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Mappings preserving sum of products $\alpha_1 ab^* + \alpha_2 b^* a + \alpha_3 ba^*$ (resp., $\alpha_1 ab^* + \alpha_2 b^* a + \alpha_3 a^* b$) on $*$-algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital prime complex $*$-algebras such that $\mathcal{A}$ has a nontrivial projection. In this paper, we study the structure of the bijective mappings $\Phi : \mathcal{A} \to \mathcal{B}$ preserving sum of products $\alpha_1 ab^* + \alpha_2 b^* a + \alpha_3 ba^*$ (resp., $\alpha_1 ab^* + \alpha_2 b^* a + \alpha_3 a^* b$), where the scalars $\{\alpha_k\}_{k=1}^3$ are rational numbers satisfying some conditions.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathcal{A}$, and that it is *multiplicative* or that *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{A}$. Denote by $\mathbb{Q}[i] = \mathbb{Q} + \mathbb{Q}i$. We say that an additive mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{Q}[i]$-linear map (resp., conjugate $\mathbb{Q}[i]$-linear map) if $\Phi(\alpha a) = \alpha\Phi(a)$ (resp., $\Phi(\alpha a) = \overline{\alpha}\Phi(a)$), for all elements $\alpha \in \mathbb{Q}[i]$ and $a \in \mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be complex $*$-algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all elements $a \in \mathcal{A}$.

An algebra $\mathcal{A}$ is said to be *prime* if $a\mathcal{A}b = 0$ implies that $a = 0$ or $b = 0$. An element p of a $*$-algebra $\mathcal{A}$ is said to be *projection* if it is an idempotent element satisfying the condition $p^* = p$. The *opposite algebra* is a new algebra, denoted by $\mathcal{A}^{op}$, obtained from the algebra $\mathcal{A}$ by redefining multiplication by $a \odot b = ba$, for all elements $a, b \in \mathcal{A}^{op}$, called *reverse multiplication*. It is evident that $*$ is an involution on $\mathcal{A}$ if and only if $*$ is an involution on $\mathcal{A}^{op}$, that an element is the multiplicative identity (resp., a projection) of $\mathcal{A}$ if and only if it is also the multiplicative identity (resp., a projection) of $\mathcal{A}^{op}$ and that $\mathcal{A}$ is prime if and only if $\mathcal{A}^{op}$ is prime.

Recently, many mathematicians devoted themselves to study bijective mappings preserving new products on $*$-algebras (see the works [1], [2] and [3] and the references therein). These products play very important roles in some research fields. In particular, the authors in [1] showed that bijective mappings $\Phi : \mathcal{A} \to \mathcal{B}$, on factor von Neumann algebras and satisfying $\Phi(ab^* - b^*a) = \Phi(a)\Phi(b)^* - \Phi(b)^*\Phi(a)$, for all elements $a, b \in \mathcal{A}$, are of the form $\sigma + \tau$, where σ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-anti-isomorphism, or the negative of a conjugate linear $*$-anti-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and τ is a mapping of $\mathcal{A}$ into $\mathbb{C}1_{\mathcal{A}}$ which maps commutators into zero and the

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authors in [2] showed that bijective mappings $\Phi : \mathcal{A} \to \mathcal{B}$, on factor von Neumann algebras and satisfying $\Phi(ab^* + b^*a) = \Phi(a)\Phi(b)^* + \Phi(b)^*\Phi(a)$, for all elements $a, b \in \mathcal{A}$, are $*$-ring isomorphisms. The purpose of the present paper is to study the structure of bijective mappings preserving two families of sums of products, related to products $ab^* - b^*a$ and $ab^* + b^*a$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two complex $*$-algebras and $\{\alpha_k\}_{k=1}^3$ arbitrary rational numbers. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves sum of products* $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 ba^*$ (resp., $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 a^*b$) if

$$\Phi(\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 ba^*) = \alpha_1\Phi(a)\Phi(b)^* + \alpha_2\Phi(b)^*\Phi(a) + \alpha_3\Phi(b)\Phi(a)^* \tag{1}$$
$$(\text{resp., } \Phi(\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 a^*b) = \alpha_1\Phi(a)\Phi(b)^* + \alpha_2\Phi(b)^*\Phi(a) + \alpha_3\Phi(a)^*\Phi(b)),$$

for all elements $a, b \in \mathcal{A}$.

Lemma 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras, $\mathcal{A}^{op}$ and $\mathcal{B}^{op}$ their respective opposite algebras, $\{\alpha_k\}_{k=1}^3$ arbitrary rational numbers, $\Phi : \mathcal{A} \to \mathcal{B}$ a map and $\Phi^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ a map defined by $\Phi^{op}(a) = \Phi(a)$, for all elements a of $\mathcal{A}^{op}$. Then, Φ preserves sum of products $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 a^*b$ if and only if Φ^{op} preserves sum of products $\alpha_2 a \odot b^* + \alpha_1 b^* \odot a + \alpha_3 b \odot a^*$.*

Our main result reads as follows.

Theorem 1.2 (Main Theorem). *Let α_1, α_2 be two nonzero rational numbers and α_3 a rational number such that $|\alpha_1 + \alpha_2| - |\alpha_3| \neq 0$, $\mathcal{A}$ and $\mathcal{B}$ two unital prime complex $*$-algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ their multiplicative identities, respectively, and such that $\mathcal{A}$ has a nontrivial projection. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ preserving sum of products $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 ba^*$ (resp., $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 a^*b$) is additive. Moreover, Φ is a $\mathbb{Q}[i]$-linear multiplicative map preserving involution or a conjugate $\mathbb{Q}[i]$-linear multiplicative map preserving involution, if the following condition holds: $\alpha_1 - \alpha_2 + \alpha_3 \neq 0$ (resp., $-\alpha_1 + \alpha_2 + \alpha_3 \neq 0$).*

2. The proof of Main Theorem

Due to Lemma 1.1, we prove the Main Theorem only for the map preserving sums of products $\alpha_1 ab^* + \alpha_2 b^*a + \alpha_3 ba^*$. The proof is made by considering several lemmas. The first two lemmas have easy proofs, and we omit the details.

Lemma 2.1. *If $\Phi(c) = \Phi(a) + \Phi(b)$, for elements a, b, c of $\mathcal{A}$, then the following identity holds: $\Phi(\alpha_1 ct^* + \alpha_2 t^*c + \alpha_3 tc^*) = \Phi(\alpha_1 at^* + \alpha_2 t^*a + \alpha_3 ta^*) + \Phi(\alpha_1 bt^* + \alpha_2 t^*b + \alpha_3 tb^*)$, for all elements t of $\mathcal{A}$.*

Lemma 2.2. $\Phi(0) = 0$.

