On the weighted maximal operators of Marcinkiewicz type Cesàro means of two-dimensional Walsh-Fourier series

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Abstract. In this paper we investigate the behaviour of the weighted maximal operators of Marcinkiewicz type \((C, \alpha)\)-means \(\sigma_{\alpha, \ast}^p(f) := \sup_{n \in \mathbb{Z}^2} \|\sigma_{\alpha}^p(f)\|_{L^p} \) in the Hardy space \(H^p(G^2)\) \((0 < \alpha < 1\) and \(p < 2/(2 + \alpha)\)). It is showed that the maximal operators \(\sigma_{\alpha, \ast}^p(f)\) are bounded from the dyadic Hardy space \(H^p(G^2)\) to the Lebesgue space \(L^p(G^2)\), and that this is in a sense sharp. It was also proved a strong convergence theorem for the Marcinkiewicz type \((C, \alpha)\) means of Walsh-Fourier series in \(H^p(G^2)\).

1. Introduction

In 1987, Simon [26] proved a strong summation theorem for Walsh-Fourier series. Namely, he certified that for any function \(f \in H^1(G)\) the following inequality holds
\[
\frac{1}{\log N} \sum_{n=1}^{N} \frac{\| S_n(f) \|_p}{n} \leq c \| f \|_{H^1}.
\]

This result has a trigonometric analogue verified by Smith [28]. Analogical theorems with respect to Vilenkin and Vilenkin-like systems were proved by Gáth [5] (even in the unbounded case) and Blahota [1]. Later, Simon [27] proved a similar result, he showed that there exists a constant \(C_p\) depending only on \(p\), such that the inequality
\[
\sum_{k=1}^{\infty} \frac{\| S_k(f) \|_p^p}{k^{2-p}} \leq C_p \| f \|_{H^p}^p
\]
holds for all \(f \in H^p(G)\) \((0 < p < 1)\). Tephnadze in [30] proved that sequence \(\{k^{2-p} : k \in \mathbb{P}\}\) in expression (1) is sharp. The next strong summation theorem for Fejér means was proved also by Tephnadze [29]. There exists a constant \(c_p > 0\) which depends only on \(p\), that
\[
\frac{1}{\log^{1/2+p} N} \sum_{n=1}^{N} \frac{\| \sigma_{k}^p(f) \|_p^p}{k^{2-p}} \leq c_p \| f \|_{H^p}^p
\]
holds for all \( f \in H_p \) (0 < \( p \leq 1/2 \), where \([x]\) denotes the integer part of \( x \). Blahota, Tephnadze and Toledo [2, 3] generalized this result for \((C, \alpha)\) means (see later inequalities (2) and (3)).

Weisz [32] investigated the maximal operator \( \sigma^{\alpha, \ast} \) (0 < \( \alpha < 1 \)) of \((C, \alpha)\) means of one-dimensional Walsh-Fourier series. Several results were proved with respect to this operator. Weisz proved that \( \sigma^{\alpha, \ast} : H_p(G) \to L^p(G) \) is bounded for \( p > 1/(1 + \alpha) \). Later, Goginava proved that \( \sigma^{\alpha, \ast} \) is not bounded from the dyadic Hardy space \( H_{1/(1+\alpha)}(G) \) to the space \( L^{1/(1+\alpha)}(G) \) [9]. This means that the endpoint of the boundedness of the maximal operator \( \sigma^{\alpha, \ast} \) is \( p_0 := 1/(1 + \alpha) \). Weisz and Simon [24] also investigated the properties of the maximal operator \( \sigma^{\alpha, \ast} \) in this endpoint. They showed that the maximal operator is bounded from the dyadic Hardy space \( H_{1/(1+\alpha)}(G) \) to the space weak-\( L^{1/(1+\alpha)}(G) \). Blahota and Tephnadze [2] continued the investigations of this topic. In 2014, they proved that the exact rate of the deviant behaviour of the \( n \)-th \((C, \alpha)\) means is \( \log^{1-\alpha}(n + 1) \). In addition they proved the next strong summation theorem. Let 0 < \( \alpha < 1 \), then there exists a positive constant \( c(\alpha) \) depending only on \( \alpha \), such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \left\| \sigma_{m}^{\alpha}(F) \right\|_{H_{1/(1+\alpha)}(G)}^{1/(1+\alpha)} \leq c(\alpha) \left\| F \right\|_{H_{1/(1+\alpha)}(G)}
\]

