aligned}$$

for $i = 1, 2$ and $\lambda = 0, 1, 2$.

Definition 2.24. *Let (M, g) be a Riemannian manifold. We can define a natural diagonal metric G on the second tangent bundle T^2M of (M, g) by*

$$\begin{cases} G_p(X^{(0)}, Y^{(0)}) = & b_1 g_x(X, Y) + d_1 g_x(X, u) g_x(Y, u) + c_1 g_x(X, \omega) g_x(Y, \omega), \\ G_p(X^{(1)}, Y^{(1)}) = & b_2 g_x(X, Y) + d_2 g_x(X, u) g_x(Y, u), \\ G_p(X^{(2)}, Y^{(2)}) = & b_3 g_x(X, Y) + d_3 g_x(X, u) g_x(Y, u), \\ G_p(X^{(i)}, Y^{(j)}) = & 0, \qquad i \neq j = 0, 1, 2 \end{cases} \tag{6}$$

where $p = S^{-1}(x, u, \omega)$, d_1, b_2, d_2 (resp. c_1, b_3, d_3) are smooth functions depending on $r_1 = g(u, u)$ (resp $r_2 = g(\omega, \omega)$) and b_1 is a smooth function depending on (r_1, r_2), such that $b_1, b_2, b_3 > 0$ and $b_1 + r_1 d_1,\ b_2 + r_1 d_2,\ b_3 + r_2 d_3 > 0$.

The Levi-Civita connection of G denoted by $\widehat{\nabla}$ has the following expressions on the horizontal and respectively on the vertical distributions of $T(T^2M)$

$$\begin{aligned} \widehat{\nabla}_{X^{(0)}}Y^{(0)} = \ & (\nabla_X Y)^{(0)} - \frac{d_1}{2b_1}\Big[g(X, u)Y^{(1)} + g(Y, u)X^{(1)}\Big] - \frac{\partial_1 b_1}{b_2 + r_1 d_2} g(X, Y) u^{(1)} \\ & - \frac{b_2 d_1' - d_1 d_2}{b_2(b_2 + r_1 d_2)} g(X, u) g(Y, u) u^{(1)} - \frac{1}{2}\Big(R(X, Y)u\Big)^{(1)} \\ & - \frac{c_1}{2b_1}\Big[g(X, \omega)Y^{(2)} + g(Y, \omega)X^{(2)}\Big] - \frac{\partial_2 b_1}{b_3 + r_2 d_3} g(X, Y)\omega^{(2)} \\ & - \frac{b_3 c_1' - c_1 d_3}{b_3(b_3 + r_2 d_3)} g(X, \omega) g(Y, \omega)\omega^{(2)} - \frac{1}{2}\Big(R(X, Y)\omega\Big)^{(2)}, \end{aligned} \tag{7}$$

$$\begin{aligned}
\widehat{\nabla}_{X^{(1)}}Y^{(1)} &= \frac{b_2'}{b_2}\Big[g(X,u)Y^{(1)} + g(Y,u)X^{(1)}\Big] - \frac{b_2' - d_2}{b_2 + r_1 d_2} g(X,Y)u^{(1)} \\
&\quad + \frac{b_2 d_2' - b_2' d_2}{b_2(b_2 + r_1 d_2)} g(X,u)g(Y,u)u^{(1)}, \\
\widehat{\nabla}_{X^{(2)}}Y^{(2)} &= \frac{b_3'}{b_3}\Big[g(X,\omega)Y^{(2)} + g(Y,\omega)X^{(2)}\Big] - \frac{b_3' - d_3}{b_3 + r_2 d_3} g(X,Y)u^{(2)} \\
&\quad + \frac{b_3 d_3' - b_3' d_3}{b_3(b_3 + r_2 d_3)} g(X,\omega)g(Y,\omega)\omega^{(2)},
\end{aligned}$$

where $\partial_1 b_1 = \frac{\partial b_1}{\partial r_1}$ and $\partial_2 b_1 = \frac{\partial b_1}{\partial r_2}$.

