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Relative controllability of conformable delay differential systems with linear parts defined by permutable matrices

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Abstract. We study relative controllability of linear and nonlinear conformable delay differential systems with linear parts defined by permutable matrices. By using a notion of delay Grammian matrix, we give a sufficient and necessary condition to examine that a linear delay controlled systems is relatively controllable. Thereafter, we construct a suitable control function for nonlinear delay controlled system, which admits us to adopt the framework of fixed point methods to investigate the same issue. More precisely, we apply Krassnoselskii's fixed point theorem to derive a relative controllability result. Finally, two examples are presented to illustrate our theoretical results with the help of computing the desired control functions and inverse of delay Grammian matrix as well.

1. Introduction

It is well known that Khusainov and Shuklin [1] and Diblík and Khusainov [2] introduced the notation of delayed exponential matrix functions to derive the representation of solutions of continuous and discrete linear delay differential systems with permutable matrices, respectively. Based on these foundation contribution, representation of solutions, stability analysis of solutions and controllability problems for linear and nonlinear delay systems have been studied extensively, see [3–22].

Recently, Khalil et al. [23], Chung [24] and Ortega and Rosales [25] studied the conformable type calculus, which provided a possible alternative approach to enrich the standard Newton mechanics. Further, Wang et al. [26] give the sufficient and necessary conditions for the complete controllability of systems governed by linear and nonlinear conformable differential equations. Xiao and Wang [27] proposed the conformable type delayed exponential matrix function to derive the representations of solutions for homogeneous and nonhomogeneous differential equations. To the best of our knowledge, controllability of linear and nonlinear conformable delay differential systems with linear parts defined by permutable matrices has not been studied until now.

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Motivated by [15, 26, 27], we study relative controllability of linear case

$$\begin{cases}\mathfrak{D}_{\alpha}^{0}y(t)=Gy(t)+Ry(t-\tau)+Ku(t),\ t\in J:=[0,t_1],\ \tau\geq 0,\\ y(t)=\varphi(t),\ -\tau\leq t\leq 0,\end{cases}\tag{1}$$

and nonlinear case

$$\begin{cases}\mathfrak{D}_{\alpha}^{0}y(t)=Gy(t)+Ry(t-\tau)+g(t,y(t))+Ku(t),\ t\in J,\ \tau\geq 0,\\ y(t)=\varphi(t),\ -\tau\leq t\leq 0,\end{cases}\tag{2}$$

where $\mathfrak{D}_{\alpha}^{0}(0<\alpha<1)$ denotes the conformable derivative with lower index zero (see [23, Definition 1]), $G,R,K\in\mathbb{R}^{n\times n}$, $GR=RG$, $g:J\times\mathbb{R}^n\to\mathbb{R}^n$. The state y takes values from $\mathbb{R}^n$ and the control function $u(\cdot)$ takes value from $L^2(J,\mathbb{R}^n)$ and $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$.

The main contributions are stated as follows:

(i) Concerning on linear conformable delay differential controlled system, instead of seeking Kalman type criterion, we introduce a notion of delay Grammian matrix and reveal the relationship between non-singular property of delay Grammian matrix and relative controllability for linear conformable delay controlled system. Meanwhile, we also give an algorithm for constructing a control.

(ii) Except for a criterion of relative controllability for linear conformable delay differential controlled system we construct a suitable control function for nonlinear conformable delay differential controlled system and adopt the framework of fixed point methods to derive relative controllability via Krasnoselskii's fixed point theorem as well.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, concepts and lemmas. In Section 3, we firstly investigate relative controllability of linear conformable delay controlled differential system and give a delay Grammian matrix criterion. Secondly, we turn relative controllability of nonlinear conformable delay controlled differential system into a fixed point problem, which admit us to take advantage of Krasnoselskii's fixed point to verify our main theorem. Two examples are given in final section to illustrate our theoretical results.

2. Preliminary

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the vector norm $\|\cdot\|$ and $\mathbb{R}^{n\times n}$ be the $n\times n$ matrix space with real value elements. The Banach space $C(J,\mathbb{R}^n)$ of vector-value continuous functions from $J\to\mathbb{R}^n$ endowed with the norm $\|y\|_C=\sup_{t\in J}\|y(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. Let X,Y be two Banach spaces, $L_b(X,Y)$ denotes the space of bounded linear operators from X to Y. Next, $L^p(J,Y)$ denotes the Banach space of functions $f:J\to Y$ which are Bochner integrable normed by $\|f\|_{L^p(J,Y)}$ for some $1<p<\infty$. For $G:\mathbb{R}^n\to\mathbb{R}^n$, we consider its matrix norm $\|G\|=\sup_{\|y\|=1}\|Gy\|$ generated by $\|\cdot\|$. In addition, $\|\varphi\|_C=\sup_{s\in[-\tau,0]}\|\varphi(s)\|$.

Set

$$\mathcal{P}(t):=e^{G\frac{t^{\alpha}}{\alpha}}e_{\tau,\alpha}^{R_1 t},\ t\geq 0,\tag{3}$$

where $R_1=e^{G\frac{(t-\tau)^{\alpha}-t^{\alpha}}{\alpha}}R$ and $e_{\tau,\alpha}^{R_1\cdot}$ is defined as follows: (see also [27, (2)])

$$e_{\tau,\alpha}^{R_1 t}=\begin{cases}\Theta,\ -\infty<t<-\tau,\\ \mathbb{I},\ -\tau\leq t\leq 0,\\ \mathbb{I}+R_1\dfrac{t^{\alpha}}{\alpha}+R_1^2\dfrac{1}{2!}\left(\dfrac{(t-\tau)^{\alpha}}{\alpha}\right)^2+R_1^3\dfrac{1}{3!}\left(\dfrac{(t-2\tau)^{\alpha}}{\alpha}\right)^3\\ +\cdots+R_1^k\dfrac{1}{k!}\left(\dfrac{(t-(k-1)\tau)^{\alpha}}{\alpha}\right)^k,(k-1)\tau\leq t<k\tau,k=1,2,\cdots,\end{cases}$$

where Θ is zero matrix, $\mathbb{I}$ is the identity matrix.