holds for all \( F \in H_{1/(1+\alpha)}(G) \). Analogical theorems for \( p < p_0 = 1/(1 + \alpha) \) are discussed in [3] by the first author, Tephnadze and Toledo. Namely, the following result was proved. There exists a positive constant \( c_{\alpha,p} \) which depends only on \( \alpha \) and \( p \) such that

\[
\sum_{m=1}^{\infty} \left\| \sigma_{m}^{\alpha}(F) \right\|_{H_{p}}^{p} \leq c_{\alpha,p} \left\| F \right\|_{H_{p}}^{p}
\]

holds for all \( F \in H_{p} \) (\( p < p_0 = 1/(1 + \alpha) \)). The case of Fejér means (setting \( \alpha = 1 \)) was proved by Tephnadze [29]. The properties of the maximal operator of Fejér means were investigated by several authors [4, 12, 13, 23, 25, 31, 35].

Goginava discussed the two-dimensional situation. He investigated the maximal operator \( \sigma^{\alpha, \ast} \) (0 < \( \alpha < 1 \)) of Marcinkiewicz type \((C, \alpha)\) means [9]. Namely, he proved that the maximal operator

\[
\sigma^{\alpha, \ast}(f) := \sup_{A \in \mathbb{P}} \left| \sigma_{A}^{\alpha}(f) \right| = \sup_{A \in \mathbb{P}} \left| \frac{1}{A_{\alpha-1}} \sum_{j=1}^{n} A_{n-j}^{-1} \chi_{A}^{j}(f) \right|
\]

is bounded from the two-dimensional dyadic martingale Hardy space \( H_{p}(G^2) \) to the space \( L^p(G^2) \) for \( p > 2/(2 + \alpha) \). Moreover, he showed that the assumption \( p > 2/(2 + \alpha) \) is essential for the boundedness of this operator, which means that the endpoint of the boundedness of two-dimensional Marcinkiewicz type maximal operator \( \sigma^{\alpha, \ast} \) is \( p_0 := 2/(2 + \alpha) \).

The maximal operator \( \sigma^{\alpha, \ast} \) is not bounded from the dyadic Hardy space \( H_{p}(G^2) \) to the space weak-\( L^p(G^2) \) for \( 0 < p < 2/(2 + \alpha) \), it can be proved by interpolation. That is why the behaviour of this maximal operator in the endpoint case (when \( p_0 = 2/(2 + \alpha) \)) interesting. Goginava [11] proved the boundedness of the maximal operator \( \sigma^{\alpha, \ast} \) from the dyadic Hardy space \( H_{2/(2+\alpha)}(G^2) \) to the space weak-\( L^{2/(2+\alpha)}(G^2) \). In the special case, while \( \alpha = 1 \) we get the so-called Marcinkiewicz means of Walsh-Fourier series. This case was discussed by several authors [12, 14, 18, 20–22, 36]. For recent investigations connected to this topic with respect to almost everywhere convergence see [6], [7] and [15], to strong convergence see [17].

