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# Additive property for the generalized Zhou inverse in a Banach algebra

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**Abstract.** Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has the generalized Zhou inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a^n - ab \in J^{\#}(\mathcal{A}), for\ some\ n \in \mathbb{N}.$$

We find some new conditions under which the generalized Zhou inverse of the sum $a + b$ can be explicitly expressed in terms of $a, b, a^z, b^z$. In particular, necessary and sufficient conditions for the existence of the generalized Zhou inverse of the sum $a + b$ are obtained.

## 1. Introduction

Throughout the paper, $\mathcal{A}$ is a complex Banach algebra. The symbols $J(\mathcal{A}), \mathcal{A}^D, \mathcal{A}^d, \mathcal{A}^{nil}, \mathcal{A}^{qnil}$ denote, respectively, the Jacobson radical, the sets of all Drazin invertible, generalized Drazin invertible, nilpotent and quasi nilpotent elements of $\mathcal{A}$. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$ and the double commutant of $a \in \mathcal{A}$ is defined by

$$comm^2(a) = \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in comm(a)\}.$$

Also we define $J^{\#}(\mathcal{A}) = \{a \in \mathcal{A} \mid a^n \in J(\mathcal{A})\ for\ some\ n \in \mathbb{N}\}$.

Let us recall that the Drazin inverse [4] of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$ which satisfies

$$b = bab, ab = ba\ and\ a - a^2b \in \mathcal{A}^{nil}. \tag{1}$$

The element $b$ above is unique if it exists and is denoted by $a^D$.
The generalized Drazin inverse [5] of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$ which satisfies

$$b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}. \tag{2}$$

Such $b$ is unique if it exits and is denoted by $a^d$. In 2012, Wang and Chen [10] introduced the notation of the pseudo Drazin inverse (or p-Drazin inverse for short) in associative rings and Banach algebras. An element $a$ in $\mathcal{A}$ has p-Drazin inverse if and only if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a^n - a^{n+1}b \in J(\mathcal{A}), for\ some\ n \in \mathbb{N}. \tag{3}$$

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We always use $\mathcal{A}^{\ddagger}$ to denote the set of all p-Drazin invertible elements in $\mathcal{A}$. Any element $b \in \mathcal{A}$ satisfying the above conditions is called p-Drazin inverse of $a$ and is denoted by $a^{\ddagger}$. The p-Drazin and generalized Drazin inverses were extensively studied in matrix theory and Banach algebras (see [3, 10–13]).
An element $a \in \mathcal{A}$ is said to be Zhou invertible [2] if there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm(a), a^n - ab \in \mathcal{A}^{nil}, for\ some\ n \in \mathbb{N}. \quad (4)$$

The preceding $b$ is unique, if such an element exists. The generalized Zhou inverse [2] of $a \in \mathcal{A}$ is an element $b \in \mathcal{A}$ which satisfies

$$b = bab, ab = ba, a^n - ab \in J^{\#}(\mathcal{A}), for\ some\ n \in \mathbb{N}. \quad (5)$$

In this case, $b$ is unique if it exists and is denoted by $a^z$. The set of all generalized Zhou invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^z$. The smallest integer $n$ which satisfies the above equation is called the generalized Zhou index of $a$, which is denoted by $ind(a)$.

It was proved that $a \in \mathcal{A}^z$ if and only if there exists an idempotent $p \in comm(a)$ such that $a^n - p \in J^{\#}(\mathcal{A})$ for some $n \in \mathbb{N}$. (see [2, Theorem 2.6]).

In Section 2, we investigate some elementary properties of generalized Zhou inverses. The multiplication of the two generalized Zhou invertible elements is studied. We prove that for any $a, b \in \mathcal{A}^z$, if $ab = ba$ then $(ab)^z$ exists and $(ab)^z = b^z a^z$. In Section 3, we apply matrix representation for the generalized Zhou inverse relative to idempotent $p \in \mathcal{A}$. Let $\mathcal{A}$ be a Banach algebra, $x \in \mathcal{A}$. Then we write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p),$$

and induce a representation given by the matrix

$$x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p,$$

so we may regard such matrix as an element in $\mathcal{A}$. Let $\mathcal{A}_1 = p\mathcal{A}p, \mathcal{A}_2 = (1-p)\mathcal{A}(1-p)$. We prove that for any $a \in \mathcal{A}, a \in \mathcal{A}^z$ if and only if there exists an idempotent $p \in \mathcal{A}$ such that

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p$$

where $a_1 \in \mathcal{A}_1^z$ and $a_2 \in J^{\#}(\mathcal{A}_2)$.

In Section 4, additive property of the two generalized Zhou invertible elements is studied. For any $a, b \in \mathcal{A}^z$, we investigate, the representations of $(a+b)^z$ under conditions $ab^2 = 0, aba = 0$ and various conditions.

## 2. The generalized Zhou inverse

In this section, some elementary results, which will be used in sequel are presenetd.

**Lemma 2.1.** *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}$ and $ab = ba$;*

(1) If $a, b \in J^{\#}(\mathcal{A})$, then $a + b \in J^{\#}(\mathcal{A})$.
(2) If $a$ *or* $b \in J^{\#}(\mathcal{A})$, then $ab \in J^{\#}(\mathcal{A})$.

*Proof.* (1) See [13, Lemma 2.4].
(2) If $a \in J^{\#}(\mathcal{A})$, then, $a^k \in J(\mathcal{A})$, for some $k \in \mathbb{N}$. As $ab = ba$, we have $(ab)^k = a^k b^k$, thus by [7, Corollary 4.2], we see that $(ab)^k \in J(\mathcal{A})$ which implies that $ab \in J^{\#}(\mathcal{A})$. □

**Theorem 2.2.** *Let $\mathcal{A}$ be a Banach algebra and $a, b \in \mathcal{A}^z$, if $ab = ba$, then $(ab)^z$ exists and $(ab)^z = b^z a^z$.*