Proposition 2.25. *Let (M,g) be a Riemannian manifold, (T^2M,G) be its second order tangent bundle and let F be a (1,1)-tensor field on M. If $C(t) = S^{-1}(x(t),y(t),z(t))$ is the horizontal lift of a curve $x(t)$, then we have*

- *(i) An integral curve of any vector field X on M is an F-geodesic with respect to the Levi-Civita connection ∇ of g if and only if the integral curve of the horizontal lift $X^{(0)}$ is an $F^{(0)}$-geodesic with respect to the Levi-Civita connection $\widehat{\nabla}$ of G defined by (6), provided $b_1 = const.$ and $d_1 = c_1 = 0$.*
- *(ii) The curve $C(t)$ is an $F^{(0)}$-geodesic with respect to the Levi-Civita connection $\widehat{\nabla}$ if and only if the curve $x(t)$ is an F-geodesic with respect to the Levi-Civita connection ∇, provided $b_1 = const.$ and $d_1 = c_1 = 0$.*
- *(iii) The above assertions (i) and (ii) remain true, if instead of an $F-$ geodesic (resp., $F^{(0)}-$ geodesic), we take an F-geodesic up to a reparameterization (resp. an $F^{(0)}$-geodesic up to a reparameterization).*

Proof. Let γ be an F-geodesic up to a reparameterization with respect to ∇, i.e.,

$$\nabla_{\dot\gamma}\dot\gamma = \alpha\,\dot\gamma + \beta\,F\dot\gamma, \tag{8}$$

where α and β are some smooth functions on the curve. For $X = Y = \dot\gamma$ the relation (7) becomes

$$\begin{aligned}
\widehat{\nabla}_{\dot\gamma^{(0)}}\dot\gamma^{(0)} &= (\nabla_{\dot\gamma}\dot\gamma)^{(0)} - \frac{d_1}{b_1} g(\dot\gamma,u)\dot\gamma^{(1)} - \frac{\partial_1 b_1}{b_2 + r_1 d_2} g(\dot\gamma,\dot\gamma)u^{(1)} \\
&\quad - \frac{b_2\,d_1' - d_1\,d_2}{b_2(b_2 + r_1\,d_2)} g(\dot\gamma,u)^2 u^{(1)} - \frac{c_1}{b_1} g(\dot\gamma,\omega)\dot\gamma^{(2)} \\
&\quad - \frac{\partial_2 b_1}{b_3 + r_2 d_3} g(\dot\gamma,\dot\gamma)\omega^{(2)} - \frac{b_3\,c_1' - c_1\,d_3}{b_3(b_3 + r_2\,d_3)} g(\dot\gamma,\omega)^2 \omega^{(2)}.
\end{aligned}$$

Using the formula (8), we have that $\widehat{\nabla}_{\dot\gamma^{(0)}}\dot\gamma^{(0)} = \alpha \circ P\,\dot\gamma^{(0)} + \beta \circ P\,F^{(0)}\dot\gamma^{(0)}$ if and only if

$$\begin{aligned}
0 &= -\frac{d_1}{b_1} g(\dot\gamma,u)\dot\gamma^{(1)} - \frac{\partial_1 b_1}{b_2 + r_1 d_2} g(\dot\gamma,\dot\gamma)u^{(1)} \\
&\quad - \frac{b_2\,d_1' - d_1\,d_2}{b_2(b_2 + r_1\,d_2)} g(\dot\gamma,u)^2 u^{(1)} - \frac{c_1}{b_1} g(\dot\gamma,\omega)\dot\gamma^{(2)} \\
&\quad - \frac{\partial_2 b_1}{b_3 + r_2 d_3} g(\dot\gamma,\dot\gamma)\omega^{(2)} - \frac{b_3\,c_1' - c_1\,d_3}{b_3(b_3 + r_2\,d_3)} g(\dot\gamma,\omega)^2 \omega^{(2)}.
\end{aligned}$$

Then, we get $d_1 = c_1 = \partial_1 b_1 = \partial_2 b_1 = 0$. If we replace $\gamma(t)$ by $x(t)$, from the formula (4) we have $\dot C(t) = (x(t))^{(0)}$. Similarly, the item (iii) can be proved. In the particular case of $\alpha = 0$ and $\beta = 1$, we deduce that the items (i) and (ii) are also true. □

3. F-Geodesics of the hypersurface $T^2_{1,1}M$

Let $T^2_{1,1}M$ be the hypersurface in T^2M defined by

$$T^2_{1,1}M = \{p = S^{-1}(x,u,w) \in T^2M,\ |u| = |\omega| = 1\}. \tag{9}$$

The unit normal vector fields to $T^2_{1,1}M$ are given by

$$\begin{array}{rcl} \mathcal{U} : T^2M & \rightarrow & T(T^2M) \\ p = S^{-1}(x,u,\omega) & \mapsto & \mathcal{U}_p = (u)^{(1)} \end{array} \tag{10}$$

$$\begin{array}{rcl} \mathcal{W} : T^2M & \rightarrow & T(T^2M) \\ p = S^{-1}(x,u,\omega) & \mapsto & \mathcal{W}_p = (\omega)^{(2)}. \end{array} \tag{11}$$