In the paper [19], Nagy and Salim defined the weighted maximal operator \( \tilde{\sigma}^{\alpha, \ast} \) by

\[
\tilde{\sigma}^{\alpha, \ast}(f) := \sup_{n \in \mathbb{P}} \left| \sigma_{n}^{\alpha}(f) \right| \frac{1}{\log^{(2+\alpha)/2}(n + 1)}
\]

and they showed that the weighted maximal operator is bounded from the dyadic Hardy space \( H_{2/(2+\alpha)}(G^2) \) to the space \( L^{2/(2+\alpha)}(G^2) \). Moreover, they proved the sharpness of the sequence \( \log^{(2+\alpha)/2}(n + 1) : n \in \mathbb{P} \).
That is, they stated that the exact rate of deviant behaviour of \( n \)th Marcinkiewicz type \((C, \alpha)\) mean, it is \( \log^{(2+\alpha)}(n+1) \) in the end point case \( p_0 = 2/(2 + \alpha) \). They also proved a two-dimensional analogue of the strong summation theorem given in the inequality (2). Namely, there exists a positive constant \( c(\alpha) \) depending only on \( \alpha \), such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \sigma_m^\alpha(f) \|_{H^2/(2+\alpha)}}{m} \leq c(\alpha) \| f \|_{H^2/(2+\alpha)},
\]

holds for all \( f \in H^2/(2+\alpha)(G^2) \).

In this paper, we deal with the case \( 0 < p < p_0 = 2/(2 + \alpha) \), while \( 0 < \alpha < 1 \). We define the maximal operator

\[
\sigma_p^\alpha(f) := \sup_{n \in \mathbb{P}} \frac{|\sigma_n^\alpha(f)|}{n^{p/(2+\alpha)}}.
\]

We prove its boundedness from the martingale Hardy space \( H_p(G^2) \) to the Lebesgue space \( L^p(G^2) \). We prove the sharpness of the sequence \( \{n^{2-p/(2+\alpha)}\} \), as well. At the end of this article, we prove a strong summation theorem for the Marcinkiewicz type \((C, \alpha)\) means of Walsh-Fourier series in the Hardy space \( H_p \) \((0 < p < p_0 = 2/(2 + \alpha)) \). That is, we prove the two-dimensional version of the strong summation theorem given in the inequality (3).

2. Definitions and notation

The Marcinkiewicz type \((C, \alpha)\) means of the two-dimensional Walsh-Fourier series in the Hardy space \( H_p \) \((0 < p < p_0 = 2/(2 + \alpha)) \). That is, we prove the two-dimensional version of the strong summation theorem given in the inequality (3).
The kernel function is given by
\[ K_n^a(x, y) := \frac{1}{A_n^{a-1}} \sum_{k=1}^{n} A_{n-k}^{a-1} D_k(x)D_k(y). \]

For trigonometric system the behaviour of Marcinkiewicz means \((a = 1)\) was investigated by Marcinkiewicz [16] and the properties of Marcinkiewicz type \((C_n)\) means was discussed by Zhizhiashvili [37].

It is well-know that the \(L^1\) norm of the kernels \(K_n^a (0 < a \leq 1)\) are uniformly bounded [8]. That is, there exists a positive constant \(c\) such that
\[ \|K_n^a\|_1 \leq c \quad \text{for all } n \in \mathbb{N}. \] (4)

3. Auxiliary results

Our proofs are based on two lemmas of Goginava [11, page 20, 22] and some results of Weisz [34].

**Lemma 3.1 (Goginava [11]).** Let \((x, y) \in I_N \times (I_b \setminus I_b+1)\), where \(0 \leq b < N, n > 2^N\). Then
\[ \int_{I_b \times I_b} \left| K_n^a(x + u, y + v) \right| d\mu(u, v) \leq \frac{c(a)}{2^{N-a} n} \sum_{j=0}^{N} D_2(y + e_b) \]
holds.

In order to apply Lemma 3.1, we decompose the set \(I_N\) in the following way.
\[ I_N = \bigcup_{b=0}^{N-1} (I_b \setminus I_{b+1}) \quad \text{and} \quad I_b \setminus I_{b+1} = \bigcup_{m=b+1}^{N} I_m, \]
where \(I_s := I_s(0, \ldots, y_s = 1, 0, \ldots, y_{s} = 1)\) for \(s < N\) and \(I^N := I_N(0, \ldots, y_b = 1, 0, \ldots, 0)\). We use it later.