Lemma 2.3. *If α_1, α_2 are two nonzero rational numbers and α_3 a rational number such that $|\alpha_1 + \alpha_2| - |\alpha_3| \neq 0$, then Φ is an additive mapping.*

We will establish the proof of Lemma 2.3 in a series of Properties. We begin, though, with a well-known result that will be used throughout this paper: Let p_i be an arbitrary nontrivial projection of $\mathcal{A}$ and write $p_j = 1_{\mathcal{A}} - p_i$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{ii} \oplus \mathcal{A}_{ij} \oplus \mathcal{A}_{ji} \oplus \mathcal{A}_{jj}$, where $\mathcal{A}_{ij} = p_i\mathcal{A}p_j$ $(i, j = 1, 2)$.

Property 2.4. *For arbitrary elements $a_{ii} \in \mathcal{A}_{ii}$, $b_{ij} \in \mathcal{A}_{ij}$ and $c_{ji} \in \mathcal{A}_{ji}$ $(i \neq j; i, j = 1, 2)$ the following hold: (i) $\Phi(a_{ii} + b_{ij}) = \Phi(a_{ii}) + \Phi(b_{ij})$ and (ii) $\Phi(a_{ii} + c_{ji}) = \Phi(a_{ii}) + \Phi(c_{ji})$.*

Proof. According to the hypothesis on Φ there exists $f = f_{ii} + f_{ij} + f_{ji} + f_{jj} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{ii}) + \Phi(b_{ij})$. Hence, by Lemma 2.1, we have

$$\Phi(\alpha_1 fp_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^*) = \Phi(\alpha_1 a_{ii}p_j^* + \alpha_2 p_j^* a_{ii} + \alpha_3 p_j a_{ii}^*)$$
$$+ \Phi(\alpha_1 b_{ij}p_j^* + \alpha_2 p_j^* b_{ij} + \alpha_3 p_j b_{ij}^*) = \Phi(\alpha_1 b_{ij} + \alpha_3 b_{ij}^*).$$

This implies that $\alpha_1 fp_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^* = \alpha_1 b_{ij} + \alpha_3 b_{ij}^*$ which leads to $\alpha_1 f_{ij} + \alpha_2 f_{ji} + \alpha_3 f_{ij}^* + (\alpha_1 + \alpha_2) f_{jj} + \alpha_3 f_{jj}^* = \alpha_1 b_{ij} + \alpha_3 b_{ij}^*$. It follows from this last identity that (†1) $\alpha_1 f_{ij} = \alpha_1 b_{ij}$, (†2) $\alpha_2 f_{ji} + \alpha_3 f_{ij}^* = \alpha_3 b_{ij}^*$ and (†3)

$(\alpha_1+\alpha_2)f_{jj}+\alpha_3 f_{jj}^* = 0$, by directness of the Peirce decomposition, and (†4) $\alpha_3 f_{jj}+(\alpha_1+\alpha_2)f_{jj}^* = 0$, by applying involution on (†3). From (†1) and (†2), we obtain $f_{ij} = b_{ij}$ and $f_{ji} = 0$. Next, multiplying (†3) by $\alpha_1+\alpha_2$, (†4) by α_3 and subtracting from each other we arrive at $(|\alpha_1+\alpha_2|^2 - |\alpha_3|^2)f_{jj} = 0$ which shows that $f_{jj} = 0$. This shows that $\Phi(f_{ii}+b_{ij}) = \Phi(a_{ii})+\Phi(b_{ij})$. Hence, for an arbitrary element $d_{ji}\in\mathcal{A}_{ji}$, we have

$$\Phi(\alpha_1(f_{ii}+b_{ij})d_{ji}^*+\alpha_2 d_{ji}^*(f_{ii}+b_{ij})+\alpha_3 d_{ji}(f_{ii}+b_{ij})^*) = \Phi(\alpha_1 a_{ii}d_{ji}^*+\alpha_2 d_{ji}^* a_{ii}+\alpha_3 d_{ji}a_{ii}^*)+\Phi(\alpha_1 b_{ij}d_{ji}^*+\alpha_2 d_{ji}^* b_{ij}+\alpha_3 d_{ji}b_{ij}^*)$$

which yields that $\Phi(\alpha_1 f_{ii}d_{ji}^*+\alpha_3 d_{ji}f_{ii}^*) = \Phi(\alpha_1 a_{ii}d_{ji}^*+\alpha_3 d_{ji}a_{ii}^*)$. As consequence, we obtain $\alpha_1 f_{ii}d_{ji}^*+\alpha_3 d_{ji}f_{ii}^* = \alpha_1 a_{ii}d_{ji}^*+\alpha_3 d_{ji}a_{ii}^*$ which allows to conclude that $\alpha_1 f_{ii}d_{ji}^* = \alpha_1 a_{ii}d_{ji}^*$. Therefore, $f_{ii} = a_{ii}$.

Similarly, we prove the case (ii). □

Property 2.5. *For arbitrary elements $b_{ij}\in\mathcal{A}_{ij}$ and $c_{ji}\in\mathcal{A}_{ji}$ $(i\neq j; i,j = 1,2)$ the following holds: $\Phi(b_{ij}+c_{ji}) = \Phi(b_{ij})+\Phi(c_{ji})$.*

Proof. Choose $f = f_{ii}+f_{ij}+f_{ji}+f_{jj}\in\mathcal{A}$ such that $\Phi(f) = \Phi(b_{ij})+\Phi(c_{ji})$. Hence, for an arbitrary element $d_{ij}\in\mathcal{A}_{ij}$, we have

$$\Phi(\alpha_1 f d_{ij}^*+\alpha_2 d_{ij}^* f+\alpha_3 d_{ij}f^*) = \Phi(\alpha_1 b_{ij}d_{ij}^*+\alpha_2 d_{ij}^* b_{ij}+\alpha_3 d_{ij}b_{ij}^*) + \Phi(\alpha_1 c_{ji}d_{ij}^*+\alpha_2 d_{ij}^* c_{ji}+\alpha_3 d_{ij}c_{ji}^*) = \Phi(\alpha_1 b_{ij}d_{ij}^*+\alpha_2 d_{ij}^* b_{ij}+\alpha_3 d_{ij}b_{ij}^*).$$