Indeed, for $p = S^{-1}(x,u,\omega) \in T^2_{1,1}M$, we have

$$\begin{array}{rcl} G_S(\mathcal{U},\mathcal{U})_p & = & g(u,u) = 1, \\ G_S(\mathcal{W},\mathcal{W})_p & = & g(w,w) = 1, \\ G_S(\mathcal{U},\mathcal{W})_p & = & 0. \end{array}$$

On the other hand, if we set

$$F_1 : T^2M \to \mathbb{R},\ \ p = S^{-1}(x,u,\omega) \mapsto g(u,u),$$

$$F_2 : T^2M \to \mathbb{R},\ \ p = S^{-1}(x,u,\omega) \mapsto g(\omega,\omega),$$

$$F : T^2M \to \mathbb{R}^2,\ \ p \mapsto (F_1(p), F_2(p)),$$

then the hypersurface $T^2_{1,1}M$ is given by

$$T^2_{1,1}M = \{p = S^{-1}(x,u,\omega) \in T^2M, \quad (F_1(p), F_2(p)) = (1,1)\},$$

where $grad_{G_S}(F_1)$ and $grad_{G_S}(F_2)$ are vector fields normal to $T^2_{1,1}M$. From Lemma 1.6, for any vector field X on M, we get

$$\begin{array}{rcl} G_S(X^{(0)}, grad_{G_S}(F_1)) & = & X^{(0)}(F_1) = X^{(0)}(g(u,u)) \\ & = & 0 = G_S(X^{(0)},\mathcal{U}), \\ G_S(X^{(1)}, grad_{G_S}(F_1)) & = & X^{(1)}(F_1) = X^{(1)}(g(u,u)) \\ & = & 2g(X,u) = 2G_S(X^{(1)},\mathcal{U}), \\ G_S(X^{(2)}, grad_{G_S}(F_1)) & = & X^{(2)}(F_1) = X^{(2)}(g(u,u)) \\ & = & 0 = 2G_S(X^{(2)},\mathcal{U}). \end{array}$$

So $\mathcal{U} = \frac{1}{2} grad_{G_S}(F_1)$. By the same way, we obtain $\mathcal{W} = \frac{1}{2} grad_{G_S}(F_2)$, therefore $\mathcal{U}$ and $\mathcal{W}$ are vector fields orthonormal to $T^2_{1,1}M$. If B (resp. $\dddot{\nabla}$) denotes the second fundamental form (resp. the Levi-Civita connection on $T^2_{1,1}M$), then we have

$$B(\widetilde{X},\widetilde{Y}) = G_S(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\mathcal{U})\mathcal{U} + G_S(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\mathcal{W})\mathcal{W}, \tag{12}$$

$$\dddot{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \rho_1(\widetilde{X},\widetilde{Y})\mathcal{U} - \rho_2(\widetilde{X},\widetilde{Y})\mathcal{W} \tag{13}$$

for all vector fields $\widetilde{X},\widetilde{Y}$ on $T^2_{1,1}M$.

Subsequently, we denote $x' = \dot{x}$, $x'' = \nabla_{\dot{x}}\dot{x}$, $y' = \nabla_{\dot{x}}y$ and $y'' = \nabla_{\dot{x}}\nabla_{\dot{x}}y$, $z' = \nabla_{\dot{x}}z$ and $z'' = \nabla_{\dot{x}}\nabla_{\dot{x}}z$.

Lemma 3.1. *Let (M, g) be a Riemannian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the Sasaki metric and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t), z(t)$ are vector fields along $x(t)$. Then, we have*

$$
\begin{array}{rrcl}
(1) & g(y, y) & = & 1 = g(z, z),\\
(2) & g(y', y) & = & 0 = g(z', z),\\
(3) & g(y'', y) & = & -|y'|^2 = -g(y', y'),\\
(4) & g(z'', z) & = & -|z'|^2 = -g(z', z').
\end{array}
$$

As $T^2_{1,1}M$ is the hypersurface in T^2M, a curve on $T^2_{1,1}M$ is a geodesic if and only if its second covariant derivative in T^2M is collinear to the unit normal vectors $(y)^{(1)}$ and $(z)^{(2)}$. From Theorem 2.13, the formula (12) and Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Let (M, g) be a Riemannian manifold and (T^2M, G_S) be its second order tangent bundle equipped with the Sasaki metric and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. Then, C is an $F^{(0)}$-geodesic on $T^2_{1,1}M$ if and only if*

$$x'' = -\Big[R(y, y') + R(z, z')\Big]x' + F(x'), \tag{14}$$

$$y'' = F(y') + \rho_1 y, \tag{15}$$

$$z'' = F(z') + \rho_2 z, \tag{16}$$

where ρ_1, ρ_2 are some functions.