**Lemma 3.2 (Goginava [11]).** Let \((x, y) \in (I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1})\), \(0 \leq a \leq b < N\) and \(n \geq 2^N\). Then
\[ \int_{I_a \times I_a} \left| K_n^a(x + u, y + v) \right| d\mu(u, v) \leq \frac{c(a)}{2^{N-a} n^a} \left( 2^{b-aa} \sum_{j=a}^{b} D_2(x + e_a) + 2^{a+b} \sum_{j=a+1}^{N} 2^{(a-1)} D_2(y + e_b + x_{b+1,j-1}) \sum_{m=a+1}^{b+1} 2^m D_2(z + e_a + e_m) \right), \]
where \(x_{i,j} := \sum_{s=1}^{j} x_s e_s, x_{i,j-1} := 0\).

For the simplicity we introduce the notation
\[ K_{n,1}^{a,b}(x, y) := \frac{c(a) 2^{b+aa}}{2^{N-a} n^a} \sum_{j=a}^{b} D_2(x + e_a) \]
and
\[ K_{n,2}^{a,b}(x, y) := \frac{c(a) 2^{a+b}}{2^{N-a} n^a} \sum_{j=a+1}^{N} \sum_{m=a+1}^{b+1} 2^m D_2(z + e_a + e_m). \]
The $\sigma$-algebra generated by the dyadic 2-dimensional cubes $I^2_k$ is denoted by $F_k (k \in \mathbb{N})$. $f = (f_n : n \in \mathbb{N})$ denotes a one-parameter martingale with respect to the sequence of $\sigma$-algebras $(F_n, n \in \mathbb{N})$. The maximal function of a martingale $f$ is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f_n|.$$ 

For $0 < p \leq \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$ 

For $f \in L^1(G^2)$ the sequence $(S_{2^n,2^n} f) : n \in \mathbb{N})$ is a martingale. The maximal function can be given in the form

$$f^*(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x) \times I_n(y))} \left| \int_{I_n(x) \times I_n(y)} f(u,v) d\mu(u,v) \right|.$$ 

The concept of Walsh-Fourier coefficient of a function $f$ can be extended to martingales in the usual way (see Weisz [33, 34]). The Walsh-Fourier coefficients of $f \in L^1(G^2)$ are the same as the ones of the martingale $(S_{2^n,2^n} f) : n \in \mathbb{N})$ obtained from $f$. Consequently, partial sums, Marcinkiewicz means and $(C,\alpha)$ means of quadratic partial sums are defined for martingales, as well.

A useful property of the Hardy spaces $H_p(G^2)$ is the atomic structure. A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic two-dimensional cube $I^2_N$, such that

a) $\int_{I^2_N} ad\mu = 0$,

b) $\|a\|_{\infty} \leq \mu(I^2_N)^{-1/p}$,

c) $\text{supp} \ a \subset I^2_N$.

The operator $T$ is said to be $p$-quasilocal if there exists a positive constant $c_p$, such that

$$\int_{I_N \times I_N} |T(a)|^p d\mu \leq c_p$$

holds for all arbitrary $p$-atom $a$ with support $I_N \times I_N$.

Lemma 3.3 (Weisz [33]). Let $0 < p \leq 1$. Suppose that the operator $T$ is $\sigma$-sublinear and $p$-quasilocal. If $T$ is bounded from $L_{\infty}$ to $L_{\infty}$, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$ 

For the martingale

$$f = \sum_{n=0}^{\infty} (f_n - f_{n-1})$$

the conjugate transforms are defined as

$$\widehat{f}^0 = \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in G$ is fixed. We note that $\widehat{f}^0 = f$. It is well-known (see [33]) that

$$\|\widehat{f}^0\|_{H_p} = \|f\|_{H_p}, \quad \|f\|_{H_p}^p \sim \int_G \|\widehat{f}^0\|^p d\mu(t),$$

and

$$(\sigma^m_{\alpha}(f))^{(0)} = \alpha_m^{\alpha}(\widehat{f}^0).$$
4. The properties of the weighted maximal function $\sigma_p^{α,*}(f)$