This implies that $\alpha_1 f d_{ij}^*+\alpha_2 d_{ij}^* f+\alpha_3 d_{ij}f^* = \alpha_1 b_{ij}d_{ij}^*+\alpha_2 d_{ij}^* b_{ij}+\alpha_3 d_{ij}b_{ij}^*$ which results in (†1) $\alpha_1 f_{ij}d_{ij}^*+\alpha_3 d_{ij}f_{ij}^*+\alpha_3 d_{ij}f_{jj}^*+\alpha_1 f_{jj}d_{ij}^*+\alpha_2 d_{ij}^* f_{ii}+\alpha_2 d_{ij}^* f_{ij} = \alpha_1 b_{ij}d_{ij}^*+\alpha_3 d_{ij}b_{ij}^*+\alpha_2 d_{ij}^* b_{ij}$. By directness of the Peirce decomposition, we obtain that $\alpha_2 d_{ij}^* f_{ij} = \alpha_2 d_{ij}^* b_{ij}$ which shows that $f_{ij} = b_{ij}$. As a consequence (†1) becomes (†2) $\alpha_3 d_{ij}f_{jj}^*+\alpha_1 f_{jj}d_{ij}^*+\alpha_2 d_{ij}^* f_{ii} = 0$. Now, for an arbitrary element $d_{ji}\in\mathcal{A}_{ji}$, we have

$$\Phi(\alpha_1 f d_{ji}^*+\alpha_2 d_{ji}^* f+\alpha_3 d_{ji}f^*) = \Phi(\alpha_1 b_{ij}d_{ji}^*+\alpha_2 d_{ji}^* b_{ij}+\alpha_3 d_{ji}b_{ij}^*) + \Phi(\alpha_1 c_{ji}d_{ji}^*+\alpha_2 d_{ji}^* c_{ji}+\alpha_3 d_{ji}c_{ji}^*) = \Phi(\alpha_1 c_{ji}d_{ji}^*+\alpha_2 d_{ji}^* c_{ji}+\alpha_3 d_{ji}c_{ji}^*)$$

that shows that $\alpha_1 f d_{ji}^*+\alpha_2 d_{ji}^* f+\alpha_3 d_{ji}f^* = \alpha_1 c_{ji}d_{ji}^*+\alpha_2 d_{ji}^* c_{ji}+\alpha_3 d_{ji}c_{ji}^*$. This results in $\alpha_2 d_{ji}^* f_{ji}+\alpha_1 f_{ii}d_{ji}^*+\alpha_2 d_{ji}^* f_{jj}+\alpha_3 d_{ji}f_{ii}^*+\alpha_1 f_{ji}d_{ji}^*+\alpha_3 d_{ji}f_{ji}^* = \alpha_2 d_{ji}^* c_{ji}+\alpha_1 c_{ji}d_{ji}^*+\alpha_3 d_{ji}c_{ji}^*$ which shows that $\alpha_2 d_{ji}^* f_{ji} = \alpha_2 d_{ji}^* c_{ji}$ and which leads to $f_{ji} = c_{ji}$. It therefore follows that

$$\Phi(\alpha_1 f p_i^*+\alpha_2 p_i^* f+\alpha_3 p_i f^*) = \Phi(\alpha_1 b_{ij}p_i^*+\alpha_2 p_i^* b_{ij}+\alpha_3 p_i b_{ij}^*) + \Phi(\alpha_1 c_{ji}p_i^*+\alpha_2 p_i^* c_{ji}+\alpha_3 p_i c_{ji}^*) = \Phi(\alpha_2 b_{ij})+\Phi(\alpha_1 c_{ji}+\alpha_3 c_{ji}^*)$$

which implies that $\Phi((\alpha_1+\alpha_2)f_{ii}+\alpha_3 f_{ii}^*+\alpha_2 b_{ij}+\alpha_1 c_{ji}+\alpha_3 c_{ji}^*) = \Phi(\alpha_2 b_{ij})+\Phi(\alpha_1 c_{ji}+\alpha_3 c_{ji}^*)$. Define $r_{ii} = (\alpha_1+\alpha_2)f_{ii}+\alpha_3 f_{ii}^*$, $r_{ij} = \alpha_2 b_{ij}+\alpha_3 c_{ji}^*$ and $r_{ji} = \alpha_1 c_{ji}$, $r = r_{ii}+r_{ij}+r_{ji}$, $s_{ij} = \alpha_2 b_{ij}$ and $t_{ji}+t_{ij} = \alpha_1 c_{ji}+\alpha_3 c_{ji}^*$, then $r_{ij} = s_{ij}+t_{ij}$, $r_{ji} = t_{ji}$ and $\Phi(r) = \Phi(s_{ij})+\Phi(t_{ji}+t_{ij})$. Hence, for an arbitrary element $d_{ji}\in\mathcal{A}_{ji}$, we have

$$\Phi(\alpha_1 r d_{ji}^*+\alpha_2 d_{ji}^* r+\alpha_3 d_{ji}r^*) = \Phi(\alpha_1 s_{ij}d_{ji}^*+\alpha_2 d_{ji}^* s_{ij}+\alpha_3 d_{ji}s_{ij}^*) + \Phi(\alpha_1(t_{ji}+t_{ij})d_{ji}^*+\alpha_2 d_{ji}^*(t_{ji}+t_{ij})+\alpha_3 d_{ji}(t_{ji}+t_{ij})^*) = \Phi(\alpha_1 t_{ji}d_{ji}^*+\alpha_2 d_{ji}^* t_{ji}+\alpha_3 d_{ji}t_{ji}^*).$$

This shows that $\alpha_1 r d_{ji}^*+\alpha_2 d_{ji}^* r+\alpha_3 d_{ji}r^* = \alpha_1 t_{ji}d_{ji}^*+\alpha_2 d_{ji}^* t_{ji}+\alpha_3 d_{ji}t_{ji}^*$ which leads to $\alpha_1 r_{ii}d_{ji}^*+\alpha_1 r_{ji}d_{ji}^*+\alpha_2 d_{ji}^* r_{ji}+\alpha_3 d_{ji}r_{ii}^*+\alpha_3 d_{ji}r_{ji}^* = \alpha_1 t_{ji}d_{ji}^*+\alpha_2 d_{ji}^* t_{ji}+\alpha_3 d_{ji}t_{ji}^*$. As a result, we have $\alpha_1 r_{ii}d_{ji}^* = 0$ which implies that $r_{ii} = 0$, that is $(\alpha_1+\alpha_2)f_{ii}+\alpha_3 f_{ii}^* = 0$. By using similar reasoning to that in the proof of Property 2.4, we arrive at $(|\alpha_1+\alpha_2|^2-|\alpha_3|^2)f_{ii} = 0$ which shows that $f_{ii} = 0$. It hence follows that $\alpha_1 f_{jj}d_{ij}^* = 0$, by identity in (†2), which allows to conclude that $f_{jj} = 0$. □

Property 2.6. *For arbitrary elements $a_{ii}\in\mathcal{A}_{ii}$, $b_{ij}\in\mathcal{A}_{ij}$, $c_{ji}\in\mathcal{A}_{ji}$ and $d_{jj}\in\mathcal{A}_{jj}$ $(i\neq j; i,j = 1,2)$ the following hold: (i) $\Phi(a_{ii}+b_{ij}+c_{ji}) = \Phi(a_{ii})+\Phi(b_{ij})+\Phi(c_{ji})$ and (ii) $\Phi(b_{ij}+c_{ji}+d_{jj}) = \Phi(b_{ij})+\Phi(c_{ji})+\Phi(d_{jj})$.*