By [27, Theorem 3.4] and [27, Theorem 3.5], the solution $y \in C([-\tau, t_1], R^n)$ of system (1) can be formulated by

$$\begin{aligned} y(t) &= \mathcal{P}(t)e^{G\frac{\tau^\alpha}{\alpha}}\varphi(-\tau) + \int_{-\tau}^{0} \mathcal{P}(t-\tau-s)e^{G\frac{\tau^\alpha}{\alpha}}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &\quad + \int_0^t \mathcal{P}(t-\tau-s)e^{G\frac{\tau^\alpha}{\alpha}}Ku(s)s^{\alpha-1}ds, \quad t \geq 0, \end{aligned} \tag{4}$$

where $\mathcal{P}$ is given in (3).

Definition 2.1. *(see [1, Definition 4]) System (1)/(2) is called relatively controllable, if for an arbitrary initial vector function $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, the final state of the vector $y_1 \in \mathbb{R}^n$ and time t_1, there exists a control $u \in L^2(J, \mathbb{R}^n)$ such that the system (1)/(2) has a solution $y \in C([-\tau, t_1], \mathbb{R}^n)$ that satisfies the boundary conditions $y(t) = \varphi(t), -\tau \leq t \leq 0$ and $y(t_1) = y_1$.*

Similar to the proof in ([4, Lemma 3]), one can get a useful norm estimation of $e_{\tau,\alpha}^{R_1 \cdot}$.

Lemma 2.2. *The following inequality holds true for all $t \geq 0$,*

$$\|e_{\tau,\alpha}^{R_1 t}\| \leq e^{\|R_1\|\frac{(t+\tau)^\alpha}{\alpha}}.$$

3. Main results

3.1. Linear case

We investigate relative controllability of system (1). We introduce a notation of a new delay Grammian matrix as follows:

$$W_\tau[0, t_1] = \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} KK^\top e_{\tau,\alpha}^{R_1^\top(t_1-\tau-s)} e^{G^\top\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1}ds. \tag{5}$$

Now we are ready to state a sufficient and necessary condition to guarantee (1) is relatively controllable.

Theorem 3.1. *System* (1) *is relatively controllable, if and only if $W_\tau[0, t_1]$ defined in* (5) *is non-singular.*

Proof. **Sufficiency**. Since $W_\tau[0, t_1]$ is non-singular, its inverse $W_\tau^{-1}[0, t_1]$ is well-defined. One can select a control function as follows:

$$u(t) = K^\top e_{\tau,\alpha}^{R_1^\top(t_1-\tau-t)} e^{G^\top\frac{(t_1-\tau-t)^\alpha+\tau^\alpha}{\alpha}} W_\tau^{-1}[0, t_1]\eta, \tag{6}$$

where

$$\begin{aligned} \eta &= y_1 - e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) \\ &\quad - \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds, \end{aligned} \tag{7}$$

and the vector $y_1 \in \mathbb{R}^n$ is arbitrarily before it is chosen.

Inserting (6) in (4), we have

$$\begin{aligned} y(t_1) &= e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) + \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &\quad + \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} KK^\top \\ &\quad \cdot e_{\tau,\alpha}^{R_1^\top(t_1-\tau-s)} e^{G^\top\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1}ds \cdot W_\tau^{-1}[0, t_1]\eta. \end{aligned} \tag{8}$$

Linking (5) and (8) via (7), it is not difficult to derive that

$$\begin{aligned} y(t_1) &= e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) + \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &\quad +\eta = y_1. \end{aligned} \tag{9}$$

The boundary condition $y(t) = \varphi(t), -\tau \le t \le 0$ holds via (4). Combining the formula (9) via Definition 2.1, the system (1) is relatively controllable.

Necessity. We adopt proof by contradiction to prove our result. Assume $W_\tau[0, t_1]$ is singular, i.e., there exists at least one nonzero state $\widetilde{y} \in R^n$ such that

$$\widetilde{y}^\top W_\tau[0, t_1]\widetilde{y} = 0.$$

Further, one obtain

$$\begin{aligned} 0 &= \widetilde{y}^\top W_\tau[0, t_1]\widetilde{y} \\ &= \int_0^{t_1} \widetilde{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} KK^\top e_{\tau,\alpha}^{R_1^\top(t_1-\tau-s)} e^{G^\top\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \widetilde{y} s^{\alpha-1}ds \\ &= \int_0^{t_1} \left[\widetilde{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K\right] \\ &\quad \left[\widetilde{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K\right]^\top s^{\alpha-1}ds \\ &= \int_0^{t_1} \left\|\widetilde{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K\right\|^2 s^{\alpha-1}ds, \end{aligned}$$

which implies that

$$\widetilde{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K = (\underbrace{0, \cdots, 0}_{n}), \quad \forall s \in J. \tag{10}$$

Since system (1) is relatively controllable, according to Definition 2.1, there exists a control $u_1(t)$ that drives the initial state to zero at t_1, that is

$$\begin{aligned} y(t_1) &= e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) + \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &\quad + \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K u_1(s)s^{\alpha-1}ds \\ &= \mathbf{0}, \end{aligned} \tag{11}$$

where $\mathbf{0}$ denotes the n dimensional zero vector.