**Theorem 4.1.** Let $0 < α < 1$ and $0 < p < 2/(2 + α)$.

a) Then the maximal operator

$$\sigma_p^{α,*}(f) := \sup_{n \in \mathbb{N}} \frac{|a_n^{p}(f)|}{n^{2/p - (2 + α)}}$$

is bounded from the dyadic Hardy space $H_p(G^2)$ to the space $L^p(G^2)$.

b) Let $φ : \mathbb{P} \to [1, \infty)$ be a non-decreasing function satisfying the condition

$$\limsup_{n \to \infty} \frac{n^{2/p - (2 + α)}}{φ(n)} = \infty.$$  \hspace{1cm} (7)

Then the weighted maximal operator

$$\sigma_p^{α,*}(f) := \sup_{n \in \mathbb{N}} \frac{|a_n^{p}(f)|}{φ(n)}$$

is not bounded from the Hardy space $H_p(G^2)$ to the space weak-$L^p(G^2)$.

**Proof.** First, we prove part a).

Inequality (4) yields the boundedness of the operator $\sigma_p^{α,*}$ from the space $L^∞$ to the space $L^∞$. Applying Lemma 3.3, we have to show that the maximal operator $\sigma_p^{α,*}$ is $p$-quasilocal. That is, there exists a constant $c(α) > 0$ such that

$$\int_{I_p} |\sigma_p^{α,*}(a)|^p dμ \leq c(α) < \infty$$

for every $p$-atom $a$, where the dyadic cube $I^2$ is the support of the $p$-atom $a$.

Let $a$ be an arbitrary $p$-atom with support $I^2$ and $μ(I^2) = 2^{-2N}$. Without loss of generality, we may assume that $I^2 := I_N × I_N$. It is easily seen that $σ_p^{α}(a) = 0$ if $n ≤ 2^N$. Therefore, we set $n > 2^N$. We know that $||a||_∞ ≤ 2^{2N/p}$. Thus,

$$|σ_n^{α}(a; x, y)| \leq \int_{I_n × I_n} |a(u, v)| |K_n^{α}(x + u, y + v)| dμ(u, v) \leq c(α)2^{2N/p} \int_{I_n × I_n} |K_n^{α}(x + u, y + v)| dμ(u, v)$$

and

$$|σ_p^{α,*}(a)| \leq c(α)2^{2N/p} \sup_{n > 2^N} \int_{I_n × I_n} \frac{|K_n^{α}(x + u, y + v)|}{n^{2/p - (2 + α)}} dμ(u, v).$$ \hspace{1cm} (8)

We decompose the set $I_N × I_N$ as

$$I_N × I_N = (I_N × I_N) ∪ (I_N × I_N) ∪ (I_N × I_N).$$

This yields that

$$\int_{I_n × I_n} |σ_n^{α,*}(a)|^p dμ = \int_{I_n × I_n} |σ_p^{α,*}(a)|^p dμ + \int_{I_n × I_n} |σ_p^{α,*}(a)|^p dμ + \int_{I_n × I_n} |σ_p^{α,*}(a)|^p dμ =: L_1 + L_2 + L_3.$$
First, we discuss the expression $L_1$ (the expression $L_2$ is discussed analogously). Lemma 3.1 and decomposition (5) imply that

$$L_1 \leq \sum_{b=0}^{N-1} \sum_{s=1}^{N} \int_{J_{b}\times I_a} \left( \frac{c(a)2^{2N/p}}{2^{N(2p-2-2a)}2^{N+at(N-b)}} \sum_{j=1}^{s} D_2(y + e_b) \right)^p d\mu(x, y)$$

$$\leq c(a) \sum_{b=0}^{N-1} \sum_{s=1}^{N} \frac{2^{2N-a}sN}{2^{N(2-2p-a)+N+pa(N-b)}}$$

$$= c(a) \sum_{b=0}^{N-1} \sum_{s=1}^{N} \frac{2^{2N-a}s}{2^{N(1-p)}}$$

$$\leq c(a) \sum_{b=0}^{N-1} \frac{2^{2p(a+1)-1}}{2^{N(1-p)}}.$$ 

There are three cases. $0 < p < 1/(1 + a)$, $p = 1/(1 + a)$ and $1/(1 + a) < p < 2/(2 + a)$.