*Proof.* It is obvious by [2, Theorem 2.2] that every generalized Zhou invertible element is pseudo Drazin invertible. Then by [10, Proposition 3.4], every generalized Zhou invertible element is generalized Drazin invertible. Now we have $a^z \in comm^2(a)$, $b^z \in comm^2(b)$ and $ab = ba$, then $a^z, b^z, a, b$ commute with each other and so $b^z a^z \in comm(ab)$, $(b^z a^z)^2(ab) = b^z a^z$. We may assume that $a^{k_1} - aa^z \in J^\#(\mathcal{A})$ and $b^{k_2} - bb^z \in J^\#(\mathcal{A})$. Let $k = k_1 k_2$, then we see that $a^k - aa^z = (a^{k_1})^{k_2} - (aa^z)^{k_2} = (a^{k_1} - aa^z)(a^{k_1(k_2-1)} + a^{k_1(k_2-2)}aa^z + \cdots + a^{k_1}(aa^z)^{k_2-2} + (aa^z)^{k_2-1})$. Then by Lemma 2.1, we have $a^k - aa^z \in J^\#(\mathcal{A})$. Likewise, $b^k - bb^z \in J^\#(\mathcal{A})$. Hence $(ab)^k - (ab)b^z a^z = -(a^k - aa^z)(b^k - bb^z) + (a^k - aa^z)b^k + a^k(b^k - bb^z)$. By Lemma 2.1, we obtain $(ab)^k - (ab)(ab)^z \in J^\#(\mathcal{A})$. This completes the proof. □

**Corollary 2.3.** *Let $a \in \mathcal{A}^z$ and $n \in \mathbb{N}$. Then*

(1) $(a^n)^z = (a^z)^n$.
(2) $(a^z)^z = a^2 a^z$.
(3) $((a^z)^z)^z = a^z$.

*Proof.* (1) It is obvious by induction and Theorem 2.2.
(2) It is easy to check $a^z a^2 a^z = a^2 a^z a^z$ and $a^2 a^z a^z a^2 a^z = a^2 a^z$. Since $a \in \mathcal{A}^z$ by [2, Theorem 2.9], we see that $a - a^{n+1} \in J^\#(\mathcal{A})$ for some $n \in \mathbb{N}$. Now by Lemma 2.1(2), we have $(a^z)^{n+1}(a - a^{n+1}) \in J^\#(\mathcal{A})$. Thus $(a^z)^{n+1}a - (a^z)^{n+1}a^{n+1} = (a^z)^n - aa^z \in J^\#(\mathcal{A})$, it follows that $(a^z)^n - a^z a^2 a^z = (a^z)^n - aa^z \in J^\#(\mathcal{A})$, then $(a^z)^z = a^2 a^z$.
(3) It is clear by (2) and Theorem 2.2. □

**Proposition 2.4.** *Let $p \in \mathcal{A}$ be an idempotent and $a \in p\mathcal{A}p$. Then $a \in \mathcal{A}^z$ if and only if $a \in (p\mathcal{A}p)^z$, moreover $a^z_{\mathcal{A}} = a^z_{p\mathcal{A}p}$.*

*Proof.* (⇒) Let $a^z_{\mathcal{A}} = x$, then we have $x^2 a = ax^2 = x$ and $ax^3 a = ax^2 xa = x^2 a = x$, which imply that, $x = ax^3 a \in p\mathcal{A}p$. Since $a^z_{\mathcal{A}} = x$, there exists $k \in \mathbb{N}$ such that $a^k - aa^z \in J^\#(\mathcal{A})$, so $(a^k - aa^z)^n \in J(\mathcal{A})$ for some $n \in \mathbb{N}$. Otherwise, $a^k - aa^z \in p\mathcal{A}p$. Thus by [7, Theorem 2.10], $(a^k - aa^z)^n \in (p\mathcal{A}p) \bigcap J(\mathcal{A}) = J(p\mathcal{A}p)$, it follows that $a^k - aa^z \in J^\#(p\mathcal{A}p)$. Also, $ax = xa, xax = x$ then $a \in (p\mathcal{A}p)^z$.
(⇐) Suppose $a \in (p\mathcal{A}p)^z$ and let $a^z_{p\mathcal{A}p} = y$. The condition $a^z_{p\mathcal{A}p} = y$ ensures that, (a) $yay = y$, (b) $ya = ay$, (c) $a^k - aa^z \in J^\#(p\mathcal{A}p)$ for some $k \in \mathbb{N}$. Applying [7, Theorem 2.10], we have $(a^k - aa^z)^n \in J(p\mathcal{A}p) = (p\mathcal{A}p) \bigcap J(\mathcal{A})$ for some $n \in \mathbb{N}$, then $(a^k - aa^z)^n \in J(\mathcal{A})$. Hence $a \in \mathcal{A}^z$ and $a^z_{\mathcal{A}} = y$. This completes the proof. □

**Corollary 2.5.** *Let $a \in \mathcal{A}$. Then the following conditions are equivalent.*

(1) $a \in \mathcal{A}^z$.
(2) $a^n \in \mathcal{A}^z$ *for any* $n \in \mathbb{N}$.
(3) $a^n \in \mathcal{A}^z$ *for some* $n \in \mathbb{N}$.

*Proof.* (1) ⇒ (2) It was proved in Corollary 2.3.
(2) ⇒ (3) It is obvious.
(3) ⇒ (1) Let $y = (a^n)^z a$. A direct calculation shows that $ya^{n-1}y = y$, $ya = ay$. Since $a^n \in \mathcal{A}^z$, there exists $k \in \mathbb{N}$ such that $(a^n)^k - a^n(a^n)^z \in J^\#(\mathcal{A})$. Then in light of Theorem 2.2, we have $(a^{n-1})^{nk} - a^{n-1}y = (a^{n-1})^{nk} - a^{n-1}a(a^n)^z = (a^n)^{(n-1)k} - a^n(a^n)^z \in J^\#(\mathcal{A})$, which implies that $a^{n-1} \in \mathcal{A}^z$. Thus $a^n \in \mathcal{A}^z \Longrightarrow a^{n-1} \in \mathcal{A}^z \Longrightarrow a^{n-2} \in \mathcal{A}^z \Longrightarrow \cdots \Longrightarrow a \in \mathcal{A}^z$. By induction we get $a \in \mathcal{A}^z$. This completes the proof. □

## 3. Matrix representation

For any Banach algebra $\mathcal{A}$ and any idempotent $p \in \mathcal{A}$,

$$M_2(\mathcal{A}, p) = \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}(1-p) \\ (1-p)\mathcal{A}p & (1-p)\mathcal{A}(1-p) \end{pmatrix},$$

is a Banach algebra with

$$I = \begin{pmatrix} p & 0 \\ 0 & (1-p) \end{pmatrix}_p.$$

**Lemma 3.1.** *Let $p$ be an idempotent element in $\mathcal{A}$. Then,*
$J(M_2(\mathcal{A}))\bigcap M_2(\mathcal{A},p) = J(M_2(\mathcal{A},p))$.