Similarly, there also exists a control $\overline{u}(\cdot)$ that drives the initial state to $\overline{y}$ at t_1, i.e.,

$$\begin{aligned} y(t_1) &= e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) + \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &\quad + \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K\overline{u}(s)s^{\alpha-1}ds \\ &= \overline{y}. \end{aligned} \tag{12}$$

Then by (11) and (12), we have

$$\overline{y} = \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K[\overline{u}(s) - u_1(s)]s^{\alpha-1}ds. \tag{13}$$

Multiplying both sides of (13) by $\overline{y}^\top$, we obtain

$$\overline{y}^\top\overline{y} = \int_0^{t_1} \overline{y}^\top e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} K[\overline{u}(s)-u_1(s)]s^{\alpha-1}ds.$$

Note that the fact (10), one can obtain $\overline{y}^\top\overline{y}=0$, i.e., $\overline{y}=\mathbf{0}$, which conflicts with $\overline{y}$ being nonzero. Thus, $W_\tau[0,t_1]$ is non-singular. The proof is finished. □

3.2. Nonlinear case

We need the following hypothesis:

$[H_1]$: The operator $W: L^2(J,\mathbb{R}^n)\to\mathbb{R}^n$ defined by

$$Wu = \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} Ku(s)s^{\alpha-1}ds$$

has an inverse operator W^{-1} which takes values in $L^2(J,\mathbb{R}^n)/kerW$ and there exists a constant $M>0$ such that

$$\|W^{-1}\|_{L_b(\mathbb{R}^n,L^2(J,\mathbb{R}^n)/kerW)} \leq M.$$

Remark 3.2. *By [15, Remark 2], we have*

$$M = \sqrt{\|W_\tau[0,t_1]^{-1}\|}. \tag{14}$$

$[H_2]$: The function $g: J\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and there exists a constant $q>1$ and $L_g(\cdot)\in L^q(J,\mathbb{R}^+)$ such that

$$\|g(t,y_1)-g(t,y_2)\| \leq L_g(t)\|y_1-y_2\|,\ y_i\in\mathbb{R}^n,\ i=1,2.$$

In viewing of $[H_1]$, for arbitrary $y(\cdot)\in C(J,\mathbb{R}^n)$, consider a control function $u_y(\cdot)$ given by

$$\begin{aligned} u_y(t) &= W^{-1}\Big[y_1 - e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1t_1}\varphi(-\tau) \\ &\quad - \int_{-\tau}^0 e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s)-G\varphi(s)]s^{\alpha-1}ds \\ &\quad - \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} g(s,y(s))s^{\alpha-1}ds\Big](t),\ t\in J. \end{aligned} \tag{15}$$

Now we state our main idea to prove our main result via fixed point method. We firstly show that, using control (15), the operator $\mathcal{F}: C(J,\mathbb{R}^n)\to C(J,\mathbb{R}^n)$ defined by

$$\begin{aligned} (\mathcal{F}y)(t) &= e^{G\frac{t^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1t}\varphi(-\tau) \\ &\quad + \int_{-\tau}^0 e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s)-G\varphi(s)]s^{\alpha-1}ds \\ &\quad + \int_0^t e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)} g(s,y(s))s^{\alpha-1}ds \\ &\quad + \int_0^t e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)} Ku_y(s)s^{\alpha-1}ds \end{aligned}$$

has a fixed point y, which is just a solution of system (2). Secondly, we check that $(\mathcal{F}y)(t_1)=y_1$ and $(\mathcal{F}y)(0)=\varphi(0)=y_0$, which means that u_y steers the system (2) from y_0 to y_1 in finite time t_1. This implies system (2) is relatively controllable on J.

For each positive number r, define $\mathcal{B}_r=\{y\in C(J,\mathbb{R}^n): \|y\|_C\leq r\}$. Then, for each r, $\mathcal{B}_r$ is obviously a bound, closed and convex set of $C(J,\mathbb{R}^n)$. For brevity, we set $R_g=\sup_{t\in J}\|g(t,0)\|$ and $N=\|G\|+\|R_1\|$.

In what follows, we apply Krasnoselskii's fixed point theorem to derive the relative controllability result for system (2).

Theorem 3.3. *Suppose that* $[H_1]$ *and* $[H_2]$ *are satisfied. Then system* (2) *is relatively controllable provided that*

$$M_1\Big(1+\frac{bM}{R_g}\|K\|\Big)<1, \tag{16}$$

where $M_1 = e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|L_g\|_{L^q(J,\mathbb{R}^+)}\Big[\frac{1}{p\alpha-p+1}t_1^{p\alpha-p+1}\Big]^{\frac{1}{p}}$, $\frac{1}{p}+\frac{1}{q}=1$, $p,\ q>1$, $\alpha>\frac{1}{q}$, *and*

$$\begin{aligned} b &= e^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi\|_C + e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\Big(\|y_0\|+\|\varphi\|_C+N\int_{-\tau}^0\|\varphi(s)s^{\alpha-1}\|ds\Big) \\ &\quad + e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha}R_g \end{aligned} \tag{17}$$

and M is given by (14).

Proof. We divide our proof into several steps.

