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On the continuity of the solution to the Minkowski problem for L_p torsional measure

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Abstract. This paper deals with on the continuity of the solution to the Minkowski problem for L_p torsional measure. For $p \in (1, n+2) \cup (n+2, \infty)$, we show that a sequence of convex bodies in $\mathbb{R}^n$ is convergent in Hausdorff metric if the sequence of the L_p torsional measures (associated with these convex bodies) is weakly convergent. Moreover, we also prove that the solution to the Minkowski problem for L_p torsional measure is continuous with respect to p.

1. Introduction

The surface area measure of a convex body (compact convex set with non-empty interior) and it's L_p extension (see [25]) is important concept in convex geometry, and received a great attention. For a convex body K in $\mathbb{R}^n$ and any Borel set ω on the unit sphere S^{n-1}, the L_p surface area measure of K with $p \in \mathbb{R}$ is given by

$$S_p(K, \omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where ν_K is the Gauss map from the boundary of K (denoted by ∂K) to S^{n-1}, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Especially, when $p = 1$, $S_1(K, \omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$ is the classical surface area measure of convex body K on Borel set $\omega \subset S^{n-1}$. For any sequence of convex bodies $\{K_i\}_{i \geq 1}$ and a convex body K containing the origin in their interiors, it has been proved that $S_p(K_i, \cdot) \to S_p(K, \cdot)$ weakly as $K_i \to K$ in Hausdorff metric. However, the opposite problem is also very interest: Does $K_i \to K$ hold in Hausdorff metric as $S_p(K_i, \cdot) \to S_p(K, \cdot)$ weakly? For all real number $p \in \mathbb{R}$, it may not always be positive. It's lucky that the opposite problem is correct if $p = 1$ (see [29]) and if $p > 1$ and $p \neq n$ by Zhu (see [49]).

Associated with the L_p surface area measure, there is a hot topic in convex geometry, i.e., the L_p Minkowski problem which aims to find the conditions for a finite measure μ on S^{n-1} such that there exists a convex body with L_p surface area measure being μ (see [25]): *For real number $p \in \mathbb{R}$ and a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions of μ such that μ is the L_p surface area measure of a convex*

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body? In recent decades, the L_p Minkowski problem has been a core object of interest in convex geometric analysis and receives a great attention (see e.g., [2, 4, 5]). The existence and uniqueness of the solution to the L_p Minkowski problem were studied (see e.g., [8, 10, 19, 20, 23, 27, 30–32, 46–48, 50]).

The solution to the L_p Minkowski problem have many important applications on the affine isoperimetric inequalities (see e.g., [11, 14–16, 26, 37, 38, 42]). Recently, the study on the solutions to the L_p Minkowski problem have also been considered by Zhu [49]. He proved that the continuity of the solution to the L_p Minkowski problem with $p > 1$ but $p \neq n$. When $p = 0$ and $0 < p < 1$, some results about the continuity associated with the L_p Minkowski problem were obtained (see e.g., [34, 35]).

As a new central object of dual Brunn-Minkowski theorem, the qth dual curvature measure has been introduced by Huang, Lutwak, Yang and Zhang [17]. They also posed the dual Minkowski problem for the qth dual curvature measure: *Given a nonzero finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, can we find a convex body K such that the qth dual curvature measure of K is μ?* The existence of solutions to the dual Minkowski problem for even measure μ and $q \in (0, n]$ has been proved in [17]. Later, the existence and uniqueness of the solution to the dual Minkowski problem for $q < 0$ were provided by Zhao [43]. One can refer [1, 3, 6, 7, 18, 22, 28, 44, 45] and reference therein for more works on the solution to the dual Minkowski problem. Motivated by the continuity of the solution to the L_p Minkowski problem [49], the authors considered the continuity of the solution to the dual Minkowski problem for $q < 0$ [33, 36]. The continuity of the solution to the Minkowski problem associated with the L_p $\mathbf{p}$-capacitary measure for $1 < p < \infty$ and $1 < \mathbf{p} < n$ is also considered [39] (see also [24, 40, 41, 51] for more information).