Let us set $0 < p < 1/(1 + a)$. Then

$$L_1 \leq \frac{c(a)}{2^{N(1-p)}} \leq c(a).$$

Now, we set $p = 1/(1 + a)$. We get that

$$L_1 \leq \frac{c(a)N}{2^{N(1-p)}} \leq c(a).$$

Let $1/(1 + a) < p < 2/(2 + a)$. In this case, we immediately write that

$$L_1 \leq c(a) \frac{2^{N(p+a-1)}}{2^{N(1-p)}} \leq c(a) \frac{2^{N(2-2p+a-1)}}{2^{N(1-p)}} = c(a)$$

Now, we discuss the expression $L_3$. We introduce the notation $I_a := I_a \setminus I_{a+1}$. We write that

$$L_3 = \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{J_{b}\times I_a} |c_{p}^{a}(a)|^p d\mu$$

$$= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{I_{b}\times I_a} |c_{p}^{a}(a)|^p d\mu + \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{I_{b}\times I_a} |c_{p}^{a}(a)|^p d\mu$$

$$=: L_{3,1} + L_{3,2}.$$

We discuss $L_{3,2}$ (by symmetry the discussion of $L_{3,1}$ is analogous). Inequality (8) and Lemma 3.2 yield that

$$L_{3,2} \leq c(a) \frac{2^{2N}}{2^{N(1-p)}} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{J_{b}\times I_a} \left( \sup_{n>2N} \frac{\mathcal{K}_{n,b}^{a,b}}{n^{2h(2-2a)}2^{N(2-2p-a)}} \right)^p d\mu$$

$$+ c(a) \frac{2^{2N}}{2^{N(1-p)}} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{J_{b}\times I_a} \left( \sup_{n>2N} \frac{\mathcal{P}_{n,b}^{a,b}}{n^{2h(2-2p-a)}} \right)^p d\mu$$

$$=: L_{3,2,1} + L_{3,2,2}.$$
Decomposition (5), Lemma 3.2 and $p < 2/(2 + \alpha)$ give that

$$L_{3,2}^1 \leq c(a) \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{r=1}^{N-1} \int_{x=0}^{N-a-b} \left( \frac{2^{b+aa}}{2^{N+a+b}2^{N(2/p-2+a)}} \right) d\mu(x, y)$$

$$\leq c(a) \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{r=1}^{N-1} \int_{x=0}^{N-a-b} \left( \frac{2^{a(a+b)N/p-(2+a)}}{2^{N(2/p)-2(a)}} \right) d\mu(x, y)$$

$$= c(a) \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{r=1}^{N-1} \int_{x=0}^{N-a-b} \left( \frac{2^{a(a+b)N/p-(2+a)}}{2^{N(2/p)-2(a)}} \right) d\mu(x, y)$$

$$\leq c(a) \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{r=1}^{N-1} \int_{x=0}^{N-a-b} \left( \frac{2^{a(a+b)N/p-(2+a)}}{2^{N(2/p)-2(a)}} \right) d\mu(x, y)$$