Step 1. We prove that there exists a positive number r such that $\mathcal{F}(\mathcal{B}_r)\subseteq\mathcal{B}_r$.

In light of $[H_2]$ and Hölder inequality, we obtain that

$$\int_0^t s^{\alpha-1}L_g(s)ds \le \Big[\frac{1}{p\alpha-p+1}t_1^{p\alpha-p+1}\Big]^{\frac{1}{p}}\|L_g\|_{L^q(J,\mathbb{R}^+)},$$

and

$$\begin{aligned} \int_0^t e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}\|g(s,0)\|s^{\alpha-1}ds &\le e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}R_g\int_0^t s^{\alpha-1}ds \\ &\le e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha}R_g. \end{aligned}$$

Taking into account of (15), using $[H_1]$, $[H_2]$, we have

$$\begin{aligned} \|u_y(t)\| &\le \|W^{-1}\|_{L(\mathbb{R}^n,L^2(J,\mathbb{R}^n)/kerW)}\Big(\|y_1\|+e^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\ &\quad +\Big\|\int_{-\tau}^0 e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}[\mathfrak{D}_\alpha^0\varphi(s)-G\varphi(s)]s^{\alpha-1}ds\Big\| \\ &\quad +\int_0^{t_1}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}\|g(s,y(s))\|s^{\alpha-1}ds\Big) \\ &\le M\|y_1\|+Me^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\ &\quad +Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\Big\|\int_{-\tau}^0\mathfrak{D}_\alpha^0\varphi(s)s^{\alpha-1}ds-\int_{-\tau}^0 G\varphi(s)s^{\alpha-1}ds\Big\| \\ &\quad +M\int_0^{t_1}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}L_g(s)\|y(s)\|s^{\alpha-1}ds \\ &\quad +M\int_0^{t_1}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}\|g(s,0)\|s^{\alpha-1}ds \\ &\le M\|y_1\|+Me^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\ &\quad +Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\Big[\|y_0\|+\|\varphi(-\tau)\|+N\int_{-\tau}^0\|\varphi(s)s^{\alpha-1}\|ds\Big] \\ &\quad +Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|L_g\|_{L^q(J,\mathbb{R}^+)}\|y\|_C\Big[\frac{1}{p\alpha-p+1}t_1^{p\alpha-p+1}\Big]^{\frac{1}{p}} \\ &\quad +Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha}R_g \\ &\le M\|y_1\|+Mb+MM_1r, \end{aligned}$$

where b and M_1 is defined in the above.

It comes from $[H_1]$ and $[H_2]$ that we have

$$\begin{aligned}
\|(\mathcal{F}y)(t)\| \leq\ & e^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\
&+\Big\|\int_{-\tau}^{0}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}[\mathfrak{D}_\alpha^0\varphi(s)-G\varphi(s)]s^{\alpha-1}ds\Big\| \\
&+\int_0^{t_1}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}\|g(s,y(s))\|s^{\alpha-1}ds \\
&+\int_0^{t_1}e^{N\frac{(t_1-s)^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1-s)^\alpha}{\alpha}}\|K\|\|u_y(s)\|s^{\alpha-1}ds \\
\leq\ & e^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\Big\|\int_{-\tau}^{0}\mathfrak{D}_\alpha^0\varphi(s)s^{\alpha-1}ds-\int_{-\tau}^{0}G\varphi(s)s^{\alpha-1}ds\Big\| \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|y\|_C\int_0^{t_1}L_g(s)s^{\alpha-1}ds \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}R_g\int_0^{t_1}s^{\alpha-1}ds \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^{t_1}\|K\|(M\|y_1\|+Mb+MM_1\|y\|_C)s^{\alpha-1}ds \\
\leq\ & e^{N\frac{t_1^\alpha+\tau^\alpha}{\alpha}}e^{N\frac{(t_1+\tau)^\alpha}{\alpha}}\|\varphi(-\tau)\| \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\Big[\|y_0\|+\|\varphi(-\tau)\|+N\int_{-\tau}^{0}\|\varphi(s)s^{\alpha-1}\|ds\Big] \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|y\|_C\|L_g\|_{L^q(J,\mathbb{R}^+)}\Big[\frac{1}{p\alpha-p+1}t_1^{p\alpha-p+1}\Big]^{\frac{1}{p}} \\
&+e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha}R_g \\
&+\|K\|\|y_1\|Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha} \\
&+M\|K\|be^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha} \\
&+MM_1\|K\|\|y\|_Ce^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\frac{t_1^\alpha}{\alpha} \\
\leq\ & b\Big[1+\frac{bM}{R_g}\|K\|+\frac{M}{R_g}\|K\|\|y_1\|\Big]+M_1\Big[1+\frac{bM}{R_g}\|K\|\Big]r=r
\end{aligned}$$

for

$$r=\frac{b\Big[1+\frac{bM}{R_g}\|K\|+\frac{M}{R_g}\|K\|\|y_1\|\Big]}{1-M_1\Big[1+\frac{bM}{R_g}\|K\|\Big]}.$$

Hence, we obtain $\mathcal{F}(\mathcal{B}_r)\subseteq\mathcal{B}_r$ for such an r.

Now, we divide $\mathcal{F}$ into two operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{B}_r$ as

$$\begin{aligned}(\mathcal{F}_1 y)(t) &= e^{G\frac{t^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t}\varphi(-\tau)\\ &\quad+\int_{-\tau}^{0} e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s)-G\varphi(s)]s^{\alpha-1}ds\\ &\quad+\int_0^t e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)}Ku_y(s)s^{\alpha-1}ds,\\ (\mathcal{F}_2 y)(t) &= \int_0^t e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)}g(s,y(s))s^{\alpha-1}ds,\end{aligned}$$

for $t\in J$, respectively.