*Proof.* See [13, Lemma 2.6]. □

**Theorem 3.2.** *Let $\mathcal{A}$ be a Banach algebra, $x, y \in \mathcal{A}$, let*

$$x = \begin{pmatrix} a & d \\ 0 & b \end{pmatrix}, y = \begin{pmatrix} b & 0 \\ d & a \end{pmatrix}.$$

*If $a, b \in \mathcal{A}^z$, then $x, y \in \mathcal{A}^z$ and*

$$x^z = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix}, y^z = \begin{pmatrix} b^z & 0 \\ u & a^z \end{pmatrix},$$

*where $u = \sum_{i=0}^{\infty} (a^z)^{i+2} db^i b^\pi + \sum_{i=0}^{\infty} a^\pi a^i d\, (b^z)^{i+2} - a^z db^z$.*

*Proof.* Suppose that $a, b \in \mathcal{A}^z$. Let

$$w = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix},$$

where $u = \sum_{i=0}^{\infty} (a^z)^{i+2} db^i b^\pi + \sum_{i=0}^{\infty} a^\pi a^i d\, (b^z)^{i+2} - a^z db^z$. Then

$$I - xw = \begin{pmatrix} a^\pi & -au - db^z \\ 0 & b^\pi \end{pmatrix}.$$

Here $a^\pi = 1 - aa^z$ and $b^\pi = 1 - bb^z$. We have

$$w(I - xw) = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix}\begin{pmatrix} a^\pi & -au - db^z \\ 0 & b^\pi \end{pmatrix} =$$

$$\begin{pmatrix} a^z a^\pi & -a^z au - a^z db^z + ub^\pi \\ 0 & b^z b^\pi \end{pmatrix}.$$

Note that $a^z a^\pi = 0$ and $b^z b^\pi = 0$, then

$$-a^z au = -a^z a(\sum_{i=0}^{\infty} (a^z)^{i+2} db^i)b^\pi + a^z db^z = -\sum_{i=0}^{\infty} (a^z)^{i+2} db^i b^\pi + a^z db^z,$$

$$ub^\pi = (\sum_{i=0}^{\infty} (a^z)^{i+2} db^i)b^\pi$$

and so $-a^z au - a^z db^z + ub^\pi = 0$. This shows that $w = wxw$. Let $r = ind(a), s = ind(b)$, then, $a^r - aa^z \in J^\#(\mathcal{A}), b^s - bb^z \in J^\#(\mathcal{A})$. Let $k = rs$, $f_k = \sum_{i=0}^{k-1} a^i db^{k-1-i}$, we have

$$x^k = (\begin{pmatrix} a & d \\ 0 & b \end{pmatrix})^k = \begin{pmatrix} a^k & f_k \\ 0 & b^k \end{pmatrix}.$$

$$x^k - xw = \begin{pmatrix} a^k & f_k \\ 0 & b^k \end{pmatrix} - \begin{pmatrix} a & d \\ 0 & b \end{pmatrix}\begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix} = \begin{pmatrix} a^k - aa^z & f_k - au - db^z \\ 0 & b^k - bb^z \end{pmatrix}.$$

As $a^k - aa^z \in J^\#(\mathcal{A}), b^k - bb^z \in J^\#(\mathcal{A})$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $(a^k - aa^z)^{n_1} \in J(\mathcal{A}), (b^k - bb^z)^{n_2} \in J(\mathcal{A})$. Let $n = \max(n_1, n_2)$ and let $x_1 = a^k - aa^z$, $x_2 = f_k - au - db^z$, $x_3 = b^k - bb^z$, then we have $t_n = \sum_{i=0}^{n-1} x_1^i x_2 x_3^{n-1-i}$,

$$(x^k - xw)^n = \begin{pmatrix} (a^k - aa^z)^n & t_n \\ 0 & (b^k - bb^z)^n \end{pmatrix}.$$

Note that, $(x^k - xw)^{2n}=$

$$\begin{pmatrix} (a^k - aa^z)^n & t_n \\ 0 & (b^k - bb^z)^n \end{pmatrix}\begin{pmatrix} (a^k - aa^z)^n & t_n \\ 0 & (b^k - bb^z)^n \end{pmatrix}$$

$$= \begin{pmatrix} (a^k - aa^z)^{2n} & (a^k - aa^z)^n t_n + t_n(b^k - bb^z)^n \\ 0 & (b^k - bb^z)^{2n} \end{pmatrix}.$$

As $(a^k - aa^z)^n, (b^k - bb^z)^n \in J(\mathcal{A})$, by [7, Corollary 4.2] and [7, page 57 Example(7)], we have $(x^k - xw)^{2n} \in J(M_2(\mathcal{A}))$. Finally we need to show that $xw = wx$. We have

$$au - ub = \sum_{i=0}^{\infty} (a^z)^{i+1}\, db^i b^\pi + aa^\pi (\sum_{i=0}^{\infty} a^i d\, (b^z)^{i+2}$$

$$-aa^z db^z - \sum_{i=0}^{\infty} (a^z)^{i+2}\, db^i b^\pi b - a^\pi \sum_{i=0}^{\infty} a^i d\, (b^z)^{i+1} + a^z db^z b$$

$$= (\sum_{i=0}^{\infty} (a^z)^{i+1}\, db^i b^\pi - \sum_{i=0}^{\infty} (a^z)^{i+2}\, db^i b^\pi)$$

$$+(a^\pi (\sum_{i=0}^{\infty} a^{i+1} d\, (b^z)^{i+2}) - a^\pi (\sum_{i=0}^{\infty} a^i d\, (b^z)^{i+1}) - aa^z db^z + a^z db^z b$$

$$= a^z db^\pi - a^\pi db^z - aa^z db^z + a^z db^z b = a^z d - db^z,$$

then $au + db^z = a^z d + ub$. This implies that $xw = wx$. Since $M_2(\mathcal{A})$ is also a Banach algebra,we can prove this conditions in the similar way for $y$. □

