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The (l, r)-Stirling numbers: a combinatorial approach

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Abstract. This work deals with a new generalization of r-Stirling numbers using l-tuple of permutations and partitions called (l, r)-Stirling numbers of both kinds. We study various properties of these numbers using combinatorial interpretations and symmetric functions. Also, we give a limit representation of the multiple zeta function using (l, r)-Stirling of the first kind.

1. Introduction

Let σ be a permutation of the set $[n] = \{1, 2, \ldots, n\}$ having k cycles $c_1, c_2, \ldots, c_k$. A *cycle leaders set* of σ, denoted $cl(\sigma)$, is the set of the smallest elements on their cycles, i. e.

$$cl(\sigma) = \{\min c_1, \min c_2, \ldots, \min c_k\}.$$

As the same way, let π be a partition of the set $[n] = \{1, 2, \ldots, n\}$ into k blocks $b_1, b_2, \ldots, b_k$. A *block leaders set* of π, denoted $bl(\pi)$, is the set of the smallest elements on their blocks, i. e.

$$bl(\pi) = \{\min b_1, \min b_2, \ldots, \min b_k\}.$$

Example 1.1.

- *For* $n = 6$, *the permutation* $\sigma = (13)(245)(6)$ *have the set of cycle leaders* $cl(\sigma) = \{1, 2, 6\}$.
- *For* $n = 7$, *the partition* $\pi = 1, 2, 4|3, 5, 7|6$ *have the set of block leaders* $bl(\pi) = \{1, 3, 6\}$.

It is well known that the **Stirling numbers of the first kind**, denoted $\left[{n \atop k}\right]$, count the number of all permutations of $[n]$ having exactly k cycles, and **Stirling numbers of the second kind**, denoted $\left\{{n \atop k}\right\}$, count the number of all partitions of $[n]$ having exactly k blocks.

One of the most interesting generalization of Stirling numbers was the r-Stirling numbers of both kind introduced By Broder [6]. Analogously to the classical Stirling numbers of both kinds, the author considered

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that r-Stirling numbers of the first kind $\left[{n \atop k}\right]_r$ (resp. the second kind $\left\{{n \atop k}\right\}_r$) counts the number of permutations σ (resp. partitions π) having exactly k cycles (resp. k blocks) such that the r first elements $1, 2, \ldots, r$ lead.

Dumont, in [8], gives the first interpretation for the "central factorial" numbers of the second kind $U(n, k)$ given by the recurrence

$$U(n,k) = U(n-1,k-1) + k^2 U(n-1,k), \qquad \text{for } 0 < k \leq n. \tag{1}$$

Then, using the notion of *quasi-permutations*, Foata and Han [9] , showed that $U(n, k)$ counts the number of pair (π_1, π_2)-partitions of $[n]$ into k blocks such that $bl(\pi_1) = bl(\pi_2)$.

In this work, we give an extension of the r-Stirling numbers of both kinds with considering l-tuple partitions (and permutations) of Dumont's partition model [8, 9].

This paper is organized as follows. In Section 2 and Section 4, we introduce the (l, r)-Stirling numbers of both kinds. Some properties are given as recurrences, orthogonality, generating functions, a relation between (l, r)-Stirling numbers and Bernoulli polynomials via Faulhaber sums and symmetric functions. In Section 7, we show the relations between multiple-zeta function and the (l, r)-Stirling numbers. Finally, in Section 8, we discuss some remarks which connect this numbers to the rooks polynomials [3].

2. The (l, r)-Stirling numbers of both kinds

Let us consider the following generalization,

Definition 2.1. *The (l, r)-Stirling number of the first kind $\left[{n \atop k}\right]_r^{(l)}$ counts the number of l-tuple of permutations $(\sigma_1, \sigma_2, \ldots, \sigma_l)$ of $[n]$ having exactly k cycles such that $1, 2, \ldots, r$ first elements lead, and*

$$cl(\sigma_1) = cl(\sigma_2) = \cdots = cl(\sigma_l).$$

Definition 2.2. *The (l, r)-Stirling number of the second kind $\left\{{n \atop k}\right\}_r^{(l)}$ counts the number of l-tuple of partitions $(\pi_1, \pi_2, \ldots, \pi_l)$ of $[n]$ having exactly k blocks such that $1, 2, \ldots, r$ first elements lead, and*

