Filomat 37:9 (2023), 2961–2968
https://doi.org/10.2298/FIL2309961B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New efficient and accurate bounds for zeros of a polynomial based on similarity of companion complex matrices

Aliaa Burqan[a], Ahmad Alsawaftah[a], Zeyad Al-Zhour[b,*]

[a] *Department of Mathematics, Zarqa University, Zarqa, Jordan*
[b] *Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia*

Abstract. In this paper, we present new bounds for the zeros of polynomials with numerical and matrix coefficients and show that these bounds are effective and more accurate for polynomials that have small differences between their coefficients. To get our main results, we apply the similarity of matrices and matrix inequalities including the numerical radius and matrix norms. Finally, some illustrated examples are given and discussed.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices, the eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$ are denoted by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, and are arranged so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots \geq |\lambda_n(A)|$. The singular values of $A \in \mathbb{M}_n(\mathbb{C})$ are the eigenvalues of $(A^*A)^{\frac{1}{2}}$ denoted by $s_1(A), s_2(A), \ldots, s_n(A)$, arranged in decreasing order and repeated according to multiplicity as $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$. For $A \in \mathbb{M}_n(\mathbb{C})$, let $r(A)$, $w(A)$, and $\|A\|$, be the spectral radius, the numerical radius and the spectral norm of A, respectively. It is well known that

$$\left|\lambda_j(A)\right| \leq r(A) \leq w(A) \leq \|A\| = s_1(A) \text{ for } j = 1, 2, \ldots, n.$$

The Frobenius companion matrix plays an important role between matrix analysis and the geometry of polynomials. It has been used for locating the zeros of polynomials by matrix methods. Some classical bounds for the zeros of polynomials are Cauchy's bound [15], Carmichael and Mason's bound [5], Montel's bound [5] and Fujii and Kubo's bound [6]. Others have provided bounds for the zeros of polynomials that relied on matrix inequalities using the Frobenius companion matrix such as [1], [2], [3], [4], [11], and [13].

On the other hand, locating the zeros of polynomials with matrix coefficients is a very important topic, which has attracted the attention of many researchers. Many bounds for the zeros of matrix polynomials can be found in [5], [9], [10], [12], and [14].

2020 *Mathematics Subject Classification*. Primary 15A60; Secondary 26C10, 30C15
Keywords. Frobenius companion matrix, similarity, inequality, zeros of polynomials, bounds.
Received: 27 April 2022; Revised: 15 September 2022; Accepted: 03 December 2022
Communicated by Fuad Kittaneh
* Corresponding author: Zeyad Al-Zhour
Email addresses: aliaaburqan@zu.edu.jo (Aliaa Burqan), ahmadwtubasi@gmail.com (Ahmad Alsawaftah), zalzhour@iau.edu.sa (Zeyad Al-Zhour)

In this paper, we employ the similarity of matrices and matrix inequalities including the numerical radius and matrix norms to derive new bounds for the zeros of polynomials.These new bounds are effective and more accurate for polynomials that have small differences between their coefficients.

2. Bounds for the zeros of scalar polynomials

Throughout this section, let $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_2 z + a_1$ be a monic polynomial of degree $n \geq 2$ with complex coefficients $a_1, a_2, \ldots, a_n$, where $a_1 \neq 0$.

The Frobenius companion matrix of p is the $n \times n$ matrix given by

$$F(p) = \begin{bmatrix} -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}.$$

It is well known that the zeros of p are exactly the eigenvalues of $F(p)$[15]. Consequently, if z is a zero of p, then $|z| \leq r(F(p))$.

Consider the invertible matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \text{ where } B^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & \ddots & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Define the matrix A as $A = BF(p)B^{-1}$, which will be called the A-companion matrix of p. Thus, the matrix

$$A = \begin{bmatrix} 1-a_n & a_n - a_{n-1} & a_{n-1} - a_{n-2} & \ldots & a_3 - a_2 & a_2 - a_1 - 1 \\ 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{bmatrix}$$

is similar to $F(p)$ and so they have the same eigenvalues.

To obtain new bounds for zeros of polynomials, we employ the similarity of matrices and matrix inequalities. This will be accomplished by applying the following lemmas which can be found in [7] and [8]. Here, $\mathbb{M}_{n\times m}(\mathbb{C})$ denotes the space of all $n \times m$ complex matrices.