**Lemma 3.3.** *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^z$ if and only if there exists an idempotent $p \in \mathcal{A}$ such that*

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p$$

*where $a_1 \in \mathcal{A}^z$ and $a_2 \in J^\#(\mathcal{A})$. In this case*

$$a^z = \begin{pmatrix} a_1^z & 0 \\ 0 & 0 \end{pmatrix}_p, \tag{6}$$

*and $p = aa^z$*

*Proof.* (⇒) Let

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p.$$

Let $p = aa^z$. Obviously,

$$pa(1 - p) = aa^z a(1 - aa^z) = 0, (1 - p)ap = (1 - aa^z)aaa^z = 0.$$

Thus $a_{12} = 0, a_{21} = 0$. Let $a_{11} = a_1, a_{22} = a_2$ . Since $a \in \mathcal{A}^z$, there exists $k \in \mathbb{N}$ such that $a^k - aa^z \in J^\#(\mathcal{A})$. We have $a_1 = aa^z aaa^z = aa^z aa^z a = aa^z a$, so by Corollay 2.3,

$$a_1 a_1^z = aa^z (aa^z)^z = a^2 a^z (a^2)^z a^2 a^z = a^4 (a^z)^4 = aa^z.$$

Hence,

$$a_1^k = (aa^z a)^k = aa^z a^k, aa^z(a^k - aa^z)$$

$$= aa^z a^k - aa^z aa^z = aa^z a^k - aa^z = a_1^k - a_1 a_1^z.$$

Thus by Lemma 2.1(2),

$$a_1^k - a_1 a_1^z \in J^\#(\mathcal{A}).$$

Therefore there exists $n \in \mathbb{N}$ such that $(a_1^k - a_1 a_1^z)^n \in J(\mathcal{A})$. Otherwise $(a_1^k - a_1 a_1^z)^n \in p\mathcal{A}p$, by [7, Theorem 2.10], we have $a_1 \in \mathcal{A}_1^z$. As $a^k - aa^z \in J^\#(\mathcal{A})$ in light of [2, Theorem 2.2],we get $a^k(1 - aa^z) \in J^\#(\mathcal{A})$. Then there exists $m \in \mathbb{N}$ such that $(a^k(1-aa^z))^m \in J(\mathcal{A})$ as $(a^k(1-aa^z))^m \in (1-p)\mathcal{A}(1-p)$ by [7, Theorem 2.10], we obtain $(a^k(1 - aa^z))^m \in J(\mathcal{A}_2)$. Then, wev have $(a(1 - aa^z))^{mk} \in J(\mathcal{A}_2)$. So, $a_2 = a(1 - aa^z) \in J^\#(\mathcal{A}_2)$.
($\Leftarrow$) Let,

$$x = \begin{pmatrix} a_1^z & 0 \\ 0 & 0 \end{pmatrix}_p.$$

A direct calculation shows that $xax = x, ax = xa$. Since $a_2 \in J^\#(\mathcal{A}_2)$, there exists $k_2 \in \mathbb{N}$ such that $a_2^{k_2} \in J(\mathcal{A}_2)$. As $a_1 \in \mathcal{A}_1^z$ , $(a_1^{k_1} - a_1 a_1^z)^{k_3} \in J(\mathcal{A}_1)$ for some $k_1, k_3 \in \mathbb{N}$. Let $k = max(k_2, k_3)$, we have $a_2^k \in J(\mathcal{A}_2) \subset J(\mathcal{A})$ and $(a_1^k - a_1 a_1^z)^k \in J(\mathcal{A}_1) \subset J(\mathcal{A})$ thus, we get

$$a^{k_1} - ax = \begin{pmatrix} a_1^{k_1} - a_1 a_1^z & 0 \\ 0 & a_2^{k_1} \end{pmatrix}_p,$$

$$(a^{k_1} - ax)^k = \begin{pmatrix} (a_1^{k_1} - a_1 a_1^z)^k & 0 \\ 0 & a_2^{k_1 k} \end{pmatrix}_p \in J(M_2(\mathcal{A})).$$

Using [7, Theorem 2.10], so $a \in \mathcal{A}^z$ □

**Theorem 3.4.** *. Let $\mathcal{A}$ be a Banach algebra, $x, y \in \mathcal{A}$, and p be an idempotent element in Banach algebra $\mathcal{A}$. Assume that*

$$x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p, y = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}_{1-p}.$$

*Then,*

(1) If $a \in \mathcal{A}_1^z$ and $b \in \mathcal{A}_2^z$, then $x, y \in \mathcal{A}^z$ and

$$x^z = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix}_p, y^z = \begin{pmatrix} b^z & 0 \\ u & a^z \end{pmatrix}_{1-p}. \tag{7}$$

Where $u = \sum_{i=0}^{\infty} (a^z)^{i+2} cb^i b^\pi + \sum_{i=0}^{\infty} a^\pi a^i c (b^z)^{i+2} - a^z cb^z$. (8)

(2) If $x \in \mathcal{A}^z$ and $a \in \mathcal{A}_1^z$ then $b \in \mathcal{A}_2^z$ and $x^z$ [resp. $y^z$] is given by (7) and (8).