$$bl(\pi_1) = bl(\pi_2) = \cdots = bl(\pi_l).$$

Theorem 2.3. *The (l, r)-Stirling numbers of the first satisfy the following recurrences*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r^{(l)} + (n-1)^l \begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(l)}, \qquad \text{for } n > r \tag{2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} = \frac{1}{(r-1)^l} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_{r-1}^{(l)} - \begin{bmatrix} n \\ k-1 \end{bmatrix}_r^{(l)} \right), \qquad \text{for } n \geq r > 1. \tag{3}$$

with boundary conditions $\left[{n \atop k}\right]_r^{(l)} = 0$, for $n < r$; and $\left[{n \atop k}\right]_r^{(l)} = \delta_{k,r}$, for $n = r$.

Proof. The $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutations of the set $[n]$ having k cycles such that $1, 2, \ldots, r$ first elements are in distinct cycles and $cl(\sigma_1) = cl(\sigma_2) = \cdots = cl(\sigma_l)$ is either obtained from:

- Inserting the *nth* elements after any element in each permutation of $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutations of the set $[n-1]$ having k cycles such that $1, 2, \ldots, r$ first elements are in distinct cycles and $cl(\sigma_1) = cl(\sigma_2) = \cdots = cl(\sigma_l)$, hence there are $(n-1)^l \left[{n-1 \atop k}\right]_r^{(l)}$ choices.

- The *nth* element forms a cycle in each permutation of $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutations, the remaining $[n-1]$ have to be $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permuted in $(k-1)$ cycles under the preceding conditions, hence there are $\left[{n-1 \atop k-1}\right]_r^{(l)}$.

This correspondence yields the first recurrence.

For the second recurrence, we use the double counting principle. Let us count the numbers of $(\sigma_1, \sigma_2, \dots, \sigma_l)$-permutations of the set $[n]$ having $(k-1)$ cycles such that $1, \dots, r-1$ are cycle leaders but r is not, with $cl(\sigma_1) = cl(\sigma_2) = \dots = cl(\sigma_l)$, this is either obtained from:

- We count the $(\sigma_1, \sigma_2, \dots, \sigma_l)$-permutations of the set $[n]$ having $(k-1)$ cycles such that $1, \dots, r-1$ are cycle leaders then we exclude from them the $(\sigma_1, \sigma_2, \dots, \sigma_l)$-permutations having r as cycle leader. That gives
$$\begin{bmatrix} n \\ k-1 \end{bmatrix}_{r-1}^{(l)} - \begin{bmatrix} n \\ k-1 \end{bmatrix}_{r}^{(l)},$$
- Or we count the $(\sigma_1, \sigma_2, \dots, \sigma_l)$-permutations of the set $[n]$ having k cycles such that $1, \dots, r$ are cycle leaders then we appending the cycle having r as leader at the end of a cycle having a smaller leader. We have $(r-1)$ choices to do in each permutation. That gives
$$(r-1)^l \begin{bmatrix} n \\ k \end{bmatrix}_{r}^{(l)},$$

from the two ways of counting we get the result. □

Theorem 2.4. *The (l, r)-Stirling numbers of the second satisfy the following recurrences*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{r}^{(l)} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{r}^{(l)} + k^l \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{r}^{(l)}, \qquad \text{for } n > r \tag{4}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{r}^{(l)} = \begin{Bmatrix} n \\ k \end{Bmatrix}_{r-1}^{(l)} - (r-1)^l \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{r-1}^{(l)}, \qquad \text{for } n \geq r > 1. \tag{5}$$

with boundary conditions $\left\{ {n \atop k} \right\}_r^{(l)} = 0$, for $n < r$; and $\left\{ {n \atop k} \right\}_r^{(l)} = \delta_{k,r}$, for $n = r$.