Lemma 2.1. *Let $T = \left[T_{ij}\right]$ be an $n \times n$ block matrix with $T_{ij} \in \mathbb{M}_{m_i \times m_j}(\mathbb{C})$ and $\sum_{i=1}^{k} m_i = n$. Then*

$$w(T) \leq \frac{1}{2} \sum_{i=1}^{k} \left(w(T_{ii}) + \sqrt{(w^2(T_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{k} \left\|T_{ij}\right\|^2} \right).$$

Lemma 2.2. *Let L_n be the $n \times n$ matrix given by*

$$L_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then $w(L_n) = \cos \frac{\pi}{n+1}$.

Theorem 2.3. *Let z be a zero of $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_2 z + a_1$. Then*

$$|z| \le \frac{1}{2}\left(1 + \sqrt{2} + |1 - a_n| + \cos\frac{\pi}{n-1} + \sqrt{|a_2 - a_1 - 1|^2 + \sum_{j=3}^{n+1} \left|a_j - a_{j-1}\right|^2} + \sqrt{n - 1 + \cos^2\frac{\pi}{n-1}}\right),$$

where $a_{n+1} = 1$.

Proof. *Partition the A-companion matrix of p as*

$$A = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix},$$

where

$$T_{11} = [1 - a_n]\,, T_{12} = \begin{bmatrix} a_n - a_{n-1} & a_{n-1} - a_{n-2} & \dots & \dots & a_3 - a_2 \end{bmatrix}, T_{13} = [a_2 - a_1 - 1]\,,$$

$$T_{21} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, T_{22} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, T_{23} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix},$$

and

$$T_{31} = [0]\,, T_{32} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}, T_{33} = [-1]\,.$$

Now, the definitions of the numerical radius and spectral norm with Lemma 2.2 *imply*

$$w(T_{11}) = |1 - a_n|\,, w(T_{22}) = \cos\frac{\pi}{n-1}, w(T_{33}) = 1,$$

$$\|T_{12}\|^2 = \sum_{j=3}^{n} \left|a_j - a_{j-1}\right|^2, \|T_{13}\|^2 = |a_2 - a_1 - 1|^2\,, \|T_{21}\|^2 = 1,$$

$$\|T_{23}\|^2 = n - 2, \|T_{31}\|^2 = 0, \|T_{32}\|^2 = 1.$$

Using Lemma 2.1, *then we have*

$$\begin{aligned} w(A) &\le \frac{1}{2}\begin{pmatrix} w(T_{11}) + w(T_{22}) + w(T_{33}) + \sqrt{w^2(T_{11}) + \|T_{12}\|^2 + \|T_{13}\|^2} \\ + \sqrt{w^2(T_{22}) + \|T_{21}\|^2 + \|T_{23}\|^2} + \sqrt{w^2(T_{33}) + \|T_{31}\|^2 + \|T_{32}\|^2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} |1 - a_n| + \cos\frac{\pi}{n-1} + 1 + \sqrt{|1 - a_n|^2 + \sum_{j=3}^{n} \left|a_j - a_{j-1}\right|^2 + |a_2 - a_1 - 1|^2} \\ + \sqrt{\cos^2\frac{\pi}{n-1} + n - 1} + \sqrt{2} \end{pmatrix}, \end{aligned}$$

Using the fact that $|z| \leq r(F(p)) = r(A) \leq w(A)$, *we get*

$$|z| \leq \frac{1}{2}\left(\begin{array}{c} 1 + \sqrt{2} + |1 - a_n| + \cos\frac{\pi}{n-1} + \sqrt{|a_2 - a_1 - 1|^2 + \sum_{j=3}^{n+1} \left|a_j - a_{j-1}\right|^2} \\ + \sqrt{n - 1 + \cos^2\frac{\pi}{n-1}} \end{array} \right),$$

where $a_{n+1} = 1$. □

Using the spectral norm of the A-companion matrix of p, the following theorem produces another bound for the zeros of polynomials.

Theorem 2.4. *Let* z *be a zero of* $p(z) = z^n + a_n z^{n-1} + a_{n-1}z^{n-2} + \ldots + a_2 z + a_1$. *Then* $|z| \leq \sqrt{1 + \sqrt{n-1} + |a_2 - a_1 - 1|^2 + \sum_{j=3}^{n+1} \left|a_j - a_{j-1}\right|^2}$, *where* $a_{n+1} = 1$.