Step 2. We show that $\mathcal{F}_1$ is a contraction mapping.

Let $y,z\in\mathcal{B}_r$. In viewing of $[H_1]$ and $[H_2]$, for each $t\in J$, we have

$$\begin{aligned}\|u_y(t)-u_z(t)\| &\le Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^{t_1}L_g(s)s^{\alpha-1}ds\|y-z\|_C\\ &\le Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|y-z\|_C\Big[\int_0^{t_1}L_g^q(s)ds\Big]^{\frac1q}\Big[\int_0^{t_1}(s^{\alpha-1})^p ds\Big]^{\frac1p}\\ &\le Me^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\|y-z\|_C\|L_g\|_{L^q(J,\mathbb{R}^+)}\Big[\frac{1}{p\alpha-p+1}t_1^{p\alpha-p+1}\Big]^{\frac1p}\\ &\le MM_1\|y-z\|_C.\end{aligned}$$

Using the above fact, we derive that

$$\begin{aligned}\|(\mathcal{F}_1y)(t)-(\mathcal{F}_1z)(t)\| &\le \|K\|e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\|u_y(s)-u_z(s)\|s^{\alpha-1}ds\\ &\le \frac{t_1^\alpha}{\alpha}\|K\|e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}MM_1\|y-z\|_C\\ &\le \frac{b}{R_g}MM_1\|K\|\|y-z\|_C,\end{aligned}$$

which gives that

$$\|\mathcal{F}_1y-\mathcal{F}_1z\|_C\le V\|y-z\|_C,\qquad V:=\frac{b}{R_g}MM_1\|K\|.$$

By virtue of (16), we conclude that $V<1$, which implies $\mathcal{F}_1$ is a contraction.

Step 3. We show that $\mathcal{F}_2$ is a compact and continuous operator.

Let $y_n\in\mathcal{B}_r$ with $y_n\to y$ in $\mathcal{B}_r$. Denote $\mathcal{G}_n(\cdot)=g(\cdot,y_n(\cdot))$ and $\mathcal{G}(\cdot)=g(\cdot,y(\cdot))$. Using $[H_2]$, we have $\mathcal{G}_n\to\mathcal{G}$ in $C(J,\mathbb{R}^n)$ and thus

$$\|(\mathcal{F}_2y_n)(t)-(\mathcal{F}_2y)(t)\| \le e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\|\mathcal{G}_n(s)-\mathcal{G}(s)\|s^{\alpha-1}ds\to 0 \text{ as } n\to\infty$$

uniformly for $t\in J$, which implies that $\mathcal{F}_2$ is continuous on $\mathcal{B}_r$.

To check the compactness of $\mathcal{F}_2$, we prove that $\mathcal{F}_2(\mathcal{B}_r)\subset C(J,\mathbb{R}^n)$ is equicontinuous and bounded.

In fact, for any $y \in \mathcal{B}_r$, $t_1 \geq t + h \geq t > 0$, it holds

$$\begin{aligned}(\mathcal{F}_2 y)(t+h) - (\mathcal{F}_2 y)(t) &= \int_0^t e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t+h-\tau-s)} g(s,y(s)) s^{\alpha-1} ds \\ &\quad - \int_0^t e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)} g(s,y(s)) s^{\alpha-1} ds \\ &\quad + \int_t^{t+h} e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t+h-\tau-s)} g(s,y(s)) s^{\alpha-1} ds \\ &\quad + \int_0^t e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)} g(s,y(s)) s^{\alpha-1} ds \\ &\quad - \int_0^t e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t-\tau-s)} g(s,y(s)) s^{\alpha-1} ds \\ &:= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_t^{t+h} e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t+h-\tau-s)} \mathcal{G}(s) s^{\alpha-1} ds, \\ I_2 &= \int_0^t e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big[e_{\tau,\alpha}^{R_1(t+h-\tau-s)} - e_{\tau,\alpha}^{R_1(t-\tau-s)} \Big] \mathcal{G}(s) s^{\alpha-1} ds, \\ I_3 &= \int_0^t \Big[e^{G\frac{(t+h-\tau-s)^\alpha+\tau^\alpha}{\alpha}} - e^{G\frac{(t-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big] e_{\tau,\alpha}^{R_1(t-\tau-s)} \mathcal{G}(s) s^{\alpha-1} ds. \end{aligned}$$

From above, we derive that

$$\|(\mathcal{F}_2 y)(t+h) - (\mathcal{F}_2 y)(t)\| \leq \|I_1\| + \|I_2\| + \|I_3\|.$$

Next, we check $\|I_i\| \to 0$ as $h \to 0$, $i = 1, 2, 3$ uniformly for t.