Proof. As in Theorem 2.3, the $(\pi_1, \pi_2, \dots, \pi_l)$-partitions of the set $[n]$ into k blocks such that $1, 2, \dots, r$ first elements are in distinct blocks and $bl(\pi_1) = bl(\pi_2) = \dots = bl(\pi_l)$ is either obtained from:

- Inserting the *nth* elements in a block of each partition of $(\pi_1, \pi_2, \dots, \pi_l)$-partitions of the set $[n-1]$ into k blocks such that $1, 2, \dots, r$ first elements are in distinct blocks and $bl(\pi_1) = bl(\pi_2) = \dots = bl(\pi_l)$, hence there are $k^l \left\{ {n-1 \atop k} \right\}_r^{(l)}$ choices (the position of the *nth* element in a block doesn't matter).
- The *nth* element forms a block in each partition of $(\pi_1, \pi_2, \dots, \pi_l)$-partitions, the remaining $[n-1]$ have to be $(\pi_1, \pi_2, \dots, \pi_l)$-partitioned into $(k-1)$ blocks under the preceding conditions, hence there are $\left\{ {n-1 \atop k-1} \right\}_r^{(l)}$.

For the Identity (5), we use the double counting principle to count the numbers of $(\pi_1, \pi_2, \dots, \pi_l)$-partitions of $[n]$ into k blocks such that $1, 2, \dots, (r-1)$ are block leaders but r is not, with $bl(\pi_1) = bl(\pi_2) = \dots = bl(\pi_l)$, this is either obtained from:

- We count the $(\pi_1, \pi_2, \dots, \pi_l)$-partitions of the set $[n]$ into k blocks such that $1, \dots, r-1$ are block leaders then we exclude from them the $(\pi_1, \pi_2, \dots, \pi_l)$-partitions having r as block leader, with $bl(\pi_1) = bl(\pi_2) = \dots = bl(\pi_l)$. That gives
$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{r-1}^{(l)} - \begin{Bmatrix} n \\ k \end{Bmatrix}_{r}^{(l)},$$

- Or we count the $(\pi_1, \pi_2, \dots, \pi_l)$-partitions of the set $[n]\backslash\{r\}$ into k blocks such that $1, \dots, r-1$ are block leaders then we include the element $\{r\}$ in any block having a smaller leader then r. We have $(r-1)$ choices to do in each partition of $(\pi_1, \pi_2, \dots, \pi_l)$-partitions, that gives
$$(r-1)^l \left\{ {n-1 \atop k} \right\}_{r-1}^{(l)},$$
from the two ways of counting we get the result.

□

Remark 2.5. *Using the previous recurrences it is easy to get the following special cases*
$$\left[{n \atop r} \right]_r^{(l)} = r^l (r+1)^l \cdots (n-2)^l (n-1)^l = (r^{\overline{n-r}})^l, \qquad \text{for } n \geq r \tag{6}$$
and
$$\left\{ {n \atop r} \right\}_r^{(l)} = r^{l(n-r)}, \qquad \text{for } n \geq r. \tag{7}$$

3. Orthogonality of (l, r)-Stirling numbers pair

Theorem 3.1. *For $n \geq k \geq 0$, for all positive integer l, we have the two orthogonality relations bellow*
$$\sum_j \left[{n \atop j} \right]_r^{(l)} \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j = \begin{cases} (-1)^n \delta_{n,k}, & \text{for } n \geq r; \\ 0, & \text{for } n < r \end{cases} \tag{8}$$
and
$$\sum_j \left[{j \atop n} \right]_r^{(l)} \left\{ {k \atop j} \right\}_r^{(l)} (-1)^j = \begin{cases} (-1)^n \delta_{n,k}, & \text{for } n \geq r; \\ 0, & \text{for } n < r. \end{cases} \tag{9}$$