Proof. *The* A*-companion matrix of* p *can be written as* $A = R + S + T$, *where*

$$R = \begin{bmatrix} 1 - a_n & a_n - a_{n-1} & a_{n-1} - a_{n-2} & \ldots & a_3 - a_2 & a_2 - a_1 - 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix}.$$

Since $R^*S = R^*T = S^*R = S^*T = T^*R = T^*S = 0$, *the triangle inequality, together with the fact that* $\|A\|^2 = \|A^*A\|$, *yield*

$$\|A\|^2 = \|R^*R + S^*S + T^*T\| \leq \|R^*R\| + \|S^*S\| + \|T^*T\|.$$

However, with a few basic calculations, we have

$$\|R^*R\| = |1 - a_n|^2 + \sum_{j=3}^{n} \left|a_j - a_{j-1}\right|^2 + |a_2 - a_1 - 1|^2, \|S^*S\| = 1 \text{and} \|T^*T\| = \sqrt{n-1},$$

So,

$$\|A\|^2 \leq |1 - a_n|^2 + \sum_{j=3}^{n} \left|a_j - a_{j-1}\right|^2 + |a_2 - a_1 - 1|^2 + 1 + \sqrt{n-1}.$$

Since $|z| \leq r(F(p)) = r(A) \leq \|A\|$, *we have*
$|z| \leq \sqrt{1 + \sqrt{n-1} + |a_2 - a_1 - 1|^2 + \sum_{j=3}^{n+1} \left|a_j - a_{j-1}\right|^2}$, *where* $a_{n+1} = 1$.

□

***Remark* 2.5.** *Despite the fact that the numerical radius of a matrix A is smaller than or equal to its spectral norm and the bound in Theorem* 2.3 *was derived based on the numerical radius while the bound in Theorem* 2.4 *was derived based on the spectral norm, the last bound is sometimes better than the first one for specific polynomials. For example, if z is a zero of the polynomial* $p(z) = z^5 + 6z^4 + 5z^3 + z + 2$, *Theorem* 2.3 *gives* $|z| \le 8.8$ *and Theorem* 2.4 *gives* $|z| \le 7.6$ *while if z is a zero of the polynomial* $p(z) = z^{10} + z^9 + 3z^8 + 6z^7 + 2z^6 + 4z^5 + 4z^3 + 2z^2 + 6z + 1$,*Theorem* 2.3 *gives* $|z| \le 8.3$ *and Theorem* 2.4 *gives* $|z| \le 10.2$.

In the following two theorems new bounds for the zeros of polynomials are obtained directly by calculating the maximum column sum matrix norm$\|.\|_1$ and the maximum row sum matrix norm$\|.\|_\infty$ for the A-companion matrix and using the fact $\left|\lambda_j(A)\right| \le N(A)$ for any matrix norm $N(.)$.

Theorem 2.6. *Let z be a zero of* $p(z) = z^n + a_n z^{n-1} + a_{n-1}z^{n-2} + \ldots + a_2 z + a_1$. *Then*

$$|z| \le \max\{1 + |1 - a_n|, 1 + |a_n - a_{n-1}|, \ldots, 1 + |a_3 - a_2|, (n-1) + |a_2 - a_1 - 1|\}.$$

Theorem 2.7. *Let z be a zero of* $p(z) = z^n + a_n z^{n-1} + a_{n-1}z^{n-2} + \ldots + a_2 z + a_1$. *Then* $|z| \le \max\left\{2, |a_2 - a_1 - 1| + \sum_{j=3}^{n+1} \left|a_j - a_{j-1}\right|\right\}$, *where* $a_{n+1} = 1$.

It should be mentioned here that these new bounds are effective and more accurate for polynomials that have small differences between their coefficients.