Proof. Choose $f = f_{ii} + f_{ij} + f_{ji} + f_{jj} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{ii}) + \Phi(b_{ij}) + \Phi(c_{ji})$. By Property 2.5 we can write $\Phi(f) = \Phi(a_{ii}) + \Phi(b_{ij} + c_{ji})$. Hence, we have

$$\Phi(\alpha_1 f p_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^*) = \Phi(\alpha_1 a_{ii} p_j^* + \alpha_2 p_j^* a_{ii} + \alpha_3 p_j a_{ii}^*) + \Phi(\alpha_1 (b_{ij} + c_{ji}) p_j^* + \alpha_2 p_j^* (b_{ij} + c_{ji}) + \alpha_3 p_j (b_{ij} + c_{ji})^*)$$

which implies that $\alpha_1 f p_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^* = \alpha_1 (b_{ij} + c_{ji}) p_j^* + \alpha_2 p_j^* (b_{ij} + c_{ji}) + \alpha_3 p_j (b_{ij} + c_{ji})^*$. This results that $\alpha_1 f_{ij} + \alpha_2 f_{ji} + \alpha_3 f_{ij}^* + (\alpha_1 + \alpha_2) f_{jj} + \alpha_3 f_{jj}^* = \alpha_1 b_{ij} + \alpha_2 c_{ji} + \alpha_3 b_{ij}^*$ which implies that (†1) $\alpha_1 f_{ij} = \alpha_1 b_{ij}$, (†2) $\alpha_2 f_{ji} + \alpha_3 f_{ij}^* = \alpha_2 c_{ji} + \alpha_3 b_{ij}^*$ and (†3) $(\alpha_1 + \alpha_2) f_{jj} + \alpha_3 f_{jj}^* = 0$, by directness of Peirce decomposition. As consequence of (†1) and (†2) we obtain $f_{ij} = b_{ij}$ and $f_{ji} = c_{ji}$, respectively. By using similar reasoning to that in the proof of Property 2.4, then (†3) becomes $(|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2) f_{jj} = 0$ which results in $f_{jj} = 0$. Hence, $\Phi(f_{ii} + b_{ij} + c_{ji}) = \Phi(a_{ii}) + \Phi(b_{ij}) + \Phi(c_{ji}) = \Phi(a_{ii} + b_{ij}) + \Phi(c_{ji})$. Next, for an arbitrary element $d_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}
&\Phi(\alpha_1 (f_{ii} + b_{ij} + c_{ji}) d_{ij}^* + \alpha_2 d_{ij}^* (f_{ii} + b_{ij} + c_{ji}) + \alpha_3 d_{ij} (f_{ii} + b_{ij} + c_{ji})^*) \\
&= \Phi(\alpha_1 (a_{ii} + b_{ij}) d_{ij}^* + \alpha_2 d_{ij}^* (a_{ii} + b_{ij}) + \alpha_3 d_{ij} (a_{ii} + b_{ij})^*) + \Phi(\alpha_1 c_{ji} d_{ij}^* + \alpha_2 d_{ij}^* c_{ji} + \alpha_3 d_{ij} c_{ji}^*) \\
&= \Phi(\alpha_1 (a_{ii} + b_{ij}) d_{ij}^* + \alpha_2 d_{ij}^* (a_{ii} + b_{ij}) + \alpha_3 d_{ij} (a_{ii} + b_{ij})^*).
\end{aligned}$$

This shows that $\alpha_1 (f_{ii} + b_{ij} + c_{ji}) d_{ij}^* + \alpha_2 d_{ij}^* (f_{ii} + b_{ij} + c_{ji}) + \alpha_3 d_{ij} (f_{ii} + b_{ij} + c_{ji})^* = \alpha_1 (a_{ii} + b_{ij}) d_{ij}^* + \alpha_2 d_{ij}^* (a_{ii} + b_{ij}) + \alpha_3 d_{ij} (a_{ii} + b_{ij})^*$ which implies that $\alpha_2 d_{ij}^* f_{ii} = \alpha_2 d_{ij}^* a_{ii}$. As consequence, we have $f_{ii} = a_{ii}$.

Similarly, we prove the case (ii). □

Property 2.7. *For arbitrary elements $a_{ii} \in \mathcal{A}_{ii}$, $b_{ij} \in \mathcal{A}_{ij}$, $c_{ji} \in \mathcal{A}_{ji}$ and $d_{jj} \in \mathcal{A}_{jj}$ $(i \neq j; i, j = 1, 2)$ the following holds: $\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj}) = \Phi(a_{ii}) + \Phi(b_{ij}) + \Phi(c_{ji}) + \Phi(d_{jj})$.*

Proof. Choose $f = f_{ii} + f_{ij} + f_{ji} + f_{jj} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{ii}) + \Phi(b_{ij}) + \Phi(c_{ji}) + \Phi(d_{jj})$. By Property 2.6(i) we have $\Phi(f) = \Phi(a_{ii} + b_{ij} + c_{ji}) + \Phi(d_{jj})$ which implies that

$$\Phi(\alpha_1 f p_i^* + \alpha_2 p_i^* f + \alpha_3 p_i f^*) = \Phi(\alpha_1 (a_{ii} + b_{ij} + c_{ji}) p_i^* + \alpha_2 p_i^* (a_{ii} + b_{ij} + c_{ji}) + \alpha_3 p_i (a_{ii} + b_{ij} + c_{ji})^*) + \Phi(\alpha_1 d_{jj} p_i^* + \alpha_2 p_i^* d_{jj} + \alpha_3 p_i d_{jj}^*).$$

It follows that $\alpha_1 f p_i^* + \alpha_2 p_i^* f + \alpha_3 p_i f^* = \alpha_1 (a_{ii} + b_{ij} + c_{ji}) p_i^* + \alpha_2 p_i^* (a_{ii} + b_{ij} + c_{ji}) + \alpha_3 p_i (a_{ii} + b_{ij} + c_{ji})^*$ which implies that (†1) $(\alpha_1 + \alpha_2) f_{ii} + \alpha_3 f_{ii}^* = (\alpha_1 + \alpha_2) a_{ii} + \alpha_3 a_{ii}^*$, (†2) $\alpha_2 f_{ij} + \alpha_3 f_{ji}^* = \alpha_2 b_{ij} + \alpha_3 c_{ji}^*$ and (†3) $\alpha_1 f_{ji} = \alpha_1 c_{ji}$, by directness of Peirce decomposition. Now, again as seen earlier (†1) becomes $(|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2) f_{ii} = (|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2) a_{ii}$ which implies that $f_{ii} = a_{ii}$ and combining (†2) and (†3) we get $f_{ij} = b_{ij}$ and $f_{ji} = c_{ji}$.