At last, we discuss $L_{3,2}^2$. $\mathcal{K}^{a,b}_{n,2}(x, y) \neq 0$ implies that

$$\mathcal{K}^{a,b}_{n,2}(x, y) \leq \frac{c(a)}{2^{2N}} 2^{a+b+qa}.$$

and

$$\mathcal{K}^{a,b}_{n,2}(x, y) \leq \frac{c(a)}{2^{2N}} 2^{a+b+qa}.$$

for some $q$ and $r$, for which $a \leq r \leq b < q < N$ (see [11]). Consequently, we have

$$\mathcal{K}^{a,b}_{n,2}(x, y) \leq \frac{c(a)}{2^{2N}} 2^{a+b+qa}.$$

Let us discuss part b) of Theorem 4.1. Let $B \in \mathbb{P}$ and

$$f_B(x_1, x_2) := (D_{2^{a+b}}(x_1) - D_{2^{a+b}}(x_1))(D_{2^{a+b}}(x_2) - D_{2^{a+b}}(x_2)).$$
In this case, we have
\[ f_b(i, j) = \begin{cases} 1 & \text{if } i, j \in \{2^{2b} + 1, \ldots, 2^{2b+1} - 1\}, \\ 0 & \text{otherwise.} \end{cases} \]

We obtain
\[ S_{ij}(f_b; x_1, x_2) = \begin{cases} (D_i(x_1) - D_{2b}(x_1)) & \text{if } i, j \in \{2^{2b} + 1, \ldots, 2^{2b+1}\}, \\ f_b(x_1, x_2) & \text{if } i, j \geq 2^{2b+1}, \\ 0 & \text{otherwise.} \end{cases} \] (10)

Using
\[ f_b(x_1, x_2) = \sup_{w \in \mathbb{N}} |S_{M_b,M_b}(f_b; x_1, x_2)| = |f_b(x_1, x_2)|, \]
we get that
\[
\|f_b\|_{l_p} = \|f_b\|_p = \|D_{2b+1} - D_{2b}\|_p^2 = \left( \left( \int_{1 \leq i,j \leq b} 2^{2b} + \int_{i \geq b+1} (2^{2b+1} - 2^{2b}p) \right)^{1/p} \right)^2 = \left( 2 \cdot \frac{2^{2b}p}{2^{2b+1}} \right)^{1/p} = 2^{2b(2-2/p)}.
\]

By equality (10) we can write
\[
|a_{2b+1}^x(f_b; x_1, x_2)| = \frac{1}{A_{2b}^2} \left| \sum_{j=1}^{2^{2b+1}} A_{2b+1-j}^{2b+1} S_{ij}(f_b; x_1, x_2) \right| \geq \frac{c}{2^{2b+x}} \quad \text{for all } (x_1, x_2) \in G^2.
\]

Using this fact, we get
\[
\frac{e^{2b+x} \mu \left\{ (x_1, x_2) : |a_{2b}^x(f_b)| \geq \frac{c}{2^{2b+x}} \right\}^{1/p}}{\|f_b\|_{l_p}} \geq \frac{c}{q(2^{2b})2^{2b+x}2^{2b(2-2/p)}} = c(2^{2b})^{2/2-\alpha}.
\]

At last, under condition (7), there exists a sequence of positive integers \(\{n_k, k \in \mathbb{N}\}\), such that
\[
\lim_{k \to \infty} \frac{\left(2^{2n_k}\right)^{2/2-\alpha}}{q(2^{2n_k})} = \infty.
\]

This completes the proof of Theorem 4.1. \(\square\)
5. Strong summation theorem

Now, we prove a strong summation theorem for the Marcinkiewicz type \((C, \alpha)\) means of Walsh-Fourier series in the Hardy space \(H_p\) \((0 < \alpha < 1, 0 < p < 2/(2 + \alpha))\).

**Theorem 5.1.** Let \(0 < \alpha < 1\) and \(0 < p < 2/(2 + \alpha)\). There exists a positive constant \(c(\alpha)\) depending only on \(\alpha\) and \(p\), such that

\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m^\alpha(f)\|_{H_p}^p}{m^{3(2+\alpha)p}} \leq c(\alpha, p) \|f\|_{H_p}^p
\]

holds for all \(f \in H_p(G^2)\).