For I_1, using $[H_2]$,

$$\begin{aligned} \|I_1\| &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \int_t^{t+h} \|\mathcal{G}(s)\| s^{\alpha-1} ds \\ &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \int_t^{t+h} (\|g(s,y(s)) - g(s,0)\| + \|g(s,0)\|) s^{\alpha-1} ds \\ &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \int_t^{t+h} L_g(s) \|y(s)\| s^{\alpha-1} ds \\ &\quad + e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \int_t^{t+h} R_g s^{\alpha-1} ds \\ &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \|y\|_C \int_t^{t+h} L_g(s) s^{\alpha-1} ds \\ &\quad + e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} R_g \int_t^{t+h} s^{\alpha-1} ds \\ &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \|y\|_C \|L_g\|_{L^q(J,\mathbb{R}^+)} \Big[\frac{1}{p\alpha - p + 1} s^{p\alpha-p+1} \big|_t^{t+h} \Big]^{\frac{1}{p}} \\ &\quad + e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} R_g \frac{s^\alpha}{\alpha} \big|_t^{t+h}. \end{aligned}$$

Let $F(x)=\frac{1}{p\alpha-p+1}x^{p\alpha-p+1}, H(x)=\frac{x^\alpha}{\alpha}$. Note that $F(x), H(x)$ are continuous on $[t,t+h]$, so

$$\lim_{h\to0}[F(x+h)-F(x)]=0,\qquad \lim_{h\to0}[H(x+h)-H(x)]=0.$$

Therefore, $\|I_1\|\to0$ as $h\to0$.

One can apply $[H_2]$ to derive that

$$\begin{aligned}\|I_2\| \le\ & e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|L_g(s)s^{\alpha-1}ds\|y\|_C\\ &+e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|\|g(s,0)\|s^{\alpha-1}ds\\ \le\ & e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|L_g(s)s^{\alpha-1}ds\|y\|_C\\ &+e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}R_g\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|s^{\alpha-1}ds.\end{aligned}$$

Note that for $t>0$,

$$\begin{aligned}e_{\tau,\alpha}^{R_1(t+h-\tau-s)} =\ & \mathbb{I}+R_1\frac{(t+h-\tau-s)^\alpha}{\alpha}+R_1^2\frac{1}{2!}\left(\frac{(t+h-2\tau-s)^\alpha}{\alpha}\right)^2\\ &+R_1^3\frac{1}{3!}\left(\frac{(t+h-3\tau-s)^\alpha}{\alpha}\right)^3+\cdots+R_1^k\frac{1}{k!}\left(\frac{(t+h-k\tau-s)^\alpha}{\alpha}\right)^k,\end{aligned}$$

and

$$\begin{aligned}e_{\tau,\alpha}^{R_1(t-\tau-s)} =\ & \mathbb{I}+R_1\frac{(t-\tau-s)^\alpha}{\alpha}+R_1^2\frac{1}{2!}\left(\frac{(t-2\tau-s)^\alpha}{\alpha}\right)^2\\ &+R_1^3\frac{1}{3!}\left(\frac{(t-3\tau-s)^\alpha}{\alpha}\right)^3+\cdots+R_1^k\frac{1}{k!}\left(\frac{(t-k\tau-s)^\alpha}{\alpha}\right)^k,\end{aligned}$$

then

$$\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|\longrightarrow0 \text{ as } h\to0.$$

So,

$$\begin{aligned}\|I_2\| \le\ & e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|L_g(s)s^{\alpha-1}ds\|y\|_C\\ &+e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}R_g\int_0^t\left\|e_{\tau,\alpha}^{R_1(t+h-\tau-s)}-e_{\tau,\alpha}^{R_1(t-\tau-s)}\right\|s^{\alpha-1}ds\longrightarrow0 \text{ as } h\to0,\end{aligned}$$

by using Lebesgue's dominated convergence theorem.

For I_3, it is easy to get that

$$\begin{aligned}\|I_3\| \le\ & e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e^{G\frac{(t+h-\tau-s)^\alpha}{\alpha}}-e^{G\frac{(t-\tau-s)^\alpha}{\alpha}}\right\|\left(L_g(s)\|y\|_C+\|g(s,0)\|\right)s^{\alpha-1}ds\\ \le\ & e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e^{G\frac{(t+h-\tau-s)^\alpha}{\alpha}}-e^{G\frac{(t-\tau-s)^\alpha}{\alpha}}\right\|\left(L_g(s)\|y\|_C+R_g\right)s^{\alpha-1}ds.\end{aligned}$$

Note that $\lim_{h\to0}\left\|e^{G\frac{(t+h-\tau-s)^\alpha}{\alpha}}-e^{G\frac{(t-\tau-s)^\alpha}{\alpha}}\right\|=0$. So

$$\begin{aligned}&\|I_3\|\\ &\le e^{N\frac{(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}}\int_0^t\left\|e^{G\frac{(t+h-\tau-s)^\alpha}{\alpha}}-e^{G\frac{(t-\tau-s)^\alpha}{\alpha}}\right\|\left(L_g(s)\|y\|_C+R_g\right)s^{\alpha-1}ds\longrightarrow0 \text{ as } h\to0,\end{aligned}$$

by using Lebesgue's dominated convergence theorem.

From above, we immediately obtain that

$$\|(\mathcal{F}_2 y)(t+h) - (\mathcal{F}_2 y)(t)\| \to 0,\ h \to 0,$$

uniformly for all t and $y \in \mathcal{B}_r$. Therefore, $\mathcal{F}_2(\mathcal{B}_r) \subset C(J, \mathbb{R}^n)$ is equicontinuous.