Definition 3.3. *Let (M, F) be an almost complex manifold. A Riemannian metric g on M such that $g(FX, FY) = g(X, Y)$ or equivalently $g(FX, Y) = -g(X, FY)$ for any vector fields X, Y is called an almost Hermitian metric. The triple (M, F, g) is called an almost Hermitian manifold [9]. Also, for any vector field X, it follows that*

$$g(X, FX) = 0. \tag{17}$$

Lemma 3.4. *Let (M, F, g) be an almost Hermitian manifold and (T^2M, G_S) be its second order tangent bundle equipped with the Sasaki metric and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', Fy)$, $c_2 = |z'|$, $\mu_2 = g(z', Fz)$, then we have*

$$
\begin{array}{rcl}
\rho_1 & = & \mu_1 - c_1^2,\\
\rho_2 & = & \mu_2 - c_2^2,\\
c_1' & = & 0 = c_2',\\
\mu_1' & = & 0 = \mu_2'.
\end{array}
$$

Proof. From the formula (15), we obtain

$$y'' = \rho_1 y + F(y')$$

$$
\begin{array}{rcl}
g(y'', y) & = & g(F(y'), y) + \rho_1 g(y, y)\\
-|y'|^2 & = & -\mu_1 + \rho_1.
\end{array}
$$

Using Lemma 3.1 (2) and the formula (17), we have

$$
\begin{array}{rcl}
\frac{1}{2}(c_1^2)' & = & g(y'', y')\\
& = & \rho_1 g(y, y') + g(F(y'), y')\\
& = & \rho_1 g(y, y')\\
& = & 0.
\end{array}
$$

By Lemma 3.1 (2), Definition 3.3 and the formula (17), we obtain

$$\begin{aligned} \mu_1' &= g(y'', F(y)) + g(y', F(y')) \\ &= g(y'', F(y)) \\ &= \rho_1\, g(y, F(y)) + g(Fy', Fy) \\ &= 0. \end{aligned}$$

Similarly, we can obtain the other formulae. □

Using Lemma 3.2 and Lemma 3.4, we get the following theorem.

Theorem 3.5. *Let (M, F, g) be an almost Hermitian manifold and (T^2M, G_S) be its second order tangent bundle equipped with the Sasaki metric and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', Fy)$, $c_2 = |z'|$, $\mu_2 = g(z', Fz)$, then the curve $C(t) = S^{-1}(x(t), y(t), z(t))$ is an $F^{(0)}$-geodesic on $T^2_{1,1}M$ if and only if*

$$\begin{aligned} c_1 &= const., \quad \mu_1 = const. \quad and \quad \rho_1 = \mu_1 - c_1^2 = const., \\ c_2 &= const., \quad \mu_2 = const. \quad and \quad \rho_2 = \mu_2 - c_2^2 = const., \\ x'' &= -\Big[R(y, y') + R(z, z')\Big]x' + F(x'), \\ y'' &= F(y') + (\mu_1 - c_1^2)y, \\ z'' &= F(z') + (\mu_2 - c_2^2)z. \end{aligned}$$

From Theorem 2.12 and Lemma 3.1, we obtain the following lemma.

Lemma 3.6. *Let (M, g) be a Riemannian manifold, (T^2M, G_S) its second order tangent bundle equipped with the Sasaki metric and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. Then, C is an $F^{(0)}$-planar curve on $T^2_{1,1}M$ if and only if*

$$\begin{aligned} x'' &= -\Big[R(y, y') + R(z, z')\Big]x' + \eta_1 x' + \eta_2 F(x'), \\ y'' &= \eta_1 y' + \eta_2 F(y') + \rho_1\, y, \\ z'' &= \eta_1 z' + \eta_2 F(z') + \rho_2\, z, \end{aligned}$$

where η_1, η_2 are smooth functions on $\mathbb{R}$ and ρ_1, ρ_2 are some functions.

Now, we will determine the functions ρ_1 and ρ_2.