Similar to the surface area measure and qth dual curvature measure, the torsional measure of a convex body K (denoted by $\mu_T(K, \cdot)$) is also an important object of interest in convex geometry. A solution to the Minkowski problem associated with the torsional measure $\mu_T(K, \cdot)$ was provided in [12]. Recently, when $p \geq 1$, the L_p torsional measure of a convex body K (see e.g., [9, 21]), denoted by $\mu_{T,p}(K, \cdot)$, was introduced as follows

$$\mu_{T,p}(K, \cdot) = h_K^{1-p}(\cdot)\mu_T(K, \cdot),$$

where $h_K(\cdot)$ is the support function of convex body K (see section 2 for unexplained definitions). Especially, $\mu_{T,1}(K, \cdot) = \mu_T(K, \cdot)$.

In this paper, we will show that the weak convergence of L_p torsional measures implies the convergence of the corresponding convex bodies.

Theorem 1.1. *Let $p > 1$ with $p \neq n + 2$ and $\Omega_i, \Omega \subset \mathbb{R}^n (i = 1, 2, \cdots)$ be convex bodies containing the origin in their interiors. If the sequence of L_p torsional measure $\mu_{T,p}(\Omega_i, \cdot)$ converges to $\mu_{T,p}(\Omega, \cdot)$ weakly, then Ω_i converges to Ω in the Hausdroff metric.*

The Minkowski problem associated with the L_p torsional measure was investigated (see [9, 21]) which can be stated as follows.

The Minkowski problem associated with L_p torsional measure: For fixed $p \geq 1$ and a given non-negative finite Borel measure μ on S^{n-1}, under what conditions there exists a unique convex body Ω such that $\mu_{T,p}(\Omega, \cdot) = \mu$?

As mentioned above, this problem was proved by Colesanti and Fimiani [12] for $p = 1$. In [9], the authors provided a solution to this problem for $p > 1$ and $p \neq n + 2$. One can refer to [21] for the solution to this problem for more general measure. Therefore, there is a natural question whether such solution is continuous with respect to p. In this paper, we also show the continuity for $p > 1$ and $p \neq n + 2$.

Theorem 1.2. *Let $p, p_i \in (1, n + 2) \cup (n + 2, \infty)$ with $p_i \to p$. Let μ be a Borel measure on S^{n-1}. If a convex body Ω containing the origin in its interior is the solution to the Minkowski problem associated with the L_p torsional measure for μ and the sequence of convex bodies Ω_i containing the origin in their interiors is the solution to the Minkowski problem associated with the L_{p_i} torsional measure for μ, then $\Omega_i \to \Omega$ as $p_i \to p$.*

2. Preliminaries and notations

In this section, we will collect some basic concepts and notations in convex geometry. For more details and more concepts on convex geometry, please refer to [13, 29].

We call a compact and convex subset with non-empty interiors as a convex body in $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the set of convex bodies containing the origin o in their interiors. The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of x. The origin-centered unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ in $\mathbb{R}^n$ and the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ are denoted by B_2^n and S^{n-1}, respectively. The volume of a convex body K is denoted by $|K|$.

For a compact convex set Ω, its support function is defined by

$$h_\Omega(x) = \max\{x \cdot y : y \in \Omega\}, \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to check that $h_{c\Omega}(x) = ch_\Omega(x)$ for $c > 0$ and $x \in \mathbb{R}^n$, here $c\Omega = \{cx : x \in \Omega\}$. Let $\mathrm{diam}(\Omega)$ be the diameter of Ω is given by

$$\mathrm{diam}(\Omega) = \sup\{|x - y| : \forall x, y \in \Omega\}.$$

Two compact convex sets Ω and Ω' in $\mathbb{R}^n$ are said to be homothetic to each other if $\Omega = c\Omega' + x_0$ for some constant $c > 0$ and any point $x_0 \in \mathbb{R}^n$. In particular, Ω and Ω' are said to be dilates to each other if x_0 is the origin. The Hausdroff metric between Ω and Ω' is defined as

$$d_H(\Omega, \Omega') = \max_{u \in S^{n-1}} |h_\Omega(u) - h_{\Omega'}(u)| = \|h_\Omega - h_{\Omega'}\|_\infty.$$