By a similar reasoning, we obtain $f_{jj} = d_{jj}$. □

Property 2.8. *For arbitrary elements $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$ $(i \neq j; i, j = 1, 2)$ the following holds: $\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$.*

Proof. Two cases are considered. First case: $\alpha_3 \neq 0$. In this case, we observe that the following identity holds

$$\alpha_1 (p_j + a_{ij})(p_i + b_{ij})^* + \alpha_2 (p_i + b_{ij})^* (p_j + a_{ij}) + \alpha_3 (p_i + b_{ij})(p_j + a_{ij})^* = \alpha_2 a_{ij} + \alpha_3 b_{ij} + \alpha_1 b_{ij}^* + \alpha_1 a_{ij} b_{ij}^* + \alpha_3 b_{ij} a_{ij}^* + \alpha_2 b_{ij}^* a_{ij}.$$

Hence, by Property 2.7 we have

$$\begin{aligned}
&\Phi(\alpha_2 a_{ij} + \alpha_3 b_{ij}) + \Phi(\alpha_1 b_{ij}^*) + \Phi(\alpha_1 a_{ij} b_{ij}^* + \alpha_3 b_{ij} a_{ij}^* + \alpha_2 b_{ij}^* a_{ij}) \\
&= \Phi(\alpha_2 a_{ij} + \alpha_3 b_{ij} + \alpha_1 b_{ij}^* + \alpha_1 a_{ij} b_{ij}^* + \alpha_3 b_{ij} a_{ij}^* + \alpha_2 b_{ij}^* a_{ij}) \\
&= \Phi(\alpha_1 (p_j + a_{ij})(p_i + b_{ij})^* + \alpha_2 (p_i + b_{ij})^* (p_j + a_{ij}) \\
&+ \alpha_3 (p_i + b_{ij})(p_j + a_{ij})^*) = \alpha_1 \Phi(p_j + a_{ij}) \Phi(p_i + b_{ij})^* \\
&+ \alpha_2 \Phi(p_i + b_{ij})^* \Phi(p_j + a_{ij}) + \alpha_3 \Phi(p_i + b_{ij}) \Phi(p_j + a_{ij})^*
\end{aligned}$$

$$= \alpha_1(\Phi(p_j) + \Phi(a_{ij}))(\Phi(p_i)^* + \Phi(b_{ij})^*)$$
$$+ \alpha_2(\Phi(p_i)^* + \Phi(b_{ij})^*)(\Phi(p_j) + \Phi(a_{ij}))$$
$$+ \alpha_3(\Phi(p_i) + \Phi(b_{ij}))(\Phi(p_j)^* + \Phi(a_{ij})^*)$$
$$= \alpha_1\Phi(p_j)\Phi(p_i)^* + \alpha_2\Phi(p_i)^*\Phi(p_j) + \alpha_3\Phi(p_i)\Phi(p_j)^*$$
$$+ \alpha_1\Phi(p_j)\Phi(b_{ij})^* + \alpha_2\Phi(b_{ij})^*\Phi(p_j) + \alpha_3\Phi(b_{ij})\Phi(p_j)^*$$
$$+ \alpha_1\Phi(a_{ij})\Phi(p_i)^* + \alpha_2\Phi(p_i)^*\Phi(a_{ij}) + \alpha_3\Phi(p_i)\Phi(a_{ij})^*$$
$$+ \alpha_1\Phi(a_{ij})\Phi(b_{ij})^* + \alpha_2\Phi(b_{ij})^*\Phi(a_{ij}) + \alpha_3\Phi(b_{ij})\Phi(a_{ij})^*$$
$$= \Phi(\alpha_1 p_j p_i^* + \alpha_2 p_i^* p_j + \alpha_3 p_i p_j^*) + \Phi(\alpha_1 p_j b_{ij}^* + \alpha_2 b_{ij}^* p_j + \alpha_3 b_{ij} p_j^*)$$
$$+ \Phi(\alpha_1 a_{ij} p_i^* + \alpha_2 p_i^* a_{ij} + \alpha_3 p_i a_{ij}^*) + \Phi(\alpha_1 a_{ij} b_{ij}^* + \alpha_2 b_{ij}^* a_{ij} + \alpha_3 b_{ij} a_{ij}^*)$$
$$= \Phi(\alpha_1 b_{ij}^*) + \Phi(\alpha_3 b_{ij}) + \Phi(\alpha_2 a_{ij}) + \Phi(\alpha_1 a_{ij} b_{ij}^* + \alpha_2 b_{ij}^* a_{ij} + \alpha_3 b_{ij} a_{ij}^*)$$

which shows that $\Phi(\alpha_2 a_{ij} + \alpha_3 b_{ij}) = \Phi(\alpha_2 a_{ij}) + \Phi(\alpha_3 b_{ij})$. Therefore, $\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$. Second case: $\alpha_3 = 0$. In this case, we observe that the following identity holds

$$\alpha_1(p_i + a_{ij})(p_j + b_{ij}^*)^* + \alpha_2(p_j + b_{ij}^*)^*(p_i + a_{ij}) + 0(p_j + b_{ij}^*)(p_i + a_{ij})^* = \alpha_1 a_{ij} + \alpha_1 b_{ij},$$

for all elements $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$. Hence, by Property 2.7 again, we have

$$\Phi(\alpha_1 a_{ij} + \alpha_1 b_{ij}) = \Phi(\alpha_1(p_i + a_{ij})(p_j + b_{ij}^*)^* + \alpha_2(p_j + b_{ij}^*)^*(p_i + a_{ij})$$
$$+ 0(p_j + b_{ij}^*)(p_i + a_{ij})^*) = \alpha_1\Phi(p_i + a_{ij})\Phi(p_j + b_{ij}^*)^*$$
$$+ \alpha_2\Phi(p_j + b_{ij}^*)^*\Phi(p_i + a_{ij}) + 0\Phi(p_j + b_{ij}^*)\Phi(p_i + a_{ij})^*$$
$$= \alpha_1(\Phi(p_i) + \Phi(a_{ij}))(\Phi(p_j)^* + \Phi(b_{ij}^*)^*)$$
$$+ \alpha_2(\Phi(p_j)^* + \Phi(b_{ij}^*)^*)(\Phi(p_i) + \Phi(a_{ij}))$$
$$+ 0(\Phi(p_j) + \Phi(b_{ij}^*))(\Phi(p_i)^* + \Phi(a_{ij})^*)$$
$$= \alpha_1\Phi(p_i)\Phi(p_j)^* + \alpha_2\Phi(p_j)^*\Phi(p_i) + 0\Phi(p_j)\Phi(p_i)^*$$
$$+ \alpha_1\Phi(p_i)\Phi(b_{ij}^*)^* + \alpha_2\Phi(b_{ij}^*)^*\Phi(p_i) + 0\Phi(b_{ij}^*)\Phi(p_i)^*$$
$$+ \alpha_1\Phi(a_{ij})\Phi(p_j)^* + \alpha_2\Phi(p_j)^*\Phi(a_{ij}) + 0\Phi(p_j)\Phi(a_{ij})^*$$
$$+ \alpha_1\Phi(a_{ij})\Phi(b_{ij}^*)^* + \alpha_2\Phi(b_{ij}^*)^*\Phi(a_{ij}) + 0\Phi(b_{ij}^*)\Phi(a_{ij})^*$$
$$= \Phi(\alpha_1 p_i p_j^* + \alpha_2 p_j^* p_i + 0 p_j p_i^*) + \Phi(\alpha_1 p_i (b_{ij}^*)^* + \alpha_2 (b_{ij}^*)^* p_i$$
$$+ 0 b_{ij}^* p_i^*) + \Phi(\alpha_1 a_{ij} p_j^* + \alpha_2 p_j^* a_{ij} + 0 p_j a_{ij}^*) + \Phi(\alpha_1 a_{ij} (b_{ij}^*)^*$$
$$+ \alpha_2 (b_{ij}^*)^* a_{ij} + 0 b_{ij}^* a_{ij}^*) = \Phi(\alpha_1 a_{ij}) + \Phi(\alpha_1 b_{ij})$$