Proof. Let us start by Identity (8). The proof goes by induction on n

- For $n < r$ the assertion is obvious.
- For $n = r$,
$$\sum_j \left[{r \atop j} \right]_r^{(l)} \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j = \left\{ {r \atop k} \right\}_r^{(l)} (-1)^r = (-1)^r \delta_{k,r}.$$
- For $n > r$, Theorem 2.3 and the induction hypothesis implies that
$$\begin{aligned}\sum_j \left[{n \atop j} \right]_r^{(l)} \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j &= \sum_j \left(\left[{n-1 \atop j-1} \right]_r^{(l)} + (n-1)^l \left[{n-1 \atop j} \right]_r^{(l)} \right) \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j \\ &= (n-1)^l (-1)^{n-1} \delta_{n-1,k} + \sum_j \left[{n-1 \atop j-1} \right]_r^{(l)} \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j,\end{aligned}$$
and from Theorem 2.4, we get
$$\sum_j \left[{n \atop j} \right]_r^{(l)} \left\{ {j \atop k} \right\}_r^{(l)} (-1)^j = (n-1)^l (-1)^{n-1} \delta_{n-1,k} - (-1)^{n-1} \delta_{n-1,k-1} - (k)^l (-1)^{n-1} \delta_{n-1,k} = (-1)^n \delta_{n,k}.$$

For the Identity (9), we go by induction on k as same as the previous proof. □

4. Properties via symmetric functions

Let $x_1, x_2, \ldots, x_n$ be n random variables. We denote, respectively, by $e_k(x_1, x_2, \ldots, x_n)$ and $h_k(x_1, x_2, \ldots, x_n)$ the elementary symmetric function and the complete homogeneous symmetric function of degree k in n-variables given for $n \geq k \geq 1$, by

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \tag{10}$$

and

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}. \tag{11}$$

In particular $e_0(x_1, x_2, \ldots, x_n) = h_0(x_1, x_2, \ldots, x_n) = \delta_{0,n}$.
The generating functions of the symmetric functions are given by

$$E(t) = \sum_{k \geq 0} e_k(x_1, x_2, \cdots, x_n) t^k = \prod_{i=1}^{n} (1 + x_i t) \tag{12}$$

and

$$H(t) = \sum_{k \geq 0} h_k(x_1, x_2, \cdots, x_n) t^k = \prod_{i=1}^{n} (1 - x_i t)^{-1}. \tag{13}$$

For more details about symmetric functions we refer readers to [2, 13, 15] and the references therein.
Let us now give some results linked to the symmetric functions and their generating functions.

Theorem 4.1. *The (l, r)-Stirling of the first kind and the elementary symmetric function are linked as*

$$\begin{bmatrix} n+1 \\ n+1-k \end{bmatrix}_r^{(l)} = e_k(r^l, \ldots, n^l), \tag{14}$$

equivalently

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} = e_{n-k}(r^l, \ldots, (n-1)^l). \tag{15}$$

Proof. It is clear that in each $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutation having $(n-k)$ cycles with $\{1, \ldots, r\}$ lead, we have $\{1, 2, \ldots, r, y_{r+1}, \ldots, y_{n-k}\}$ lead a cycle and $\{x_1, x_2, \ldots, x_k\}$ elements don't lead where $r < y_{r+1} < y_{n-k} < \cdots \leq n$ and $r < x_1 < x_2 < \cdots \leq n$.

To construct all $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutations having $(n-k)$ cycles where $\{1, \ldots, r\}$ lead, we proceed as follows

- Construct $(n-k)$ cycles having only one element from $\{1, 2, \ldots, r, y_{r+1}, \ldots, y_{n-k}\}$, i. e.

$$\sigma = (1)(2) \ldots (r)(y_{r+1}) \ldots (y_{n-k}),$$

- Insert x_1 after an element of cycles smaller than x_1, we have $(x_1 - 1)$ ways of inserting x_1. Then Insert x_2 after an element of cycles smaller than x_2, we have $(x_2 - 1)$ choices, and so on. We have $(x_1 - 1)(x_2 - 1) \cdots (x_k - 1)$ ways to construct a permutation.

- Repeat the process with each permutation $\sigma \in \{\sigma_1, \ldots, \sigma_l\}$, so we have $(x_1 - 1)^l (x_2 - 1)^l \cdots (x_k - 1)^l$ ways of construction.