Example 2.8. *Consider the polynomial* $p(z) = z^5 + 6z^4 + 7z^3 + 8z^2 + 9z + 10$. *Then the upper bounds for the zeros of this polynomial estimated by different mathematicians such as Cauchy [15], Carmichael and Mason [5], Montel [5], Fujii and Kubo [6], Kittaneh and et al. ([11], Theorem 2.9), Bhunia and Paul [4] are as shown in the following table*

Bound	*Value*
Cauchy	*10*
Carmichael and Mason	*18.1934*
Montel	*20*
Fujii and Kubo	*12.9489*
Kittaneh	*10.2647*
Bhunia and Paul	*13.09*

but if z is a zero of the polynomial $p(z)$, *then Theorem* 2.3 *gives* $|z| \le 7.9497$, *Theorem* 2.4 *gives* $|z| \le 5.9161$,*Theorem* 2.6 *and Theorem* 2.7 *give* $|z| \le 6$ *which are better than all estimates mentioned above.*

3. Bounds for the zeros of polynomials with matrix coefficients

In the preceding section, we have established bounds for zeros of scalar polynomials. In this section, we shall give bounds for zeros of polynomials with matrix coefficients using numerical radius inequalities.

Consider the monic polynomial $P(z) = Iz^m + A_m z^{m-1} + A_{m-1}z^{m-2} + \ldots + A_2 z + A_1$, where $I, A_1, A_2, \ldots, A_m \in \mathbb{M}_n(\mathbb{C})$. The Frobenius companion matrix of P is the $nm \times nm$ matrix given by

$$F(P) = \begin{bmatrix} -A_m & -A_{m-1} & \ldots & -A_2 & -A_1 \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & I & 0 \end{bmatrix}.$$

A complex number z is called a zero of $P(z)$ if there is a non-zero vector $x \in \mathbb{C}^n$ such that $P(z)x = 0$. As in the case of numerical coefficients, the zeros of $P(z)$ coincide with the eigenvalues of $F(P)$[5]. Thus, if z is a zero of $P(z)$, then $|z| \le r(F(P))$.

To obtain our estimate for the zeros of P, consider the invertible matrix

$$\widetilde{B}=\begin{bmatrix} I & I & I & I & \dots & I\\ 0 & I & I & I & \dots & I\\ 0 & 0 & I & I & \dots & I\\ 0 & 0 & 0 & I & \dots & I\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix}, \text{ where } \widetilde{B}^{-1}=\begin{bmatrix} I & -I & 0 & 0 & \dots & 0\\ 0 & I & -I & 0 & \dots & 0\\ 0 & 0 & I & -I & \ddots & 0\\ 0 & 0 & 0 & I & \ddots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & -I\\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix}.$$

Define the $\widetilde{A}$-companion matrix of P as $\widetilde{A}=\widetilde{B}F(P)\widetilde{B}^{-1}$. Thus,

$$\widetilde{A}=\begin{bmatrix} I-A_m & A_m-A_{m-1} & A_{m-1}-A_{m-2} & \dots & A_3-A_2 & A_2-A_1-I\\ I & 0 & 0 & \dots & 0 & -I\\ 0 & I & 0 & \dots & 0 & -I\\ 0 & 0 & I & \dots & 0 & -I\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & I & -I \end{bmatrix}.$$

It is well known that $\widetilde{A}$ and $F(P)$ have the same eigenvalues. So,

$$|z|\le r(\widetilde{A})\le w(\widetilde{A}).$$

To achieve our goal of finding new bounds for the zeros of P, we obtain an estimate of the numerical radius of $\widetilde{A}$ by using the following lemmas that can be found in [10].

Lemma 3.1. *Let $T=\left[T_{ij}\right]$ be an $n\times n$ block matrix with $T_{ij}\in\mathbb{M}_{m_i\times m_j}(\mathbb{C})$ and $\sum_{i=1}^{n}m_i=n$. Then*

$$w(T)\le w(\left[t_{ij}\right]),$$

where

$$t_{ij}=w\left(\begin{bmatrix} 0 & T_{ij}\\ T_{ji} & 0\end{bmatrix}\right).$$

In particular, $t_{ii}=w(T_{ii})$ for $i=1,2,\dots,n$.

Lemma 3.2. *Let L_n be the $n\times n$ block matrix given by*

$$L_n=\begin{bmatrix} 0 & \frac{1}{2}I & 0 & \dots & 0\\ \frac{1}{2}I & 0 & \frac{1}{2}I & \dots & 0\\ 0 & \frac{1}{2}I & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2}I\\ 0 & 0 & \dots & \frac{1}{2}I & 0 \end{bmatrix}.$$

Then the eigenvalues of L_n are $\lambda_j=\cos\frac{\pi j}{n+1}$ for $j=1,2,\dots,n$.