Lemma 3.7. *Let (M, g, F) be an almost Hermitian manifold, (T^2M, G_S) its second order tangent bundle equipped with the diagonal lift Sasaki metric and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', Fy)$, $c_2 = |z'|$, $\mu_2 = g(z', Fz)$, then we have*

$$\begin{aligned} c_1 &= K_1 \exp\left(\int \eta_1 dt\right), \qquad c_2 = K_3 \exp\left(\int \eta_1 dt\right), \\ \mu_1 &= K_2 \exp\left(\int \eta_1 dt\right), \qquad \mu_2 = K_4 \exp\left(\int \eta_1 dt\right), \\ \rho_1 &= \eta_2\, \mu_1 - c_1^2 = \eta_2\, K_2 \exp\left(\int \eta_1 dt\right) - K_1^2 \exp\left(2 \int \eta_1 dt\right), \\ \rho_2 &= \eta_2\, \mu_2 - c_2^2 = \eta_2\, K_3 \exp\left(\int \eta_1 dt\right) - K_4^2 \exp\left(2 \int \eta_1 dt\right), \end{aligned}$$

where η_1, η_2 are smooth functions on $\mathbb{R}$.

Proof. From the formula (15), we obtain

$$y'' = \rho_1\, y + \eta_1 y' + \eta_2 F(y'),$$

$$\begin{aligned} g(y'', y) &= \eta_1 g(y', y) + \eta_2 g(F(y'), y) + \rho_1\, g(y, y), \\ -|y'|^2 &= -\eta_2\, \mu_1 + \rho_1. \end{aligned}$$

Then $\rho_1 = \eta_2\, \mu_1 - c_1^2$.

Using the formula (17), we get

$$\begin{aligned} \frac{1}{2}(c_1^2)' &= g(y'', y') \\ &= \rho_1\, g(y, y') + \eta_1\, g(y', y') + \eta_2\, g(F(y'), y') \\ &= \eta_1\, g(y', y') \\ &= \eta_1\, c_1^2, \end{aligned}$$

from which we get $c_1 = K_1 \exp\left(\int \eta_1 dt\right)$.

On the other hand, we have

$$\begin{aligned} \mu_1' &= g(y'', Fy) + g(y', F(y')) \\ &= g(y'', Fy) \\ &= \rho_1\, g(y, Fy) + \eta_1 g(y', Fy) + \eta_2\, g(Fy', Fy) \\ &= \eta_1\, g(y', Fy) \\ &= \eta_1\, \mu_1, \end{aligned}$$

from which we get $\mu_1 = K_2 \exp\left(\int \eta_1 dt\right)$.

By the same way, we obtain the other formulas. □

Using Lemma 3.6 and Lemma 3.7, we obtain the following theorem.

Theorem 3.8. *Let (M, g, F) be an almost Hermitian manifold and let (T^2M, G_S) be its second order tangent bundle equipped with the diagonal lift Sasaki metric and let $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', Fy)$, $c_2 = |z'|$, $\mu_2 = g(z', Fz)$, then the curve $C(t) = S^{-1}(x(t), y(t), z(t))$ is an $F^{(0)}$-planar curve on $T^2_{1,1}M$ if and only if*

$$\begin{aligned} c_1 &= K_1 \exp\left(\int \eta_1 dt\right), \qquad c_2 = K_3 \exp\left(\int \eta_1 dt\right), \\ \mu_1 &= K_2 \exp\left(\int \eta_1 dt\right), \qquad \mu_2 = K_4 \exp\left(\int \eta_1 dt\right), \\ \rho_1 &= \eta_2\, \mu_1 - c_1^2 = \eta_2\, K_2 \exp\left(\int \eta_1 dt\right) - K_1^2 \exp\left(2 \int \eta_1 dt\right), \\ \rho_2 &= \eta_2\, \mu_2 - c_2^2 = \eta_2\, K_3 \exp\left(\int \eta_1 dt\right) - K_4^2 \exp\left(2 \int \eta_1 dt\right), \end{aligned}$$

$$\begin{aligned} x'' &= -\left[R(y, y') + R(z, z')\right]x' + \varrho_1(t)x' + \varrho_2(t)F(x'), \\ y'' &= \varrho_1(t)y' + \varrho_2(t)F(y') + (\eta_2\, \mu_1 - c_1^2)y, \\ z'' &= \varrho_1(t)z' + \varrho_2(t)F(z') + (\eta_2\, \mu_2 - c_2^2)z. \end{aligned}$$

Remark 3.9. *1) The Theorem 3.8 remains true if $F^{(0)}$-planar curve is replaced by $F^{(0)}$-geodesic up to reparameterization.*

2) In the case of $\eta_1 = 0$ and $\eta_2 = 1$ we obtain Theorem 3.5.

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