- Summing over all possible set of numbers $\{x_1, x_2, \ldots, x_k\}$, hence the total number of ways to construct $(\sigma_1, \sigma_2, \ldots, \sigma_l)$-permutations having $(n-k)$ cycles with $\{1, \ldots, r\}$ lead is

$$\begin{aligned}\left[{n \atop n-k}\right]_r^{(l)} &= \sum_{r<x_1<x_2<\cdots\leq n} (x_1-1)^l (x_2-1)^l \cdots (x_k-1)^l \\ &= \sum_{r\leq x_1<x_2<\cdots<n} x_1^l x_2^l \cdots x_k^l \\ &= e_k(r^l, \ldots, (n-1)^l).\end{aligned}$$

□

Theorem 4.2. *The (l, r)-Stirling of the first kind and the complete homogeneous symmetric function are linked as*

$$\left\{{n+k \atop n}\right\}_r^{(l)} = h_k(r^l, \ldots, n^l), \tag{16}$$

Proof. Let us count the number of $(\pi_1, \pi_2, \ldots, \pi_l)$-partitions of $[n+k]$ into n blocks with $\{1, 2, \ldots, r\}$ are leaders. First, we denote, $\{y_1, y_2, \ldots, y_k\}$ the elements that are not leaders where $y_1 < y_2 < \cdots < y_k$. Let x_i be the number of leaders smaller than y_i, $i \in \{1, \ldots, k\}$, it is clear that $r \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$.

The construction of such partition goes as follows

- Construct a partition of n blocks with $[n+k]\backslash\{y_1, y_2, \ldots, y_k\}$ where $1, 2, \ldots, r$ are leaders, i. e.

$$\{1\}\{2\}\ldots\{r\}\{z_{r+1}\}\ldots\{z_n\}.$$

- Insert the $\{y_1, y_2, \ldots, y_k\}$ elements to the n blocks. It is clear that y_i can belong only to a block having a leader smaller than y_i, we have $x_1 \cdot x_2 \cdots x_k$ ways to do.

- Repeat the process with each partition $\pi \in \{\pi_1, \ldots, \pi_l\}$, so we have $(x_1)^l (x_2)^l \ldots (x_k)^l$ ways of construction.

- Summing over all possible set of numbers $\{x_1, x_2, \ldots, x_k\}$, hence the total number of ways to construct $(\pi_1, \pi_2, \ldots, \pi_l)$-partitions of $[n+k]$ having n blocks with $\{1, \ldots, r\}$ lead is

$$\begin{aligned}\left\{{n+k \atop n}\right\}_r^{(l)} &= \sum_{r\leq x_1\leq x_2\leq\cdots\leq n} x_1^l x_2^l \cdots x_k^l \\ &= h_k(r^l, \ldots, n^l).\end{aligned}$$

□

5. Generating functions

Now, we can use the symmetric functions to construct the generating functions for the (l, r)-Stirling of both kinds.

Theorem 5.1. *The generating function for the (l, r)-Stirling numbers of the first kind is*

$$\sum_k \left[{n \atop k}\right]_r^{(l)} z^k = z^r \prod_{i=r}^{n-1} \left(z + i^l\right) = z^r \left(z + r^l\right)\left(z + (r+1)^l\right)\cdots\left(z + (n-1)^l\right), \tag{17}$$

Proof. From Theorem 4.1 and the generating function (12) we obtain

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} z^k = z^n \sum_k e_k(r^l, \ldots, (n-1)^l)(z^{-1})^k = z^n \prod_{i=r}^{n-1} \left(1 + \frac{i^l}{z}\right) = z^r \prod_{i=r}^{n-1} (z + i^l). \tag{18}$$

□

Theorem 5.2. *The generating function for the (l, r)-Stirling numbers of the second kind is*

$$\sum_{n=k} \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(l)} z^n = z^k \left(\prod_{i=r}^{k} (1 - zi^l) \right)^{-1} = \frac{z^k}{(1 - zr^l)(1 - z(r+1)^l)(1 - zk^l)}. \tag{19}$$

Proof. From Theorem 4.2 and the generating function of homogeneous symmetric function (13), we obtain

$$\sum_{n \geq k} \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(l)} z^n = \sum_{j \geq 0} \begin{Bmatrix} k+j \\ k \end{Bmatrix} z^{k+j} = z^k \sum_{j \geq 0} h_j(r^l, \ldots, k^l) z^j = z^k \left(\prod_{i=r}^{k} (1 - zi^l) \right)^{-1}.$$

□

In the following theorem we investigate the symmetric functions to obtain a convolution formula for the (l, r)-Stirling numbers of both kinds.