Next, repeating the above computations, we have

$$\begin{aligned}
\|(\mathcal{F}_2 y)(t)\| &\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \int_0^t \Big(rL_g(s)s^{\alpha-1} + R_g s^{\alpha-1}\Big)ds \\
&\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \left[r\left(\int_0^t L_g^q(s)ds\right)^{\frac{1}{q}} \left(\int_0^t s^{p\alpha-p}ds\right)^{\frac{1}{p}} + \frac{t_1^\alpha}{\alpha}R_g \right] \\
&\leq e^{N\frac{2(t_1+\tau)^\alpha+\tau^\alpha}{\alpha}} \Big[r\|L_g\|_{L^q(J,\mathbb{R}^+)} \Big(\frac{1}{p\alpha-p+1} s^{p\alpha-p+1}\big|_0^{t_1}\Big)^{\frac{1}{p}} + \frac{t_1^\alpha}{\alpha}R_g \Big].
\end{aligned}$$

Hence $\mathcal{F}_2(\mathcal{B}_r)$ is bounded. By Arzela-Ascoli theorem, $\mathcal{F}_2(\mathcal{B}_r) \subset C(J, \mathbb{R}^n)$ is relatively compact in $C(J, \mathbb{R}^n)$.

Thus, $\mathcal{F}_2$ is a compact and continuous operator. Hence, using Krasnoselskii's fixed point theorem, $\mathcal{F}$ has a fixed point y on $\mathcal{B}_r$. Obviously, y is a solution of the system (2) satisfying $y(t_1) = y_1$, the boundary condition $y(t) = \varphi(t),\ -\tau \leq t \leq 0$ also holds. The proof is completed. □

4. Examples

In this part, we give two examples to demonstrate the validity of our method and make some discussions.

Example 4.1. *Set $t_1 = 1$, $\tau = 0.2$. Consider the following nonlinear delay differential controlled systems*

$$\begin{cases} \mathfrak{D}_{0.8}^0 y(t) = Gy(t) + Ry(t-0.2) + g(t, y(t)) + Ku(t),\ y(t) \in \mathbb{R}^2, \\ \qquad\qquad t \in [0,1] := J_1,\ u \in L^2(J_1, R^2), \\ y(t) = \varphi(t) = e^{-2.5(-t)^{0.8}},\ -0.2 \leq t \leq 0. \end{cases} \tag{18}$$

For the sake of simplicity, we set

$$G = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},\quad R = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\quad g(t, y(t)) = \begin{pmatrix} \frac{1}{1000}(t+0.1)y_1(t) + 1 \\ \frac{1}{1000}(t+0.1)y_2(t) + 1 \end{pmatrix},\quad K = \mathbb{I}.$$

Obviously,

$$GR = RG = \begin{bmatrix} 0.06 & 0 \\ 0 & 0.06 \end{bmatrix},$$

since G, R are diagonal matrices.

By elementary calculation, when $0 \leq t \leq 1$, we have

$$\begin{aligned}
R_1(t) &= e^{G\frac{(t-0.2)^{0.8}-t^{0.8}}{0.8}} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \\
&= e^{\begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}\frac{(t-0.2)^{0.8}-t^{0.8}}{0.8}} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\quad N = \|G\| + \sup_{t\in[0,1]} \|R_1(t)\| = 0.3 + 0.2218 = 0.5218,
\end{aligned}$$

and

$$\|L_g\|_{L^2(J_1,\mathbb{R}^+)} = 0.0006658,$$

and

$$M_1 = 0.00466.$$

Now we use (14) to estimate M. For this purpose, we need to obtain $W_\tau[0,t_1]$ and then derive $W_\tau[0,t_1]^{-1}$. Obviously, $G = G^\top, R = R^\top, R_1 = R_1^T$ and $K = K^\top = \mathbb{I}$. Hence, the delay Grammian matrix (5) has the following explicit form

$$\begin{aligned} W_\tau[0,t_1] &= \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} KK^\top e_{\tau,\alpha}^{R_1^\top(t_1-\tau-s)} e^{G^\top \frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_0^{t_1} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)} e_{\tau,\alpha}^{R_1^\top(t_1-\tau-s)} e^{G^\top \frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= W_1 + W_2 + W_3 + W_4 + W_5. \end{aligned}$$

where

$$\begin{aligned} W_1 &= \int_0^{0.2} e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^\alpha}{\alpha} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^\alpha}{\alpha}\Big)^2 \\ &\quad + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^\alpha}{\alpha}\Big)^3 + R_1^4\frac{1}{4!}\Big(\frac{(0.2-s)^\alpha}{\alpha}\Big)^4\Big]^2 e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_0^{0.2} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2 \\ &\quad + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^{0.8}}{0.8}\Big)^3 + R_1^4\frac{1}{4!}\Big(\frac{(0.2-s)^{0.8}}{0.8}\Big)^4\Big]^2 e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \frac{1}{s^{0.2}} ds, \\ W_2 &= \int_{0.2}^{0.4} e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^\alpha}{\alpha} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^\alpha}{\alpha}\Big)^2 \\ &\quad + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^\alpha}{\alpha}\Big)^3\Big]^2 e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_{0.2}^{0.4} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2 \\ &\quad + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^{0.8}}{0.8}\Big)^3\Big]^2 e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \frac{1}{s^{0.2}} ds \\ W_3 &= \int_{0.4}^{0.6} e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^\alpha}{\alpha} \\ &\quad + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^\alpha}{\alpha}\Big)^2\Big]^2 e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_{0.4}^{0.6} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} \\ &\quad + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2\Big]^2 e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \frac{1}{s^{0.2}} ds, \\ W_4 &= \int_{0.6}^{0.8} e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^\alpha}{\alpha}\Big]^2 e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_{0.6}^{0.8} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8}\Big]^2 e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \frac{1}{s^{0.2}} ds, \\ W_5 &= \int_{0.8}^{1} e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} \mathbb{I}^2 e^{G\frac{(1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} s^{\alpha-1} ds \\ &= \int_{0.8}^{1} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} e^{G\frac{(0.8-s)^{0.8}+0.2^{0.8}}{0.8}} \frac{1}{s^{0.2}} ds. \end{aligned}$$