Let $W^{1,2}(\Omega)$ be the Sobolev space of those functions having weak derivatives up to the second order in $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$ is the set of functions in $W^{1,2}(\Omega)$ having compact support. We use $C_c^\infty(\mathbb{R}^n)$ to denote the class of all infinitely differentiable functions with compact support in $\mathbb{R}^n$. For a convex body Ω, the torsional rigidity $T(\Omega)$ of Ω is defined by

$$\frac{1}{T(\Omega)} = \inf\left\{\frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u| dx)^2} : u \in W_0^{1,2}(\Omega), \int_\Omega |u| dx > 0\right\},$$

where ∇u is the gradient of u. It has been proved that if u is the unique solution of the boundary-value problem

$$\begin{cases} \Delta u = -2 \text{ in } \Omega, \\ u = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

then

$$T(\Omega) = \int_\Omega |\nabla u|^2 dx.$$

Moreover, one can obtain the upper bound of $|\nabla u|$.

Lemma 2.1. *(see [12]) Let Ω be an open bounded convex subset of $\mathbb{R}^n$. If u is the solution of the problem (2.1) in Ω, then*

$$|\nabla u(x)| \leq \mathrm{diam}(\Omega), \forall x \in \Omega.$$

The definition of the torsional rigidity shows that $T(a\Omega) = a^{n+2}T(\Omega)$ for any $\Omega \in \mathcal{K}_o^n$ and $a > 0$. The torsional measure $\mu_T(\Omega, \cdot)$ is a nonnegative Borel measure on S^{n-1} which can be defined as (see [12]): for any measurable subset $\omega \subset S^{n-1}$,

$$\mu_T(\Omega, \omega) = \int_{\nu_\Omega^{-1}(\omega)} |\nabla u(x)|^2 d\mathcal{H}^{n-1}(x). \tag{2.2}$$

Obviously, for $a > 0$

$$\mu_T(a\Omega, \cdot) = a^{n+1}\mu_T(\Omega, \cdot) \text{ on } S^{n-1}.$$

In addition, $\mu_T(\Omega, \cdot)$ is not concentrated on any closed hemisphere of S^{n-1}, i.e.,

$$\int_{S^{n-1}} (v \cdot u)_+ d\mu_T(\Omega, u) > 0 \text{ for any } v \in S^{n-1},$$

where $(v \cdot u)_+ = \max\{v \cdot u, 0\}$.

From (2.2), we have the relation between $\mu_T(\Omega, \cdot)$ and $S(\Omega, \cdot)$ as follows

$$d\mu_T(\Omega, v) = |\nabla u(\nu_\Omega^{-1}(v))|^2 dS(\Omega, v) \text{ for any } v \in S^{n-1}. \tag{2.3}$$

Based on the relation (2.3), the L_p torsional measure of Ω with $p > 1$ was induced as follows (see [9, 21])

$$\mu_{T,p}(\Omega, \omega) = \int_{x \in \nu_\Omega^{-1}(\omega)} (x \cdot \nu_\Omega(x))^{1-p} |\nabla u(x)|^2 d\mathcal{H}^{n-1}(x),$$

for any Borel set ω on the unit sphere S^{n-1}. It has also been proved that the weak convergence of the L_p torsional measure, i.e., if a sequence of convex bodies $\Omega_i \in \mathcal{K}_o^n (i = 1, 2, \cdots)$ converges to a convex body Ω, then

$$\mu_{T,p}(\Omega_i, \cdot) \to \mu_{T,p}(\Omega, \cdot) \text{ weakly on } S^{n-1} \text{ as } i \to \infty. \tag{2.4}$$

Moreover, the L_p mixed torsional rigidity $T_p(\Omega_1, \Omega_2)$ of the convex bodies $\Omega_1, \Omega_2 \in \mathcal{K}_o^n$ for $p > 1$ was given by

$$T_p(\Omega_1, \Omega_2) = \frac{1}{n+2} \int_{S^{n-1}} h_{\Omega_2}(u)^p d\mu_{T,p}(\Omega_1, u).$$