Theorem 5.3. *For all positive integers l, n, k and r with $(n \geq k \geq r)$, we have*

$$\sum_{\substack{i_0 + 2i_1 \cdots + 2^l i_l = k \\ i_0, \cdots, i_l \geq 0}} \begin{Bmatrix} n + i_l \\ n \end{Bmatrix}_r^{(2^l)} \prod_{s=0}^{l-1} \begin{bmatrix} n+1 \\ n+1-i_s \end{bmatrix}_r^{(2^s)} = \begin{Bmatrix} n+k \\ n \end{Bmatrix}_r. \tag{20}$$

Proof. Let us consider the generating function of the complete homogeneous symmetric function (13). From that we have

$$\begin{aligned}
\sum_{k \geq 0} h_k(x_1, \ldots, x_n) z^k &= \prod_{i=1}^{n} \frac{1}{(1 - x_i z)} \\
&= \prod_{i=1}^{n} \frac{1}{(1 - x_i z)} \prod_{s=0}^{l-1} \left(\frac{1 + x_i^{2^s} z^{2^s}}{1 + x_i^{2^s} z^{2^s}} \right) \\
&= \prod_{i=1}^{n} \frac{1}{(1 - x_i^{2^l} z^{2^l})} \prod_{s=0}^{l-1} \left(1 + x_i^{2^s} z^{2^s}\right) \\
&= \sum_{k \geq 0} h_k(x_1^{2^l}, \ldots, x_n^{2^l}) z^{2^l k} \prod_{s=0}^{l-1} \sum_{k \geq 0} e_k(x_1^{2^s}, \ldots, x_n^{2^s}) z^{2^s k}
\end{aligned}$$

$$= \sum_{k\geq 0} h_k(x_1^{2^l},\ldots,x_n^{2^l})z^{2^l k} \sum_{k\geq 0} e_k(x_1,\ldots,x_n)z^k \sum_{k\geq 0} e_k(x_1^2,\ldots,x_n^2)z^{2k} \cdots \sum_{k\geq 0} e_k(x_1^{2^{l-1}},\ldots,x_n^{2^{l-1}})z^{2^{l-1}k}$$

$$= \sum_{k\geq 0} \left(\sum_{\substack{i_0+2i_1+\cdots+2^l i_l = k;\\ i_0,\ldots,i_l\geq 0.}} h_{i_l}(x_1^{2^l},\ldots,x_n^{2^l}) \prod_{s=0}^{l-1} e_{i_s}(x_1^{2^s},\ldots,x_n^{2^s}) \right) z^k.$$

From Theorem 4.1 and Theorem 4.2 and by comparing the coefficients of z^k of the two sides the result holds true. □

The simplest case of the previous theorem is the corollary bellow which generalize the result of Broder [6].

Corollary 5.4. *For $l = 1$, we have*

$$\sum_{i=0}^{\lfloor k/2\rfloor} \left\{ {n+i \atop n} \right\}_r^{(2)} \left[{n+1 \atop n+1+2i-k} \right]_r = \left\{ {n+k \atop n} \right\}_r. \tag{21}$$

6. The (l,r)-Stirling numbers, the sum of powers, and Bernoulli polynomials

Recall, for every integer $n \geq 0$, the Bernoulli polynomials, denoted $B_n(x)$, are defined by

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n} = \frac{te^{xt}}{e^t-1}. \tag{22}$$

The sum of the powers of natural numbers is closely related to the Bernoulli polynomials $B_n(x)$. Jacobi [12, 16] gives the following identity using the sum of powers and Bernoulli polynomials

$$\sum_{j=1}^{n} j^m = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1}. \tag{23}$$

The following theorem gives the relation between (l,r)-Stirling of both kinds and Bernoulli polynomials.