*Proof.* (1) Applying Theorem 3.3, and Proposition 2.4, we get $x \in (M_2(\mathcal{A}))^z$ and

$$x^z = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix}_p,$$

where $u = \sum_{i=0}^{\infty} (a^z)^{i+2} cb^i b^\pi + \sum_{i=0}^{\infty} a^\pi a^i c (b^z)^{i+2} - a^z cb^z$. Then there exist $k, m \in \mathbb{N}$ such that $(x^k - xx^z)^m \in J(M_2(\mathcal{A}))$. Lemma 3.1 ensures that, $(x^k - xx^z)^m \in J(M_2(\mathcal{A}, p))$. Then, $x \in (M_2(\mathcal{A}), p)^z$ which implies that $x \in \mathcal{A}^z$. Next, we consider the generalized Zhou inverse of $y$ since

$$y = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}_{1-p} = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p$$

from the first part, we obtain $y \in \mathcal{A}^z$ and

$$y^z = \begin{pmatrix} a^z & u \\ 0 & b^z \end{pmatrix}_p = \begin{pmatrix} b^z & 0 \\ u & a^z \end{pmatrix}_{1-p}.$$

We drive the result.
(2). We prove $b^z = [(1-p)x(1-p)]^z = (1-p)x^z(1-p)$. Since $x \in \mathcal{A}^z, a \in \mathcal{A}_1^z, \mathcal{A}^z \subset \mathcal{A}^d$, then $x \in \mathcal{A}^d, a \in \mathcal{A}_1^d$ and $x^d = x^z$, $a^d = a^z$. According to [1, Theorem 2.3(2)], it follows that

$$\begin{pmatrix} a^d & u \\ 0 & b^d \end{pmatrix}_p = x^d = \begin{pmatrix} px^dp & px^d(1-p) \\ (1-p)x^dp & (1-p)x^d(1-p) \end{pmatrix}_p,$$

then we have, $(1-p)x^dp = 0$, i.e. $(1-p)x^zp = 0$, which implies that $(1-p)x^z(1-p) = (1-p)x^z$. Note that $(1-p)xp = 0$ , we can get $(1-p)x(1-p) = (1-p)x$. Therefore, we need only to prove $[(1-p)x]^z = (1-p)x^z$. Let $v = (1-p)x^z$.
$(a)[(1-p)x]v = (1-p)x(1-p)x^z = (1-p)xx^z = (1-p)x^zx = (1-p)x^z(1-p)x = v[(1-p)x]$.
(b) $v[(1-p)x]v = (1-p)x^z(1-p)x(1-p)x^z = (1-p)x^z(1-p)xx^z = (1-p)x^zxx^z = v$.
(c) As $(1-p)xp = 0$,we have $(1-p)x(1-p) = (1-p)x$, thus by induction we see that, $((1-p)x)^k = (1-p)x^k, (1-p)x^k(1-p) = (1-p)x^k$. Now we prove $[(1-p)(x^k - xx^z)]^n = (1-p)(x^k - xx^z)^n$, for any $n \in \mathbb{N}$ by induction. It is obvious for $n = 1$. Assume $[(1-p)(x^k - xx^z)]^n = (1-p)(x^k - xx^z)^n$. Since $(1-p)xx^z(1-p) = (1-p)x(1-p)x^z(1-p) = (1-p)x(1-p)x^z = (1-p)xx^z$, for the $(n+1)$ case, $[(1-p)(x^k-xx^z)]^{n+1} = (1-p)(x^k-xx^z)[(1-p)(x^k-xx^z)]^n = (1-p)(x^k-xx^z)[(1-p)(x^k-xx^z)]^n = [(1-p)x^k(1-p)-(1-p)xx^z(1-p)](x^k-xx^z)^n$ $=[(1-p)x^k - (1-p)xx^z](x^k - xx^z)^n = (1-p)(x^k - xx^z)(x^k - xx^z)^n = (1-p)(x^k - xx^z)^{n+1}$. Then, we see that $b^k - bv = ((1-p)x)^k - (1-p)xv = (1-p)x^k - (1-p)x(1-p)x^z = (1-p)x^k - (1-p)xx^z = (1-p)(x^k - xx^z)$. Since $(x^k - xx^z) \in J^\#(\mathcal{A})$. Thus we have $(x^k - xx^z)^n \in J(\mathcal{A})$, for some $n \in \mathbb{N}$ , therefore by [7, Corollary 4.2(2)], $(1-p)(x^k - xx^z)^n \in J(\mathcal{A})$, which implies that $(b^k - bv)^n \in J(\mathcal{A}) \bigcap \mathcal{A}_2 = J(\mathcal{A}_2)$. Hence $b^z = (1-p)x^z$. Using part (1), we see $x^z$ is given by (7),(8). Following an analogous strategy as in the proof for $y$ of part (1), we have (2) for $y$. □

Moreover, when an element $x \in \mathcal{A}^z$ commutes with an idempotent $p \in \mathcal{A}$, the generalized Zhou inverse of $x$ has a simple form of the matrix representation relative to $p$.

**Corollary 3.5.** *Let $\mathcal{A}$ be a unital Banach algebra and let $x \in \mathcal{A}$ , $p$ is an idempotent element in $\mathcal{A}$. If*

$$x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}_p$$

*then $x \in \mathcal{A}^z$ if and only if $x_1 \in \mathcal{A}_1^z$ and and $x_2 \in \mathcal{A}_2^z$ in this situation, one has*

$$x^z = \begin{pmatrix} x_1^z & 0 \\ 0 & x_2^z \end{pmatrix}_p.$$

*Proof.* If $x_1 \in \mathcal{A}_1^z$ and $x_2 \in \mathcal{A}_2^z$, by Theorem 3.4(1), we have $x \in \mathcal{A}^z$.
Conversly if $x \in \mathcal{A}^z$ by Lemma 3.3, we see that, $x_1 \in \mathcal{A}_1^z$ , $x_2 \in J^\#(\mathcal{A}_2) \subset \mathcal{A}_2^z$, where $x_2^z = 0$ , as required. □

## 4. Additive results

In this section, we investigate the representation for generalized Zhou inverse of the sum of two elements in a Banach algebra under various conditions. In particular, necessary and sufficient conditions for the existence of generalized Zhou inverse of the sum $a + b$ are obtained under certain conditios.