which leads to $\Phi(\alpha_1 a_{ij} + \alpha_1 b_{ij}) = \Phi(\alpha_1 a_{ij}) + \Phi(\alpha_1 b_{ij})$. As consequence, we conclude that $\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$. □

Property 2.9. *For arbitrary elements $a_{ii}, b_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$ the following holds: $\Phi(a_{ii} + b_{ii}) = \Phi(a_{ii}) + \Phi(b_{ii})$.*

Proof. Choose $f = f_{ii} + f_{ij} + f_{ji} + f_{jj} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{ii}) + \Phi(b_{ii})$. Then

$$\Phi(\alpha_1 f p_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^*) = \Phi(\alpha_1 a_{ii} p_j^* + \alpha_2 p_j^* a_{ii} + \alpha_3 p_j a_{ii}^*) + \Phi(\alpha_1 b_{ii} p_j^* + \alpha_2 p_j^* b_{ii} + \alpha_3 p_j b_{ii}^*) = 0.$$

This imples that $\alpha_1 f p_j^* + \alpha_2 p_j^* f + \alpha_3 p_j f^* = 0$ which leads to $\alpha_1 f_{ij} = 0$, $\alpha_2 f_{ji} + \alpha_3 f_{ij}^* = 0$ and $(\alpha_1 + \alpha_2) f_{jj} + \alpha_3 f_{jj}^* = 0$. As the third of the last three identities turns into $(|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2) f_{jj} = 0$, then we conclude that $f_{ij} = 0$, $f_{ji} = 0$ and $f_{jj} = 0$. It therefore follows that $\Phi(f_{ii}) = \Phi(a_{ii}) + \Phi(b_{ii})$. Hence, for an arbitrary element $d_{ji} \in \mathcal{A}_{ji}$ we have

$$\Phi(\alpha_1 f_{ii} d_{ji}^* + \alpha_3 d_{ji} f_{ii}^*) = \Phi(\alpha_1 f_{ii} d_{ji}^* + \alpha_2 d_{ji}^* f_{ii} + \alpha_3 d_{ji} f_{ii}^*)$$
$$= \Phi(\alpha_1 a_{ii} d_{ji}^* + \alpha_2 d_{ji}^* a_{ii} + \alpha_3 d_{ji} a_{ii}^*) + \Phi(\alpha_1 b_{ii} d_{ji}^* + \alpha_2 d_{ji}^* b_{ii} + \alpha_3 d_{ji} b_{ii}^*)$$
$$= \Phi(\alpha_1 a_{ii} d_{ji}^* + \alpha_3 d_{ji} a_{ii}^*) + \Phi(\alpha_1 b_{ii} d_{ji}^* + \alpha_3 d_{ji} b_{ii}^*) = \Phi(\alpha_1(a_{ii} + b_{ii}) d_{ji}^* + \alpha_3 d_{ji}(a_{ii} + b_{ii})^*)$$

which shows that $\alpha_1 f_{ii} d_{ji}^* + \alpha_3 d_{ji} f_{ii}^* = \alpha_1(a_{ii} + b_{ii})d_{ji}^* + \alpha_3 d_{ji}(a_{ii} + b_{ii})^*$. This results in $\alpha_1 f_{ii} d_{ji}^* = \alpha_1(a_{ii} + b_{ii})d_{ji}^*$ which leads to $f_{ii} = a_{ii} + b_{ii}$. □

Property 2.10. *Φ is an additive mapping.*

Proof. The result is an immediate consequence of Properties 2.7, 2.8 and 2.9. □

To prove the second part of the Main Theorem, we assume that the condition $\alpha_1 - \alpha_2 + \alpha_3 \neq 0$ holds. We start by proving the following Lemma.

Lemma 2.11. $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$.

Proof. Choose an element $a \in \mathcal{A}$ such that $\Phi(a) = 1_\mathcal{B}$. Then

$$\begin{aligned}\Phi((\alpha_1 + \alpha_2)a^* + \alpha_3 a) &= \Phi(\alpha_1 1_\mathcal{A} a^* + \alpha_2 a^* 1_\mathcal{A} + \alpha_3 a 1_\mathcal{A}^*) = \alpha_1 \Phi(1_\mathcal{A})\Phi(a)^* \\ &\quad + \alpha_2 \Phi(a)^* \Phi(1_\mathcal{A}) + \alpha_3 \Phi(a)\Phi(1_\mathcal{A})^* = (\alpha_1 + \alpha_2)\Phi(1_\mathcal{A}) + \alpha_3 \Phi(1_\mathcal{A})^*\end{aligned}$$

and

$$\begin{aligned}\Phi((\alpha_1 + \alpha_2)a + \alpha_3 a^*) &= \Phi(\alpha_1 a 1_\mathcal{A}^* + \alpha_2 1_\mathcal{A}^* a + \alpha_3 1_\mathcal{A} a^*) = \alpha_1 \Phi(a)\Phi(1_\mathcal{A})^* \\ &\quad + \alpha_2 \Phi(1_\mathcal{A})^* \Phi(a) + \alpha_3 \Phi(1_\mathcal{A})\Phi(a)^* = (\alpha_1 + \alpha_2)\Phi(1_\mathcal{A})^* + \alpha_3 \Phi(1_\mathcal{A}).\end{aligned}$$