Therefore, we get that

$$W_\tau[0,1] = \begin{bmatrix} 0.0027 & 0 \\ 0 & 0.0027 \end{bmatrix}, \quad W_\tau^{-1}[0,1] = \begin{bmatrix} 370.3704 & 0 \\ 0 & 370.3704 \end{bmatrix}.$$

Consequently, we obtain

$$M = \sqrt{\|W_\tau^{-1}[0,1]\|} = 19.245.$$

Next,

$$b = 17.8441.$$

Hence, W satisfies the assumption $[H_1]$. Further, it is easy to see that for any $y(t), z(t) \in \mathbb{R}^2$ and $t \in J_1$,

$$\begin{aligned} \|g(t,y) - g(t,z)\| &= \frac{1}{1000}(t+0.1)\sqrt{(y_1(t)-z_1(t))^2 + (y_2(t)-z_2(t))^2} \\ &\leq \frac{1}{1000}(t+0.1)\|y-z\|. \end{aligned}$$

Hence, g satisfies the assumption $[H_2]$, where we set $L_g(\cdot) = \frac{\cdot+0.1}{1000} \in L^q(J_1, \mathbb{R}^+)$.

Obviously, $\|L_g\|_{L^q(J_1,\mathbb{R}^+)} = \frac{1}{1000}(\frac{1.1^{q+1}-0.1^{q+1}}{q+1})^{\frac{1}{q}}$ and $R_g = \sup_{t\in J_1}\|g(t,0)\| = 1$. Next, $\|K\| = 1$, $\|L_g\|_{L^2(J_1,\mathbb{R}^+)} = 0.0006658$ and $M_1 = 0.00466$, when we choose $p = q = 2$. Hence,

$$\begin{aligned} \Gamma &= M_1\Big(1 + \frac{bM}{R_g}\|K\|\Big) \\ &= 0.00466 \times \Big(1 + \frac{17.8441 \times 19.245}{1} \times 1\Big) \\ &= 0.00466 \times 38.0891 = 0.1775 < 1, \end{aligned}$$

which implies that the condition (17) holds.

Thus all the conditions of Theorem 3.3 are satisfied. Hence, system (1) is relatively controllable on $[0,1]$.

Example 4.2. *Consider the relative controllability of system (18) (with $g(\cdot,y) \equiv 0$) on J_1, where G, R, R_1, K are defined in Example 4.1.*

According to Theorem 3.1, we can know that system (18) is relative controllability when $g(\cdot,y) = 0$. Further, keeping in mind of (7), one can get

$$\begin{aligned} \eta &= y_1 - e^{G\frac{t_1^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1 t_1}\varphi(-\tau) \\ &\quad - \int_{-\tau}^{0} e^{G\frac{(t_1-\tau-s)^\alpha+\tau^\alpha}{\alpha}} e_{\tau,\alpha}^{R_1(t_1-\tau-s)}[\mathfrak{D}_\alpha^0\varphi(s) - G\varphi(s)]s^{\alpha-1}ds \\ &= y_1 - \begin{bmatrix} -0.7563 \\ -0.4983 \end{bmatrix}, \quad y_1 \in \mathbb{R}^2. \end{aligned}$$

By using the selection form of the control in (6), we arrive at

$$
\begin{aligned}
u(t) &= K^{\top} e_{\tau,\alpha}^{R_1^{\top}(t_1-\tau-t)} e^{G^{\top}\frac{(t_1-\tau-t)^{\alpha}+\tau^{\alpha}}{\alpha}} W_{\tau}^{-1}[0,t_1]\eta \\
&= e_{0.2,0.8}^{R_1(0.8-t)} e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta \\
&= \begin{cases}
\Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2 + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^{0.8}}{0.8}\Big)^3 \\
\quad + R_1^4\frac{1}{4!}\Big(\frac{(0.2-s)^{0.8}}{0.8}\Big)^4\Big] e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta,\ 0 \le t < 0.2, \\
\Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2 + R_1^3\frac{1}{3!}\Big(\frac{(0.4-s)^{0.8}}{0.8}\Big)^3\Big] \\
\qquad \times e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta,\ 0.2 \le t < 0.4, \\
\Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8} + R_1^2\frac{1}{2!}\Big(\frac{(0.6-s)^{0.8}}{0.8}\Big)^2\Big] \\
\qquad \times e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta,\ 0.4 \le t < 0.6, \\
\Big[\mathbb{I} + R_1\frac{(0.8-s)^{0.8}}{0.8})\Big] \times e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta,\ 0.6 \le t < 0.8, \\
\mathbb{I} \times e^{G\frac{(0.8-t)^{0.8}+0.2^{0.8}}{0.8}} W_{\tau}^{-1}[0,1]\eta,\ 0.8 \le t < 1.
\end{cases}
\end{aligned}
$$

where R_1 and $W_{\tau}^{-1}[0,1]$ are given in the above.

5. Conclusions

The purpose of this paper is to develop a controllability method for linear and nonlinear conformable delay controlled systems with linear parts defined by permutable matrices. In order to achieve this purpose, a representation of solutions is used with the help of a delayed matrix exponential. Such an approach leads to new criteria for the relative controllability of our issues by constructing delay Grammian matrix and applying fixed point method, respectively.

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