**Lemma 4.1.** *Let $\mathcal{A}$ be a Banach algebra, if $a, b \in \mathcal{A}^z$ and $ab = 0$, then, $a + b \in \mathcal{A}^z$ and $(a+b)^z = \sum_{i=0}^{\infty} (b^z)^{i+1} a^i a^\pi + b^\pi \sum_{i=0}^{\infty} b^i (a^z)^{i+1}$.*

*Proof.*

$$\text{Let } A = \begin{pmatrix} 1 \\ a \end{pmatrix}, B = \begin{pmatrix} b & 1 \end{pmatrix}. \text{ Then } AB = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix},$$

since $ab = 0$ and $BA = a + b$, also $a, b \in \mathcal{A}^z$. Then, by Theorem 3.2, we have $AB \in (M_2(\mathcal{A}))^z$ and

$$(AB)^z = \begin{pmatrix} b^z & w \\ 0 & a^z \end{pmatrix}$$

where $w = \sum_{i=0}^{\infty} (b^z)^{i+2} a^i a^\pi + b^\pi \sum_{i=0}^{\infty} b^i (a^z)^{i+2} - b^z a^z$. By Cline's formula [2, Theorem 3.1], we have $BA = a + b \in \mathcal{A}^z$,

$$(a+b)^z = (BA)^z = B((AB)^z)^2 A = \begin{pmatrix} b & 1 \end{pmatrix} (\begin{pmatrix} b^z & w \\ 0 & a^z \end{pmatrix})^2 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

$=\sum_{i=0}^{\infty} (b^z)^{i+1} a^i a^\pi + b^\pi \sum_{i=0}^{\infty} b^i (a^z)^{i+1}$. □

**Theorem 4.2.** *Let $a \in \mathcal{A}^z$ , $b \in J^\#(\mathcal{A})$. If $aba = 0, ab^2 = 0$, then $a + b \in \mathcal{A}^z$ and $(a + b)^z = (a^z + bua)(1 + a^z b)$ where $u = \sum_{i=0}^{\infty} b^{2i}(a + b) (a^z)^{2i+4}$*

*Proof.*

$$\text{Let } X_1 = \begin{pmatrix} a \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & b \end{pmatrix}. \text{ Then, } a + b = X_2 X_1.$$

Let $M = X_1 X_2 = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$, so

$$M^2 = \begin{pmatrix} a^2 + ab & a^2 b \\ a + b & ab + b^2 \end{pmatrix} = \begin{pmatrix} ab & a^2 b \\ 0 & ab \end{pmatrix} + \begin{pmatrix} a^2 & 0 \\ a + b & b^2 \end{pmatrix} := F + G.$$

The conditions $aba = 0, ab^2 = 0$ imply $FG = 0, F^2 = 0$. Since $a \in \mathcal{A}^z$ then, $a^2 \in \mathcal{A}^z$ and $(a^2)^z = (a^z)^2$. As $b \in J^\#(\mathcal{A})$, then $b^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$, which implies that $b^z = 0$. Now we have $b^\pi = 1 - bb^z = 1$ and by applying Theorem 3.2, we obtain that $G \in (M_2(\mathcal{A}))^z$ and $G = \begin{pmatrix} (a^z)^2 & 0 \\ u & 0 \end{pmatrix}$ where

$$u = \sum_{i=0}^{\infty} b^{2i}(a + b) (a^z)^{2i+4}.$$

As $F^2 = 0$ , then $F^z = 0$. By Lemma 4.1, we deduce that $M^2 \in (M_2(\mathcal{A}))^z$, and

$$(M^2)^z = G^z + (G^z)^2 F = \begin{pmatrix} (a^z)^2 + (a^z)^3 b & (a^z)^2 b \\ u + ua^z b & ua^z ab \end{pmatrix}.$$

Applying Corollary 2.5, $M \in (M_2(\mathcal{A}))^z$. Finally, according to [2, Theorem 3.1], we have $a + b \in \mathcal{A}^z$ and

$$(a + b)^z = X_2 (M^2)^z X_1.$$

Observe that $a^z ba = 0$ and by a simple computation, we obtain the result. □

**Theorem 4.3.** *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^z$ and $s = (1 - b^\pi)a(1 - b^\pi) \in \mathcal{A}^z$. If $b^\pi aba = 0, b^\pi ab^2 = 0$ and $t = (1 - b^\pi)(a + b)(1 - b^\pi) \in \mathcal{A}^z$, then $a + b \in \mathcal{A}^z$. In this case, we have*

$$(a + b)^z = t^z + (1 - t^z a)x + \sum_{i=0}^{\infty} (t^z)^{i+2} ab^\pi (a + b)^i [1 - (a + b)x] + \sum_{i=0}^{\infty} t^\pi t^i (1 - b^\pi) ax^{i+2}, \qquad (9).$$

*Where $x = \sum_{i=0}^{\infty} b^\pi b^i (a^z)^{i+1} b^\pi (1 + a^z b)$.*

*Proof.* According to Lemma 3.3, we consider the matrix representation of $a, b$ relative to the idempotent $p = bb^z$,

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p,$$

where $b_1 \in \mathcal{A}_1^z, b_2 \in J^\#(\mathcal{A}_2)$. The condition $(b^\pi)ab^2 = 0$ expressed in matrix form yields

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_p = b^\pi ab^2 = \begin{pmatrix} 0 & 0 \\ a_{21}b_1^2 & a_{22}b_2^2 \end{pmatrix}_p.$$

Then we have

$$a_{21}b_1^2 = (1 - bb^z)abb^z b^4(b^z)^2 = (1 - bb^z)abb^z b^3 b^z = 0.$$

This gives

$$(1 - bb^z)abb^z b^3 b^z (b^z)^2 = (1 - bb^z)ab^4(b^z)^4 = (1 - bb^z)abb^z = 0$$

, which implies that

$$a_{21} = 0, a_{22}b_2^2 = 0.$$

Denote $a_1 = a_{11}, a_2 = a_{22}, a_3 = a_{12}$. Thus

$$a = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}_p, a + b = \begin{pmatrix} t & a_3 \\ 0 & a_2 + b_2 \end{pmatrix}_p.$$

Since $a_1 = s \in \mathcal{A}^z$, by Proposition 2.4, we have $a_1 \in \mathcal{A}_1^z$. Also $a \in \mathcal{A}^z$. Using Theorem 3.4(2), we deduce that $a_2 \in \mathcal{A}_2^z$ and

$$a^z = \begin{pmatrix} a_1^z & u_1 \\ 0 & a_2^z \end{pmatrix}_p.$$

From the condition $b^\pi aba = 0$, we can get

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_p = b^\pi aba = b^\pi ab^2 = \begin{pmatrix} 0 & 0 \\ 0 & a_2b_2a_2 \end{pmatrix}_p,$$

which implies $a_2b_2a_2 = 0$. Hence, applying Theorem 4.2 to $a_2, b_2$, we have $a_2 + b_2 \in \mathcal{A}_2^z$ and

$$(a_2 + b_2)^z = [a_2^z + \sum_{i=0}^{\infty} (b_2)^{2i+1} (a_2 + b_2) \left(a_2^z\right)^{2i+3}](1 - p + a_2^z b_2).$$