Hence, multiplying the first identity by $\alpha_1 + \alpha_2$ and the second by α_3, we get $\Phi(|\alpha_1 + \alpha_2|^2 a^* + (\alpha_1 + \alpha_2)\alpha_3 a) = |\alpha_1 + \alpha_2|^2 \Phi(1_\mathcal{A}) + (\alpha_1 + \alpha_2)\alpha_3 \Phi(1_\mathcal{A})^*$ and $\Phi(\alpha_3(\alpha_1 + \alpha_2)a + |\alpha_3|^2 a^*) = \alpha_3(\alpha_1 + \alpha_2)\Phi(1_\mathcal{A})^* + |\alpha_3|^2 \Phi(1_\mathcal{A})$, respectively. Subtracting the last identity from the previous one, we arrive at $\Phi((|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)a^*) = (|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)\Phi(1_\mathcal{A})$ which leads to $(|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)\Phi(a^*) = (|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)\Phi(1_\mathcal{A})$. This results that $a^* = 1_\mathcal{A}$ which yields $a = 1_\mathcal{A}$. □

Lemma 2.12. *Φ preserves involution on the both sides.*

Proof. For an arbitrary element $a \in \mathcal{A}$ we have

$$\begin{aligned}\Phi((\alpha_1 + \alpha_2)a^* + \alpha_3 a) &= \Phi(\alpha_1 1_\mathcal{A} a^* + \alpha_2 a^* 1_\mathcal{A} + \alpha_3 a 1_\mathcal{A}^*) = \alpha_1 \Phi(1_\mathcal{A})\Phi(a)^* \\ &\quad + \alpha_2 \Phi(a)^* \Phi(1_\mathcal{A}) + \alpha_3 \Phi(a)\Phi(1_\mathcal{A})^* = (\alpha_1 + \alpha_2)\Phi(a)^* + \alpha_3 \Phi(a)\end{aligned}$$

and

$$\begin{aligned}\Phi((\alpha_1 + \alpha_2)a + \alpha_3 a^*) &= \Phi(\alpha_1 a 1_\mathcal{A}^* + \alpha_2 1_\mathcal{A}^* a + \alpha_3 1_\mathcal{A} a^*) = \alpha_1 \Phi(a)\Phi(1_\mathcal{A})^* \\ &\quad + \alpha_2 \Phi(1_\mathcal{A})^* \Phi(a) + \alpha_3 \Phi(1_\mathcal{A})\Phi(a)^* = (\alpha_1 + \alpha_2)\Phi(a) + \alpha_3 \Phi(a)^*.\end{aligned}$$

Using a reasoning similar to the previous proof, we arrive at $(|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)\Phi(a^*) = (|\alpha_1 + \alpha_2|^2 - |\alpha_3|^2)\Phi(a)^*$. As consequence, we obtain $\Phi(a^*) = \Phi(a)^*$. Since Φ^{-1} has the same characteristics of Φ, then Φ preserves involution on the both sides. □

Lemma 2.13. *(i) $\Phi(i1_\mathcal{A})^2 = -1_\mathcal{B}$ and (ii) if $\alpha_1 - \alpha_2 + \alpha_3 \neq 0$, then $\Phi(ia) = \Phi(i1_\mathcal{A})\Phi(a) = \Phi(a)\Phi(i1_\mathcal{A})$, for all elements $a \in \mathcal{A}$, where $\Phi(i1_\mathcal{A}) = \pm i1_\mathcal{B}$.*

Proof. By Lemmas 2.11 and 2.12 we have

$$\begin{aligned}&(\alpha_1 + \alpha_2 + \alpha_3)\Phi(1_\mathcal{A}) = \Phi((\alpha_1 + \alpha_2 + \alpha_3)1_\mathcal{A}) = \Phi(\alpha_1(i1_\mathcal{A})(i1_\mathcal{A})^* + \alpha_2(i1_\mathcal{A})^*(i1_\mathcal{A}) + \alpha_3(i1_\mathcal{A})(i1_\mathcal{A})^*) \\ &= \alpha_1 \Phi(i1_\mathcal{A})\Phi(i1_\mathcal{A})^* + \alpha_2 \Phi(i1_\mathcal{A})^* \Phi(i1_\mathcal{A}) + \alpha_3 \Phi(i1_\mathcal{A})\Phi(i1_\mathcal{A})^* = -(\alpha_1 + \alpha_2 + \alpha_3)\Phi(i1_\mathcal{A})^2,\end{aligned}$$

which implies that $\Phi(i1_{\mathcal{A}})^2 = -1_{\mathcal{B}}$. Next, for an arbitrary self-adjoint element $a \in \mathcal{A}$ we have

$$\begin{aligned}(\alpha_1+\alpha_2-\alpha_3)\Phi(ia) &= \Phi((\alpha_1+\alpha_2-\alpha_3)ia) = -\Phi(\alpha_1 a(i1_{\mathcal{A}})^* + \alpha_2(i1_{\mathcal{A}})^*a + \alpha_3(i1_{\mathcal{A}})a^*)\\ &= -(\alpha_1\Phi(a)\Phi(i1_{\mathcal{A}})^* + \alpha_2\Phi(i1_{\mathcal{A}})^*\Phi(a) + \alpha_3\Phi(i1_{\mathcal{A}})\Phi(a)^*) = \alpha_1\Phi(a)\Phi(i1_{\mathcal{A}})\\ &\quad + \alpha_2\Phi(i1_{\mathcal{A}})\Phi(a) - \alpha_3\Phi(i1_{\mathcal{A}})\Phi(a) \quad (2)\end{aligned}$$

and

$$\begin{aligned}(\alpha_1+\alpha_2-\alpha_3)\Phi(ia) &= \Phi((\alpha_1+\alpha_2-\alpha_3)ia) = \Phi(\alpha_1(i1_{\mathcal{A}})a^* + \alpha_2 a^*(i1_{\mathcal{A}}) + \alpha_3 a(i1_{\mathcal{A}})^*)\\ &= \alpha_1\Phi(i1_{\mathcal{A}})\Phi(a)^* + \alpha_2\Phi(a)^*\Phi(i1_{\mathcal{A}}) + \alpha_3\Phi(a)\Phi(i1_{\mathcal{A}})^* = \alpha_1\Phi(i1_{\mathcal{A}})\Phi(a)\\ &\quad + \alpha_2\Phi(a)\Phi(i1_{\mathcal{A}}) - \alpha_3\Phi(a)\Phi(i1_{\mathcal{A}}). \quad (3)\end{aligned}$$