Especially, for $\Omega \in \mathcal{K}_o^n$, we have

$$T(\Omega) = T_p(\Omega, \Omega) = \frac{1}{n+2} \int_{S^{n-1}} h_\Omega(u) d\mu_T(\Omega, u).$$

Obviously, it can be easily checked that T_p is homogeneous with respect to its variables, i.e., for any $p > 1$, $\Omega_1, \Omega_2 \in \mathcal{K}_o^n$ and any real numbers $s, t > 0$,

$$T_p(s\Omega_1, t\Omega_2) = s^{n+2-p} t^p T_p(\Omega_1, \Omega_2). \tag{2.5}$$

We will use the Minkowski type inequality for L_p mixed torsional rigidity (see [9, 21]): Let $p > 1$ and $\Omega_1, \Omega_2 \in \mathcal{K}_o^n$, then

$$T_p(\Omega_1, \Omega_2)^{n+2} \geq T(\Omega_1)^{n+2-p} T(\Omega_2)^p, \tag{2.6}$$

with equality if and only if K and L are dilates. This inequality plays an important role on solving the Minkowski problem for L_p torsional measure.

Lemma 2.2. *(see [9, 21]) Let $p > 1$ with $p \neq n + 2$. If μ is a finite Borel measure on S^{n-1} whose support is not concentrated on any closed hemisphere, then there exists a unique convex body $\Omega \in \mathcal{K}_o^n$ such that*

$$\mu = \mu_{T,p}(\Omega, \cdot) = h_\Omega^{1-p} \mu_T(\Omega, \cdot).$$

3. The proof of main result

In this section, we will prove our main theorems. That is the continuity of the solution to the Minkowski problem for L_p torsional measure. To do so, we firstly provide several lemmas which will be used in the proofs of our main results.

Lemma 3.1. *Let $p > 1$ and $\Omega, \Omega_i \in \mathcal{K}_o^n (i = 1, 2, \cdots)$. If the sequence of the measures $\mu_{T,p}(\Omega_i, \cdot)$ converges to $\mu_{T,p}(\Omega, \cdot)$ weakly, then for all $u \in S^{n-1}$*

$$f_i(u) = \int_{S^{n-1}} (u \cdot v)_+^p d\mu_{T,p}(\Omega_i, v)$$

converges to

$$f(u) = \int_{S^{n-1}} (u \cdot v)_+^p d\mu_{T,p}(\Omega, v)$$

uniformly on S^{n-1}.

Proof. Since $p > 1$, for any real numbers $\alpha, \beta > 0$ and $u_1, u_2 \in S^{n-1}$, we have

$$f_i^{\frac{1}{p}}(\alpha u_1 + \beta u_2) \leq \alpha f_i^{\frac{1}{p}}(u_1) + \beta f_i^{\frac{1}{p}}(u_2),$$

and

$$f^{\frac{1}{p}}(\alpha u_1 + \beta u_2) \leq \alpha f^{\frac{1}{p}}(u_1) + \beta f^{\frac{1}{p}}(u_2).$$

Thus $f^{\frac{1}{p}}(u)$ and $f_i^{\frac{1}{p}}(u)$ are support functions of convex bodies (see, e.g., Schneider [29]). Since the pointwise and uniform convergence of support functions are equivalent (also see, e.g., Schneider [29]). Thus $f_i^{\frac{1}{p}}$ converges to $f^{\frac{1}{p}}$ uniformly on S^{n-1}. This implies that f_i converges to f uniformly on S^{n-1}. □

Lemma 3.2. *Suppose $p > 1$ with $p \neq n + 2$. Let Ω be a compact convex set with $o \in \Omega$ and let μ be a Borel measure on S^{n-1} such that $T(\Omega) \cdot h_\Omega^{p-1}(\cdot)\mu = \mu_T(\Omega, \cdot)$. If there exists a constant $R_0 > 0$ such that*

$$\int_{S^{n-1}} (u \cdot v)_+^p d\mu(v) \geq \frac{n+2}{R_0^p}$$

for all $u \in S^{n-1}$, then $\Omega \subset R_0 B_2^n$.