In order to give the expression of $(a_2 + b_2)^z$ in terms of $a, a^z, b, b^z$, we calculate $b^\pi a^z$,$b^\pi b^{2i+1}(a + b)(a^z)^{2i+3}$, $b^\pi a^z b$ separately in matrix form as follows

$$b^\pi a^z = \begin{pmatrix} 0 & 0 \\ 0 & a_2^z \end{pmatrix}_p, b^\pi a^z b = \begin{pmatrix} 0 & 0 \\ 0 & a_2^z b_2 \end{pmatrix}_p,$$

$$b^\pi b^{2i+1}(a + b)(a^z)^{2i+3} = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{2i+1}(a_2 + b_2)(a_2^z)^{2i+3} \end{pmatrix}_p.$$

Thus,

$$b^\pi a^z = a_2^z, b^\pi b^{2i+1}(a + b)(a^z)^{2i+3} = b_2^{2i+1}(a_2 + b_2)(a_2^z)^{2i+3}$$

and $b^\pi a^z b = a_2^z b_2$. Write $x = (a_2 + b_2)^z$. Note that

$$a(a^z)^{2i+3} = (a^z)^{2i+2} \; for \; i \geq 0,$$

then

$$x = b^{\pi}[a^{z} + \sum_{i=0}^{\infty} b^{2i+1}(a+b)\,(a^{z})^{2i+3}]b^{\pi}(1+a^{z}b)$$

$$= b^{\pi}[a^{z} + \sum_{i=0}^{\infty} b^{2i+1}\,(a^{z})^{2i+2} + \sum_{i=0}^{\infty} b^{2i+2}\,(a^{z})^{2i+3}]b^{\pi}(1+a^{z}b)$$

$$= b^{\pi}(\sum_{i=0}^{\infty} b^{i}\,(a^{z})^{i+1})b^{\pi}(1+a^{z}b).$$

Now, by Theorem 3.4, we have $a+b \in \mathcal{A}^{z}$ if and only if $t \in \mathcal{A}^{z}$. Moreover

$$(a+b)^{z} = \begin{pmatrix} t^{z} & u \\ 0 & x \end{pmatrix}_{p},$$

where

$$u = \sum_{i=0}^{\infty} (t^{z})^{i+2}\, a_3(a_2+b_2)^{i}(a_2+b_2)^{\pi} + \sum_{i=0}^{\infty} t^{\pi}t^{i}a_3x^{i+2} - t^{z}a_3x. \qquad (10)$$

As $b^{\pi}ab^{2} = 0$, then $b^{\pi}ab^{z} = 0$. Thus

$$a_2 + b_2 = b^{\pi}(a+b)b^{\pi} = b^{\pi}ab^{\pi} + b^{\pi}b = b^{\pi}a(1-bb^{z}) + b^{\pi}b = b^{\pi}(a+b)$$

which ensures

$$(a_2+b_2)^{i} = b^{\pi}(a+b)^{i}b^{\pi} \ for\ i \in \mathbb{N}.$$

Also we can easily obtain that $b^{\pi}(a+b)^{i}b^{\pi} = b^{\pi}(a+b)^{i}$, for $i \in \mathbb{N}$ by induction. Note $a_3 = (1-b^{\pi})ab^{\pi}$. Thus (10) reduces to

$$u = \sum_{i=0}^{\infty} (t^{z})^{i+2}\, ab^{\pi}(a+b)^{i}[1-(a+b)x] + \sum_{i=0}^{\infty} t^{\pi}t^{i}(1-b^{\pi})ax^{i+2} - t^{z}ax.$$

From $(a+b)^{z} = t^{z} + u + x$, we get (9) holds. □

**Corollary 4.4.** *Let $a, b \in \mathcal{A}^{z}$, and let $s = (1-b^{\pi})a(1-b^{\pi}) \in \mathcal{A}^{z}$. If $aba = 0, ab^{2} = 0$, then $a+b \in \mathcal{A}^{z}$. In this case, we have*

$$(a+b)^{z} = b^{z}a^{\pi} + (b^{z})^{2}aa^{\pi} + \sum_{i=1}^{\infty} (b^{z})^{i+2}\,(a^{i+1}a^{\pi} - a^{i+1}a^{z}b + a^{i}b)$$

$$+ \sum_{i=0}^{\infty} b^{\pi}b^{i}(a^{z})^{i+1}(1+a^{z}b) - b^{z}a^{z}b - (b^{z})^{2}aa^{z}b. \qquad (11)$$

*Proof.* From $ab^{2} = 0$, it follows that $ab^{z} = 0$. Thus, we can get $s = (1-b^{\pi})a(1-b^{\pi}) = 0 \in \mathcal{A}^{z}, t = (1-b^{\pi})(a+b)(1-b^{\pi}) = b(bb^{z})$. Since $(bb^{z})^{z} = bb^{z}$, using Theorem 2.2, we deduce that $t \in \mathcal{A}^{z}$, $t^{z} = b^{z}$. Thus, Theorem 4.3 is applicable. Furthermore, note that $a^{z}b^{z} = 0, aba^{z} = 0$. Let

$$x = \sum_{i=0}^{\infty} b^{\pi}b^{i}(a^{z})^{i+1}b^{\pi}(1+a^{z}b).$$

We have

$$ax = a\sum_{i=0}^{\infty} b^{\pi}b^{i}(a^{z})^{i+1}b^{\pi}(1+a^{z}b)$$

$$= a(1-bb^{z})a^{z}b^{\pi}(1+a^{z}b) + a(1-bb^{z})b(a^{z})^{2}b^{\pi}(1+a^{z}b)$$

$$= aa^z b^\pi (1 + a^z b) + ab(a^z)^2 b^\pi (1 + a^z b)$$
$$= aa^z (1 - bb^z)(1 + a^z b) = aa^z + a^z b,$$
$$abx = ab[\sum_{i=0}^{\infty} b^\pi b^i \, (a^z)^{i+1} \, b^\pi (1 + a^z b)]$$
$$= ab(1 - bb^z)(a^z) b^\pi (1 + a^z b) = 0.$$