Subtracting (3) from (2), we arrive at $\alpha_1(\Phi(a)\Phi(i1_{\mathcal{A}}) - \Phi(i1_{\mathcal{A}})\Phi(a)) - \alpha_2(\Phi(a)\Phi(i1_{\mathcal{A}}) - \Phi(i1_{\mathcal{A}})\Phi(a)) + \alpha_3(\Phi(a)\Phi(i1_{\mathcal{A}}) - \Phi(i1_{\mathcal{A}})\Phi(a)) = 0$ which results in $(\alpha_1-\alpha_2+\alpha_3)(\Phi(a)\Phi(i1_{\mathcal{A}}) - \Phi(i1_{\mathcal{A}})\Phi(a)) = 0$. This shows that $\Phi(ia) = \Phi(i1_{\mathcal{A}})\Phi(a) = \Phi(a)\Phi(i1_{\mathcal{A}})$, in views of identity (2). It therefore follows that, for an arbitrary element $a \in \mathcal{A}$, write $a = a_1 + ia_2$, where a_1 and a_2 are self-adjoint elements. Then, by (i) we have

$$\begin{aligned}\Phi(ia) &= \Phi(ia_1 - a_2) = \Phi(i1_{\mathcal{A}})\Phi(a_1) + \Phi(i1_{\mathcal{A}})^2\Phi(a_2) = \Phi(i1_{\mathcal{A}})(\Phi(a_1) + \Phi(i1_{\mathcal{A}})\Phi(a_2))\\ &= \Phi(i1_{\mathcal{A}})(\Phi(a_1) + \Phi(ia_2)) = \Phi(i1_{\mathcal{A}})\Phi(a).\end{aligned}$$

Similarly, we prove that $\Phi(ia) = \Phi(a)\Phi(i1_{\mathcal{A}})$. In particular, this shows that $\Phi(i1_{\mathcal{A}})$ is a central element of $\mathcal{B}$. As a result, by part (i) again, we have $(\Phi(i1_{\mathcal{A}}) - i1_{\mathcal{B}})\mathcal{B}(\Phi(i1_{\mathcal{A}}) + i1_{\mathcal{B}}) = 0$ which implies that $\Phi(i1_{\mathcal{A}}) = i1_{\mathcal{B}}$ or $\Phi(i1_{\mathcal{A}}) = -i1_{\mathcal{B}}$, in view of the primeness of $\mathcal{B}$. □

Lemma 2.14. *Φ is a $\mathbb{Q}[i]$-linear map preserving involution or a conjugate $\mathbb{Q}[i]$-linear map preserving involution.*

Proof. By Lemmas 2.3, Φ is a $\mathbb{Q}$-linear map. Thus, by Lemmas 2.12 and 2.13(ii) Φ is a $\mathbb{Q}[i]$-linear map preserving involution or a conjugate $\mathbb{Q}[i]$-linear map preserving involution. □

Lemma 2.15. *If $\alpha_3 \neq 0$, then Φ is multiplicative.*

Proof. For arbitrary self-adjoint elements $a, b \in \mathcal{A}$ we have $\Phi(\alpha_1 ab + \alpha_2 ba + \alpha_3 ba) = \alpha_1\Phi(a)\Phi(b) + \alpha_2\Phi(b)\Phi(a) + \alpha_3\Phi(b)\Phi(a)$, and $\Phi(\alpha_1 a(ib)^* + \alpha_2(ib)^*a + \alpha_3(ib)a^*) = \alpha_1\Phi(a)\Phi(ib)^* + \alpha_2\Phi(ib)^*\Phi(a) + \alpha_3\Phi(ib)\Phi(a)^*$ which implies that $\Phi(-\alpha_1 ab - \alpha_2 ba + \alpha_3 ba) = -\alpha_1\Phi(a)\Phi(b) - \alpha_2\Phi(b)\Phi(a) + \alpha_3\Phi(b)\Phi(a)$, by Lemmas 2.12 and 2.13. Hence, adding the first identity to the third we get $\Phi(\alpha_3 ba) = \alpha_3\Phi(b)\Phi(a)$ which results in $\alpha_3\Phi(ab) = \alpha_3\Phi(a)\Phi(b)$. As consequence, we obtain $\Phi(ab) = \Phi(a)\Phi(b)$. It therefore follows that, for two arbitrary elements $a, b \in \mathcal{A}$ with $a = a_1 + ia_2$ and $b = b_1 + ib_2$, where a_1, a_2, b_1, b_2 are self-adjoint elements of $\mathcal{A}$, we have

$$\begin{aligned}\Phi(ab) &= \Phi((a_1+ia_2)(b_1+ib_2)) = \Phi(a_1b_1 + ia_1b_2 + ia_2b_1 - a_2b_2) = \Phi(a_1)\Phi(b_1) + \Phi(i1_{\mathcal{A}})\Phi(a_1)\Phi(b_2)\\ &+ \Phi(i1_{\mathcal{A}})\Phi(a_2)\Phi(b_1) + \Phi(i1_{\mathcal{A}})^2\Phi(a_2)\Phi(b_2) = \Phi(a_1)(\Phi(b_1) + \Phi(ib_2)) + \Phi(ia_2)(\Phi(b_1) + \Phi(ib_2))\\ &= \Phi(a_1 + ia_2)\Phi(b_1 + ib_2) = \Phi(a)\Phi(b).\end{aligned}$$

Thus, Φ is multiplicative. □

Lemma 2.16. *If $\alpha_1 - \alpha_2 \neq 0$, then Φ is multiplicative.*

Proof. Two cases are considered. First case: $\alpha_3 \neq 0$. In this case, the result follows directly from Lemma 2.15. Second case: $\alpha_3 = 0$. For arbitrary self-adjoint elements $a, b \in \mathcal{A}$ we have $\Phi(\alpha_1 ab + \alpha_2 ba) = \alpha_1\Phi(a)\Phi(b) + \alpha_2\Phi(b)\Phi(a)$ and, replacing a by b and b by a, $\Phi(\alpha_1 ba + \alpha_2 ab) = \alpha_1\Phi(b)\Phi(a) + \alpha_2\Phi(a)\Phi(b)$. Hence, adding and subtracting the two last identity, we arrive at $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$ and $\Phi(ab - ba) = \Phi(a)\Phi(b) - \Phi(b)\Phi(a)$, respectively. This results that $\Phi(ab) = \Phi(a)\Phi(b)$. Thus, for arbitrary elements $a, b \in \mathcal{A}$, using a reasoning similar to the previous proof, we arrive at $\Phi(ab) = \Phi(a)\Phi(b)$. This shows that Φ is multiplicative. □

Lemma 2.17. *Φ is a $\mathbb{Q}[i]$-linear multiplicative map preserving involution or a conjugate $\mathbb{Q}[i]$-linear multliplicative map preserving involution.*

Proof. First, note that $\alpha_1 - \alpha_2 + \alpha_3 \neq 0$, implies that $\alpha_3 \neq 0$ or $\alpha_1 - \alpha_2 \neq 0$. In either case we have that Φ is a multiplicative map, by Lemmas 2.15 and 2.16. Thus, the result follows from Lemma 2.14. □

The Theorem is proved.

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