Proof. Let $R := h_\Omega(v_0) = \max\{h_\Omega(u) : u \in S^{n-1}\}$ for some $v_0 \in S^{n-1}$. Since the segment $[o, Rv_0] \subset \Omega$, thus $R(u \cdot v_0)_+ \leq h_\Omega(u)$ for all $u \in S^{n-1}$, and hence

$$\begin{aligned}
\frac{R^p}{R_0^p} &\leq \frac{R^p}{n+2} \int_{S^{n-1}} (u \cdot v_0)_+^p d\mu(u) \\
&\leq \frac{1}{n+2} \int_{S^{n-1}} h_\Omega^p(u) d\mu(u) \\
&= \frac{1}{n+2} \int_{S^{n-1}} h(\Omega, u) \frac{d\mu_T(\Omega, u)}{T(\Omega)} \\
&= 1.
\end{aligned}$$

This gives $R \leq R_0$ which shows that $\Omega \subset R_0 B_2^n$. □

Lemma 3.3. *Suppose $p > 1$ with $p \neq n + 2$. Let $\Omega \in \mathcal{K}_o^n$ be a convex body and let $\{\Omega_i\}_{i=1}^\infty \subset \mathcal{K}_o^n$ be a sequence of convex bodies. If the sequence of measures $\{\mu_{T,p}(\Omega_i, \cdot)\}_{i=1}^\infty$ converges weakly to $\mu_{T,p}(\Omega, \cdot)$, then Ω_i is bounded and there exist $\eta_1, \eta_2 > 0$ with $\eta_1 < \eta_2$ and $N \in \mathbb{N}$ such that*

$$\eta_1 < T(\Omega_i) < \eta_2$$

for all $i \geq N$.

Proof. Let

$$\mu_{T,p}(\Omega, S^{n-1}) = \int_{S^{n-1}} d\mu_{T,p}(\Omega, u).$$

The inequality (2.6) gives that

$$\left[\frac{\mu_{T,p}(\Omega_i, S^{n-1})}{n+2}\right]^{n+2} \geq T(\Omega_i)^{n+2-p} T(B_2^n)^p.$$

Since $\mu_{T,p}(\Omega_i, \cdot)$ converges weakly to $\mu_{T,p}(\Omega, \cdot)$, there exist constants $c_1, c_1' > 0$ and $N_0 \in \mathbb{N}$ such that

$$T(\Omega_i) \leq c_1 \text{ for } 1 < p < n+2 \tag{3.7}$$

and

$$T(\Omega_i) \geq c_1' \text{ for } p > n+2 \tag{3.8}$$

for all $i \geq N_0$.

Since the L_p torsional measure $\mu_{T,p}$ is not concentrated on any closed hemisphere of S^{n-1}, i.e., there exists a constant $\eta > 0$ such that

$$\int_{S^{n-1}} (u \cdot v)_+ d\mu_{T,p}(\Omega, v) > \eta.$$

Combined with Lemma 3.1, this implies that there exist $R_0 > 0$ and $N_1 \in \mathbb{N}$ such that for all $u \in S^{n-1}$ and $i \geq N_1$,

$$f_i(u) = \int_{S^{n-1}} (u \cdot v)_+^p d\mu_{T,p}(\Omega_i, v) \geq \frac{n+2}{R_0^p}. \tag{3.9}$$

By (2.5) and Lemma 2.2, there exists a unique convex body $\Omega_i' \in \mathcal{K}_o^n$ such that

$$T(\Omega_i') h_{\Omega_i'}^{p-1} \cdot \mu_i = \mu_T(\Omega_i', \cdot)$$

with

$$\Omega_i' = T(\Omega_i)^{-\frac{1}{p}} \Omega_i.$$

Let $\mu_i = \mu_{T,p}(\Omega_i, \cdot)$, combined with (3.9) and Lemma 3.2, this implies that $\Omega_i' \subset R_0 B_2^n$ for all $i \geq N_1$. Thus

$$T(\Omega_i)^{\frac{p-(n+2)}{p}} = T(\Omega_i') \leq R_0^{n+2} T(B_2^n)$$

for all $i \geq N_1$. Then there exist constants $c_2, c_2' > 0$ and $N_2 \in \mathbb{N}$ such that

$$T(\Omega_i) \geq c_2 \ \text{ for } \ 1 < p < n+2 \tag{3.10}$$

and

$$T(\Omega_i) \leq c_2' \ \text{ for } \ p > n+2 \tag{3.11}$$

for all $i \geq N_2$.