Hence,

$$a[1 - (a + b)x] = a - a^2 x - abx = aa^\pi - aa^z b.$$

Note,

$$a(a + b)^i = a^i (a + b) \; for \; i \geq 1.$$

So,

$$ab^\pi (a + b)^i [1 - (a + b)x] = a(1 - bb^z)(a + b)^i [1 - (a + b)x]$$
$$= a(a + b)^i [1 - (a + b)x] = a^i (a + b)[1 - (a + b)x]$$
$$= a^i a[1 - (a + b)x] + a^i b[1 - (a + b)x]$$
$$= a^{i+1} a^\pi - a^{i+1} a^z b + a^i b.$$

Obsever that

$$t^\pi t^i (1 - b^\pi) a x^{i+2} = b^\pi (b^2 b^z)^i (1 - b^\pi) a x^{i+2} = 0 \; for \; i \geq 0.$$

Finally, by using these relations and (9) we get (11). □

**Theorem 4.5.** *Let $a, b, p \in \mathcal{A}$ be such that $a \in \mathcal{A}^z, p^2 = p, pa = ap, bp = p$ [resp. $pb = b$]. If $r = (a + b)p \in \mathcal{A}^z$, then $(a + b) \in \mathcal{A}^z$ and*

$$(a + b)^z = \sum_{i=0}^{\infty} (1 - p) a^\pi a^i b \, (r^z)^{i+3} \, (a + b) - a^z (1 - p) b (r^z)^2 (a + b) + a^z (1 - p)$$
$$+ \sum_{i=0}^{\infty} (a^z)^{i+2} \, (1 - p) b (a + b)^i [1 - r^z (a + b)] + p (r^z)^2 (a + b)$$

*[resp.*

$$(a + b)^z = r^z + \sum_{i=0}^{\infty} (r^z)^{i+2} \, b (1 - p) a^i a^\pi + (1 - r^z b)(1 - p) a^z + r^\pi \sum_{i=0}^{\infty} r^i b (1 - p) \, (a^z)^{i+2}]$$

.

*Proof.* We consider the matrix representation of $p, a, b$ relative to $p$ we have

$$p = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_p, a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p, b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_p.$$

The condition $pa = ap$ implies $a_{12} = 0, a_{21} = 0$ we denote $a_1 = a_{11}, a_2 = a_{22}$. Thus

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p.$$

Observe that $(1 - p)a = a(1 - p)$ and $(1 - p)^z = 1 - p$. By Theorem 2.2, we can conclude that $a_2 = (1 - p)a \in \mathcal{A}_2^z$ and $a_2^z = (1 - p)a^z = a^z(1 - p)$. From $bp = b$, it follows that $b_{12} = 0, b_{22} = 0$. Denote $b_1 = b_{11}, b_3 = b_{21}$. Hence

$$a + b = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p + \begin{pmatrix} b_1 & 0 \\ b_3 & 0 \end{pmatrix}_p = \begin{pmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 \end{pmatrix}_p.$$

Since $bp = b$, $a_1 + b_1 = p(a+b)p = p(a+b)$ which implies that $p(a+b)^i p = p(a+b)^i, (p(a+b))^i = p(a+b)^i$ for any $i \geq 0$. From the condition $r = (a+b)p \in \mathcal{A}^z$ and[2, Theorem 3.1], we deduce that $a_1 + b_1 \in \mathcal{A}_1^z$ and $(a_1+b_1)^z = p(r^z)^2(a+b)$. According to Theorem 3.4, we obtain $a+b \in \mathcal{A}^z$ and

$$(a+b)^z = \begin{pmatrix} (a_1+b_1)^z & 0 \\ u & a_2^z \end{pmatrix}_p.$$

Where,

$$u = \sum_{i=0}^{\infty} (a_2)^{i+2} b_3(a_1+b_1)^i(a_1+b_1)^\pi + \sum_{i=0}^{\infty} a_2^\pi a_2^i b_3 \left((a_1+b_1)^z\right)^{i+2} - a_2^z b_3(a_1+b_1)^z.$$

Note that

$$\begin{aligned} &(a_2^z)^{i+2} b_3(a_1+b_1)^i(a_1+b_1)^\pi = \\ &[a^z(1-p)]^{i+2}(1-p)bp[p(a+b)]^i[p - p(a+b)p(r^z)^2(a+b)] \\ &= (a^z)^{i+2}(1-p)bp(a+b)^i p[1-(a+b)p(r^z)^2(a+b)], a_2^\pi a_2^i b_3[(a_1+b_1)^z]^{i+2} \\ &= [(1-p) - (1-p)aa^z(1-p)][(1-p)a]^i(1-p)bp[p(r^z)^2(a+b)]^{i+2} \\ &= (1-p)(1-aa^z)(1-p)a^i(1-p)bp[p(r^z)^{i+3}(a+b)] \\ &= (1-p)a^\pi a^i b(r^z)^{i+3}(a+b), \\ &a_2^z b_3(a_1+b_1)^z = a^z(1-p)(1-p)bpp(r^z)^2(a+b) = a^z(1-p)b(r^z)^2(a+b). \end{aligned}$$

Therefore we have result. The proof for the case $pb = b$ is similar. □

**Corollary 4.6.** *Let $a \in \mathcal{A}^z, b \in \mathcal{A}$ be such that $ba^z = 0$ [resp. $a^z b = 0$], $r = (a+b)a^\pi \in \mathcal{A}^z$. Then $a+b \in \mathcal{A}^z$ and we have*

$$(a+b)^z = $$

$$\sum_{i=0}^{\infty} (a^z)^{i+2} b(a+b)^i[1 - r^z(a+b)] + a^z + [1 - a^z(a+b)](r^z)^2(a+b).$$

*[resp.$(a+b)^z = a^z + r^z + r^\pi \sum_{i=0}^{\infty} r^i b\,(a^z)^{i+2} - r^z b a^z$].*

*Proof.* By $p = a^\pi$ in Theorem 4.5, we obtain the result. □


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