Theorem 6.1. *For all positive integers n, k and l, we have*

$$\sum_{j=0}^{k} (-1)^j (j+1) \left[{n+1 \atop n-j} \right]^{(l)} \left\{ {n+k-j \atop n} \right\}^{(l)} = \frac{B_{lk+l+1}(n+1) - B_{lk+l+1}(0)}{lk+l+1}, \tag{24}$$

Proof. In the first hand we have Jacobi's Identity (23)

$$\sum_{j=1}^{n} (j^l)^k = \frac{B_{lk+1}(n+1) - B_{lk+1}(0)}{lk+1}, \tag{25}$$

in the second hand, we have

$$H(t) = \sum_{k\geq 0} h_k(1^l, 2^l, \ldots, n^l)t^k = \prod_{j=1}^{n} \frac{1}{(1-j^s t)}$$

and

$$E(t) = \sum_{k\geq 0} e_k(1^l, 2^l, \ldots, n^l)t^k = \prod_{j=1}^{n} (1+j^s t),$$

from the obvious observation that $H(t) = 1/E(-t)$, we obtain

$$\frac{d}{dt}\ln H(t) = \frac{H'(t)}{H(t)} = H(t)E'(-t) \tag{26}$$

but

$$\frac{d}{dt}\ln H(t) = \sum_{j=1}^{n} \frac{j^l}{(1-j^l t)} = \sum_{k\geq 0}\sum_{j=1}^{n} j^{s(k+1)} t^k. \tag{27}$$

Then from equations (26) and (27), we get

$$\begin{aligned}\sum_{k\geq 0}\sum_{j=1}^{n} j^{s(k+1)} t^k &= H(t)E'(-t)\\ &= \left(\sum_{k\geq 0} h_k(1^l,\ldots,n^l)t^k\right)\left(\sum_{k\geq 1} k(-1)^{k-1} e_k(1^l,\ldots,n^l)t^{k-1}\right).\end{aligned} \tag{28}$$

Cauchy product and equating coefficient of t^k gives

$$\sum_{j=1}^{n} j^{s(k+1)} = \sum_{j\geq 1}^{n} (j+1)(-1)^j e_{j+1}(1^l,\ldots,n^l) h_{k-j}(1^l,\ldots,n^l), \tag{29}$$

replacing symmetric functions by stirling numbers from Theorem 4.1 and Theorem 4.2, and comparing with Equation (25) we get the result.
□

7. Multiple zeta function and (l, r)-Stirling numbers of the first kind

For any ordered sequence of positive integers $i_1, i_2, \ldots, i_k$, the *multiple zeta function* is introduced by Hoffman [11] and independently Zagier [17] by the following infinite sums

$$\zeta(i_1, i_2, \ldots, i_k) = \sum_{0<j_1<j_2<\cdots<j_k} \frac{1}{j_1^{i_1} j_2^{i_2}\cdots j_k^{i_k}}. \tag{30}$$

Recently, the multiple zeta function has been studied quite intensively by many authors in various fields of mathematics and physics (see [4, 5, 7, 11, 17, 19]). Here we give a relation between (l, r)-Stirling numbers of the first kind and the multiple zeta function.

Theorem 7.1. *For all positive integers n, k, l and r with* $(n \geq k \geq r)$*, we have*

$$\begin{aligned}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_r^{(l)} &= \left(\frac{n!}{(r-1)!}\right)^l \sum_{j_k=k}^{n}\sum_{j_{k-1}=k-1}^{j_k-1}\cdots\sum_{j_r=r}^{j_{(r+1)}-1} \frac{1}{(j_r j_2\cdots j_k)^l}\\ &= \left(\frac{n!}{(r-1)!}\right)^l \sum_{r-1<j_1<j_2<\cdots<j_k\leq n} \frac{1}{(j_1 j_2\cdots j_k)^l}.\end{aligned} \tag{31}$$

Proof. Since $\begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r^{(l)} + (n-1)^l \begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(l)}$ from Theorem 2.3. If we proceed iteratively, we obtain that

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(l)} = ((n-1)!)^l \sum_{j=k-1}^{n-1} \frac{1}{(j!)^l}\begin{bmatrix} j \\ k-1 \end{bmatrix}_r^{(l)}. \tag{32}$$