From (3.7), (3.8), (3.10) and (3.11), there exist $\eta_1, \eta_2 > 0$ with $\eta_1 < \eta_2$ and $N = \max\{N_0, N_1, N_2\} \in \mathbb{N}$ such that

$$\eta_1 < T(\Omega_i) < \eta_2 \tag{3.12}$$

for all $i \geq N$.

Let $R_i = h(\Omega_i, u_i) = \max\{h(\Omega_i, u) : u \in S^{n-1}\}$ for some $u_i \in S^{n-1}$. Since the segment $[o, R_i u_i] \subset \Omega_i$, thus

$$R_i (u \cdot u_i)_+ \leq h(\Omega_i, u)$$

for all $u \in S^{n-1}$. Combined with (3.9) and (3.12), this proves that for all $i \geq N$

$$\begin{aligned} \frac{R_i^p}{R_0^p} &\leq \frac{R_i^p}{n+2} \int_{S^{n-1}} (u \cdot u_i)_+^p d\mu_{T,p}(\Omega_i, u) \\ &\leq \frac{1}{n+2} \int_{S^{n-1}} h^p(\Omega_i, u) d\mu_{T,p}(\Omega_i, u) \\ &= T(\Omega_i) \\ &< \eta_2. \end{aligned}$$

This gives that Ω_i is bounded. $\square$

We now prove our first main result, i.e., Theorem 1.1, we repeat it as follows.

Theorem 3.4. *Let $p > 1$ with $p \neq n+2$. Let $\Omega \in \mathcal{K}_o^n$ be a convex body and let $\{\Omega_i\}_{i=1}^\infty \subset \mathcal{K}_o^n$ be a sequence of convex bodies. If the sequence of measures $\{\mu_{T,p}(\Omega_i, \cdot)\}_{i=1}^\infty$ converges to $\mu_{T,p}(\Omega, \cdot)$ weakly, then Ω_i converges to Ω in the Hausdroff metric.*

Proof. Suppose Ω_i does not converge to Ω. That is to say, there exists a subsequence Ω_{i_j} of Ω_i and $\varepsilon_0 > 0$ such that

$$\|h(\Omega_{i_j}, u) - h(\Omega, u)\|_\infty \geq \varepsilon_0$$

for all $u \in S^{n-1}$ and $i_j \in \mathbb{N}$.

Lemma 3.3 implies that Ω_i is bounded and thus Ω_{i_j} is also bounded. By the Blaschke selection theorem, there exists a subsequence $\Omega_{i_{j_k}}$ of Ω_{i_j} converges to a compact convex set Ω_0 with $\Omega_0 \neq \Omega$.

Next we show that Ω_0 is a convex body. Indeed, the formula (2.2) and Lemma 2.1 show that $T(\Omega_{i_{j_k}}) \leq [\mathrm{diam}(\Omega_{i_{j_k}})]^2|\Omega_{i_{j_k}}|$, i.e.,

$$|\Omega_{i_{j_k}}| \geq \frac{1}{[\mathrm{diam}(\Omega_{i_{j_k}})]^2} T(\Omega_{i_{j_k}}).$$

This gives that $|\Omega_0| \geq \frac{1}{[\mathrm{diam}(\Omega_0)]^2} T(\Omega_0) > 0$ as $\Omega_{i_{j_k}} \to \Omega_0$. This further implies that Ω_0 is a convex body but $\Omega_0 \neq \Omega$.