Theorem 3.3. *Let z be a zero of $P(z)=Iz^m+A_mz^{m-1}+A_{m-1}z^{m-2}+\ldots+A_2z+A_1$. Then*

$$|z|\le\frac{1}{2}\left(\begin{array}{c} 1+\cos\frac{\pi}{m-1}+w(I-A_m)+\sqrt{w^2(I-A_m)+\sum_{j=1}^{m-1}\left|a_j\right|^2}\\ +\sqrt{1+\frac{m+1}{4}+|a_1|^2}+\sqrt{\frac{m+1}{4}+\cos^2\frac{\pi}{n-1}\sum_{j=2}^{m-1}\left|a_j\right|^2}\end{array}\right),$$

where $a_j=\frac{\|A_{j+1}-A_j\|}{2}$, $j=2,3,\dots,m-2$, $a_{m-1}=w\left(\begin{bmatrix} 0 & A_m-A_{m-1}\\ I & 0\end{bmatrix}\right)$ and $a_1=\frac{\|A_2-A_1-I\|}{2}$.

Proof. For any two matrices $X, Y \in \mathbb{M}_m(\mathbb{C})$, let $T_{(X,Y)} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$. Applying Lemma 3.1 on $\widetilde{A}$, we get

$$w(\widetilde{A}) \le w(S),$$

where S is the matrix given by

$$S = \begin{bmatrix} w(I - A_m) & w(T_{(A_m - A_{m-1}, I)}) & w(T_{(A_{m-1} - A_{m-2}, 0)}) & \cdots & \cdots & w(T_{(A_2 - A_1 - I, 0)}) \\ w(T_{(I, A_m - A_{m-1})}) & w(0) & w(T_{(0,I)}) & \cdots & \cdots & w(T_{(-I,0)}) \\ w(T_{(0, A_{m-1} - A_{m-2})}) & w(T_{(I,0)}) & w(0) & \cdots & \cdots & w(T_{(-I,0)}) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ & & & & w(0) & w(T_{(-I,I)}) \\ w(T_{(0, A_2 - A_1 - I)}) & w(T_{(0,-I)}) & w(T_{(0,-I)}) & \cdots & w(T_{(I,-I)}) & w(-I) \end{bmatrix}.$$

Using the fact that $w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = \frac{\|A\|}{2}$ for every $A \in \mathbb{M}_n(\mathbb{C})$, then we have

$$S = \begin{bmatrix} w(I - A_m) & a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_1 \\ a_{m-1} & 0 & 1/2 & 0 & \cdots & 1/2 \\ a_{m-2} & 1/2 & 0 & 1/2 & \cdots & 1/2 \\ a_{m-3} & 0 & 1/2 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ a_1 & 1/2 & 1/2 & \cdots & 1 & 1 \end{bmatrix},$$

where $a_j = \frac{\|A_{j+1} - A_j\|}{2}$, $j = 2, 3, \ldots, m-2$, $a_{m-1} = w\left(\begin{bmatrix} 0 & A_m - A_{m-1} \\ I & 0 \end{bmatrix}\right)$ and $a_1 = \frac{\|A_2 - A_1 - I\|}{2}$. To find $w(S)$ we need to partition the matrix S as

$$S = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix},$$

where

$$T_{11} = w(I - A_m), T_{12} = \begin{bmatrix} a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_1 \end{bmatrix}, T_{13} = [a_1],$$

$$T_{21} = \begin{bmatrix} a_{m-1} \\ a_{m-2} \\ a_{m-3} \\ \vdots \\ a_2 \end{bmatrix}, T_{22} = \begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1/2 \\ 0 & 0 & \cdots & 1/2 & 0 \end{bmatrix}, T_{23} = \begin{bmatrix} 1/2 \\ 1/2 \\ \vdots \\ 1/2 \\ 1 \end{bmatrix},$$

$$T_{31} = [a_1], T_{32} = \begin{bmatrix} 1/2 & 1/2 & \cdots & 1/2 & 1 \end{bmatrix}, T_{33} = [1].$$

Now, the definitions of the numerical radius and spectral norm with Lemma 3.2 imply

$$w(T_{11}) = w(I - A_m), w(T_{22}) = \cos\frac{\pi}{m-1}, w(T_{33}) = 1,$$

$$\|T_{12}\|^2 = \sum_{j=2}^{m-1} \left|a_j\right|^2, \|T_{13}\|^2 = |a_1|^2, \|T_{21}\|^2 = \sum_{j=2}^{m-1} \left|a_j\right|^2,$$

$$\|T_{23}\|^2 = \frac{m+1}{4}, \|T_{31}\|^2 = |a_1|^2, \|T_{32}\|^2 = \frac{m+1}{4}.$$