For $k = r$, from (6) and (32) we obtain

$$\begin{bmatrix} n \\ r \end{bmatrix}_r^{(l)} = (r^{\overline{n-r}})^l = \left(\frac{(n-1)!}{(r-1)!}\right)^l. \tag{33}$$

For $k = r + 1$, from (32) and (33) we obtain

$$\begin{aligned} \begin{bmatrix} n \\ r+1 \end{bmatrix}_r^{(l)} &= ((n-1)!)^l \sum_{j=r}^{n-1} \frac{1}{(j!)^l} \begin{bmatrix} j \\ r \end{bmatrix}_r^{(l)} \\ &= \left(\frac{(n-1)!}{(r-1)!}\right)^l \sum_{j=r}^{n-1} \left(\frac{(j-1)!}{j!}\right)^l \\ &= \left(\frac{(n-1)!}{(r-1)!}\right)^l \sum_{j=r}^{n-1} \frac{1}{j^l}. \end{aligned} \tag{34}$$

For $k = r + 2$, from (33) and (34) we obtain

$$\begin{bmatrix} n \\ r+2 \end{bmatrix}_r^{(l)} = \left(\frac{(n-1)!}{(r-1)!}\right)^l \sum_{j=r+1}^{n-1} \sum_{i=r}^{j-1} \frac{1}{(ij)^l}, \tag{35}$$

iterating the process with $k \in \{r + 3, r + 4, \dots\}$ and so on, then yields the result. □

Proposition 7.2. *For $r = 1$, we have*

$$\lim_{n\to\infty} \frac{1}{(n!)^l} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}^{(l)} = \zeta(\{l\}_k), \tag{36}$$

where $\{l\}_n = (\underbrace{l, l, \dots, l}_{n \text{ times}})$.

Proof. The proposition follows immediately from the definition of multiple zeta function (30) as an infinity sums and Theorem 7.1 for $r = 1$. □

Corollary 7.3. *For $k \geq 1$, we have*

- *For $l = 2$*

$$\lim_{n\to\infty} \frac{1}{(n!)^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}^{(2)} = \frac{\pi^{2k}}{(2k+1)!}. \tag{37}$$

- *For $l = 4$*

$$\lim_{n\to\infty} \frac{1}{(n!)^4} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}^{(4)} = \frac{4(2\pi)^{4k}}{(4k+2)!} \left(\frac{1}{2}\right)^{2k+1}. \tag{38}$$

- *For* $l = 6$

$$\lim_{n\to\infty}\frac{1}{(n!)^6}\begin{bmatrix}n+1\\k+1\end{bmatrix}^{(6)}=\frac{6(2\pi)^{6k}}{(6k+3)!}. \tag{39}$$

- *For* $l = 8$

$$\lim_{n\to\infty}\frac{1}{(n!)^8}\begin{bmatrix}n+1\\k+1\end{bmatrix}^{(8)}=\frac{\pi^{8k}}{(8k+4)!}2^{8k+3}\left(\left(1+\frac{1}{\sqrt{2}}\right)^{4k+2}+\left(1-\frac{1}{\sqrt{2}}\right)^{4k+2}\right). \tag{40}$$

Proof. Authors in [7] give the following special values of multiple zeta function

$$\begin{aligned}
\zeta(\{2\}_n) &= \frac{\pi^{2n}}{(2n+1)!},\\
\zeta(\{4\}_n) &= \frac{4(2\pi)^{4n}}{(4n+2)!}\left(\frac{1}{2}\right)^{2n+1},\\
\zeta(\{6\}_n) &= \frac{6(2\pi)^{6n}}{(6n+3)!},\\
\zeta(\{8\}_n) &= \frac{\pi^{8n}}{(8n+4)!}2^{8n+3}\left(\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2}+\left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right),
\end{aligned}$$

the corollary is a consequence of the previous special cases and Proposition 7.2. □

8. Remarks

- The (l, r)-Stirling gives another graphical view of Rooks polynomials of higher dimensions in triangle boards [3, 18] using set partitions.
- In this work we gives a limit representation of multiple zeta function using (l, r)-Stirling numbers.
- We can obtain the well-known Euler identity $\zeta(2) = \frac{\pi^2}{6}$ from Equation (37) for $k = 1$.

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