Let $\mu = \mu_{T,p}(\Omega, \cdot)$, which implies that $\mu_{T,p}(\Omega_{i_{j_k}}, \cdot)$ converges to μ weakly. On the other hand, the weak convergence (2.4) implies that

$$\mu_{T,p}(\Omega_{i_{j_k}}, \cdot) \to \mu_{T,p}(\Omega_0, \cdot) \text{ weakly as } \Omega_{i_{j_k}} \to \Omega_0.$$

Combined with $\mu_{T,p}(\Omega_{i_{j_k}}, \cdot) \to \mu_{T,p}(\Omega, \cdot)$ weakly, this yields that $\mu_{T,p}(\Omega_0, \cdot) = \mu = \mu_{T,p}(\Omega, \cdot)$, which further implies that $\Omega_0 = \Omega$ by the uniqueness of the solution to the Minkowski problem for L_p torsional measure in Lemma 2.2. This contradiction shows that Ω_i converges to Ω. □

Obviously, the theorem above is closely related to the Minkowski problem for the L_p torsional measure and can be described as follows:

Theorem 3.5. *Let $p > 1$ with $p \neq n+2$ and μ_i, μ be nonzero finite Borel measures on S^{n-1}. If $\Omega_i \in \mathcal{K}_o^n$ is the solution to the Minkowski problem for L_p torsional measure associated with μ_i and $\Omega \in \mathcal{K}_o^n$ is the solution to the Minkowski problem for L_p torsional measure associated with μ, then $\Omega_i \to \Omega$ as $\mu_i \to \mu$.*

Proof. From Lemma 2.2 and the uniqueness of the solution to the Minkowski problem for L_p torsional measure, we have

$$\mu_i = \mu_{T,p}(\Omega_i, \cdot) \text{ and } \mu = \mu_{T,p}(\Omega, \cdot).$$

Since $\mu_i \to \mu$, i.e.,

$$\mu_{T,p}(\Omega_i, \cdot) \to \mu_{T,p}(\Omega, \cdot).$$

This, together with Theorem 3.4, gives $\Omega_i \to \Omega$. □

To show the continuity of solution to the Minkowski problem for L_p torsional measure with respect to p, we shall use the following lemma.

Lemma 3.6. *Suppose $p, p_i \in (1, n+2) \cup (n+2, \infty)$ with $p_i \to p$. Let μ be a Borel measure on S^{n-1} that is not concentrated on a closed hemisphere. If Ω is the solution to the Minkowski problem for L_p torsional measure associated with μ and Ω_i is the solution to the Minkowski problem for L_{p_i} torsional measure associated with μ, then Ω_i is bounded from above and there exist constants $\eta_3, \eta_4 > 0$ with $\eta_3 < \eta_4$ and $N \in \mathbb{N}$ such that*

$$\eta_3 < T(\Omega_i) < \eta_4$$

for all $i \geq N$.

Proof. Let $|\mu| = \int_{S^{n-1}} d\mu(u)$. Since Ω is the solution to the Minkowski problem for L_p torsional measure associated with μ, by Lemma 2.2, we obtain

$$T_p(\Omega, B_2^n) = \frac{\mu_{T,p}(\Omega, S^{n-1})}{n+2} = \frac{|\mu|}{n+2}.$$

By the Minkowski inequality for L_{p_i} mixed torsional rigidity, we have

$$\left[\frac{|\mu|}{n+2}\right]^{n+2} \geq T(\Omega_i)^{n+2-p_i} T(B_2^n)^{p_i}.$$

By the condition that $p, p_i \in (1, n+2) \cup (n+2, \infty)$ with $p_i \to p$, there exist constants $c_3', c_3 > 0$ and $N_1 \geq \mathbb{N}$ such that

$$T(\Omega_i) \leq c_3 \ \text{ for } \ 1 < p < n+2 \tag{3.13}$$

and

$$T(\Omega_i) \geq c_3' \ \text{ for } \ p > n+2 \tag{3.14}$$

for all $i \geq N_1$.