Applying Lemma 2.1 *on the partition of S, we have*

$$w(S) \leq \frac{1}{2}\left(\begin{array}{c} w(T_{11}) + w(T_{22}) + w(T_{33}) + \sqrt{w^2(T_{11}) + \|T_{12}\|^2 + \|T_{13}\|^2} + \\ \sqrt{w^2(T_{22}) + \|T_{21}\|^2 + \|T_{23}\|^2} + \sqrt{w^2(T_{33}) + \|T_{31}\|^2 + \|T_{32}\|^2} \end{array} \right)$$

$$= \frac{1}{2}\left(\begin{array}{c} w(I - A_m) + 1 + \cos\frac{\pi}{m-1} + \sqrt{w^2(I - A_m) + \sum_{j=1}^{m-1} \left|a_j\right|^2} \\ \sqrt{\cos^2\frac{\pi}{n-1} + \sum_{j=2}^{m-1} \left|a_j\right|^2 + \frac{m+1}{4}} + \sqrt{1 + |a_1|^2 + \frac{m+1}{4}} \end{array} \right).$$

Since $|z| \leq r(F(P)) = r(\widetilde{A}) \leq w(\widetilde{A}) \leq w(S)$, *the proof is completed.* □

4. Conclusion

We have established new effective more accurate bounds for the zeros of polynomials that have small differences between their coefficients by employing the similarity of matrices and matrix inequalities including the numerical radius and matrix norms. It is worth noting that our results can be used in many applications in geometry and matrix analysis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Alpin, Y., Chien, M. T., Yeh, L., The numerical radius and bounds for zeros of a polynomial. Proceedings of the American Mathematical Society, 131(3) (2003), 725-730.
[2] Bhunia, Pintu; Bag, Santanu; Nayak, Raj Kumar; Paul, Kallol Estimations of zeros of a polynomial using numerical radius inequalities. Kyungpook Math. J. 61 (2021), no. 4, 845–858.
[3] Bhunia, P., Paul, K., Proper improvement of well-known numerical radius inequalities and their applications. Results in Mathematics, 76(4) (2021), 1–12.
[4] Bhunia, P., Paul, K., Annular bounds for the zeros of a polynomial from companion matrices. Advancess in Operator Theory, 7(1) (2022), 1–19.
[5] Fujii, M., Kubo, F., Operator norms as bounds for roots of algebraic equations. Proceedings of the Japan Academy, 49(10) (1973), 805–808.
[6] Fujii, M., Kubo, F., Buzano's inequality and bounds for roots of algebraic equations. Proceedings of the American Mathematical Society, 117(2) (1993), 359–361.
[7] Guelfen, H., Kittaneh, F., On numerical radius inequalities for operator matrices. Numerical Functional Analysis and Optimization, 40(11) (2019), 1231–1241.
[8] Gustafson, K. E., Rao, D. K., Numerical range. In Numerical range (pp. 1-26) (1997), Springer, New York, NY.
[9] Higham, N. J., Tisseur, F., Bounds for eigenvalues of matrix polynomials. Linear algebra and its applications, 358(1-3) (1997), 5–22.
[10] Jaradat, A., Kittaneh, F., Bounds for the eigenvalues of monic matrix polynomials from numerical radius inequalities. Advances in Operator Theory, (2020) 1–10.
[11] Kittaneh, F., Odeh, M., Shebrawi, K., Bounds for the zeros of polynomials from compression matrix inequalities. Filomat, 34(3) (2020), 1035–1051.
[12] Le, C.T., Du, T.H.B., Nguyen, T.D., On the location of eigenvalues of matrix polynomials. Oper. Matrices 13 (2019), 937–954.
[13] Linden, H., Bounds for zeros of polynomials using traces and determinants. Seminarberichte Fachb. Math. FeU Hagen, 69 (2000), 127–146.
[14] Melman, A., Polynomial eigenvalue bounds from companion matrix polynomials. Linear and Multilinear Algebra, 67(3) (2019), 598–612.
[15] Horn, R., Johnson, C., Matrix analysis, (2013), Cambridge University Press, Cambridge.