Since $p, p_i \in (1, n+2) \cup (n+2, \infty)$ with $p_i \to p$ and μ is not concentrated on a closed hemisphere of S^{n-1}, there exist $N_2 \in \mathbb{N}$, $R_0, m_0 > 0$ and $p_2 > p > p_1 > 1$ such that

$$p_2 \geq p_i \geq p_1, \ \ (R_0 + m_0)^{p_i} \geq R_0^{p_2}$$

for $i \geq N_2$, and

$$\int_{S^{n-1}} (u \cdot v)_+^{p_2} d\mu(v) \geq \frac{n+2}{R_0^{p_2}}$$

for all $u \in S^{n-1}$. When $i \geq N_2$, one has

$$\begin{aligned} \int_{S^{n-1}} (u \cdot v)_+^{p_i} d\mu(v) &\geq \int_{S^{n-1}} (u \cdot v)_+^{p_2} d\mu(v) \\ &\geq \frac{n+2}{(R_0 + m_0)^{p_i}} \end{aligned}$$

for all $u \in S^{n-1}$. From Lemma 3.2, it follows that

$$T(\Omega_i)^{-\frac{1}{p}} \Omega_i \subset (R_0 + m_0) B_2^n, \tag{3.15}$$

thus

$$T(\Omega_i)^{\frac{p_i - (n+2)}{p_i}} \leq (R_0 + m_0)^{n+2} T(B_2^n).$$

Since $p_i \to p$, we conclude that there exist c_4', $c_4 > 0$ and $N_3 \in \mathbb{N}$ such that

$$T(\Omega_i) \geq c_4 \ \text{ for } \ 1 < p < n+2 \tag{3.16}$$

and

$$T(\Omega_i) \leq c_4' \ \text{ for } \ p > n+2 \tag{3.17}$$

for all $i \geq N_3$.

By (3.13), (3.14), (3.16) and (3.17), there exist $\eta_3, \eta_4 > 0$ with $\eta_3 < \eta_4$ and $N = \max\{N_1, N_2, N_3\} \in \mathbb{N}$ such that

$$\eta_3 < T(\Omega_i) < \eta_4$$

for all $i \geq N$. From this and (3.15), we have

$$\Omega_i \subset \eta_4^{\frac{1}{p}} (R_0 + m_0) B_2^n.$$

This shows that Ω_i is bounded from above. □

We now prove Theorem 1.2 and repeat it again as follows.

Theorem 3.7. *Suppose $p, p_i \in (1, n+2) \cup (n+2, \infty)$ with $p_i \to p$. Let μ be a Borel measure on S^{n-1}. If $\Omega \in \mathcal{K}_o^n$ is the solution to the Minkowski problem for L_p torsional measure associated with μ and the sequence of convex bodies $\Omega_i \in \mathcal{K}_o^n$ is the solution to the Minkowski problem for L_{p_i} torsional measure associated with μ, then $\Omega_i \to \Omega$ as $p_i \to p$.*

Proof. Suppose Ω_i does not converge to Ω. That is to say, there exists a subsequence Ω_{i_j} of Ω_i and $\varepsilon_0 > 0$ such that

$$\|h(\Omega_{i_j}, u) - h(\Omega, u)\|_\infty \geq \varepsilon_0$$

for all $i_j \in \mathbb{N}$.

By Lemma 3.6 and the same argument in the proof of Theorem 3.4, this shows that there exist a convex body Ω_0 but $\Omega_0 \neq \Omega$ and a subsequence $\Omega_{i_{j_k}}$ of Ω_{i_j} such that $\lim_{k\to\infty} \Omega_{i_{j_k}} = \Omega_0$.

Since $\Omega_{i_{j_k}}$ is the solution to the Minkowski problem for $L_{p_{i_{j_k}}}$ torsional measure associated with μ, this gives that

$$h(\Omega_{i_{j_k}}, \cdot)^{p_{i_{j_k}}-1} \cdot \mu = \mu_T(\Omega_{i_{j_k}}, \cdot).$$

Taking $k \to \infty$ on both sides, we obtain

$$h(\Omega_0, \cdot)^{p-1} \cdot \mu = \mu_T(\Omega_0, \cdot).$$

By the uniqueness of the solution to the Minkowski problem for L_p torsional measure, we have $\Omega_0 = \Omega$, which is a contraction. This shows that $\Omega_i \to \Omega$. □

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