**Proof.** In the sequel, we show that there exists a positive constant \(c(\alpha, p)\) which depends only on \(\alpha\) and \(p\) such that the following inequality holds

\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m^\alpha(f)\|_{H_p}^p}{m^{3(2+\alpha)p}} \leq c(\alpha, p) \|f\|_{H_p}^p \quad (f \in H_p(G^2)). \tag{11}
\]

Inequality (4) gives that \(\sigma_n^\alpha\) is bounded from the space \(L_\infty\) to the space \(L_\infty\). By Lemma 3.3 it is enough to prove that inequality (11) holds for every arbitrary \(p\)-atom \(a\). Taking into account that \(\|a\|_{H_p} \leq 1\) we show that there exists a positive constant \(c(\alpha)\) such that

\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m^\alpha(a)\|_{H_p}^p}{m^{3(2+\alpha)p}} < c(\alpha, p) \tag{12}
\]

holds.

Let \(a\) be an arbitrary \(p\)-atom with support \(I^2\) and \(\mu(I^2) = 2^{-2N}\). Without loss of generality, we may assume that \(I^2 := I_N \times I_N\). We know that \(\|a\|_{L_\infty} \leq 2^{2N/p}\). By a simple consideration \(\sigma_n^\alpha(a) = 0\) if \(n \leq 2^N\). Therefore, we set \(n > 2^N\).

To prove inequality (12), we apply the next decomposition.

\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m^\alpha(a)\|_{H_p}^p}{m^{3(2+\alpha)p}} = \sum_{m=1}^{\infty} \frac{\|\sigma_m^\alpha(a)\|_{H_p}^p}{m^{3(2+\alpha)p}} \leq \sum_{m=1}^{\infty} \int_{I_N \times I_N} \frac{|\sigma_m^\alpha(a)|^p}{m^{3(2+\alpha)p}} d\mu + \sum_{m=1}^{\infty} \int_{I_N \times I_N} \frac{|\sigma_m^\alpha(a)|^p}{m^{3(2+\alpha)p}} d\mu
\]

\[
+ \sum_{m=1}^{\infty} \int_{I_N \times I_N} \frac{|\sigma_m^\alpha(a)|^p}{m^{3(2+\alpha)p}} d\mu + \sum_{m=1}^{\infty} \int_{I_N \times I_N} \frac{|\sigma_m^\alpha(a)|^p}{m^{3(2+\alpha)p}} d\mu =: I_1 + I_2 + I_3 + I_4.
\]

First, we apply inequality (4) and we write that

\[
I_1 \leq \sum_{m=2^N}^{\infty} \int_{I_N \times I_N} \frac{|\sigma_m^\alpha(a)|^p}{m^{3(2+\alpha)p}} d\mu \leq c(\alpha, p) \sum_{m=2^N}^{\infty} \frac{1}{m^{3(2+\alpha)p}} \|a\|_{L_\infty}^p 2^{-2N} \leq c(\alpha, p).
\]
Second, we discuss expression $I_2$. Decomposition (5) and Lemma 3.1 yield that

$$I_2 \leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{b=0}^{N-1} \sum_{a=0}^{N-1} \frac{1}{m^{1-(2+\alpha)p}} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu$$

$$\leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{b=0}^{N-1} \sum_{a=0}^{N-1} \frac{1}{m^{1-(2+\alpha)p}} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

There are 3 cases. $0 < p < 1/(1+\alpha)$, $p = 1/(1+\alpha)$ and $1/(1+\alpha) < p < 2/(2+\alpha)$.

Let us set $0 < p < 1/(1+\alpha)$. Then

$$I_2 \leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{b=0}^{N-1} \sum_{a=0}^{N-1} \frac{1}{m^{1-(2+\alpha)p}} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

Now, we set $p = 1/(1+\alpha)$.

$$I_2 \leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{b=0}^{N-1} \sum_{a=0}^{N-1} \frac{1}{m^{1-(2+\alpha)p}} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

Let us set $1/(1+\alpha) < p < 2/(2+\alpha)$.

$$I_2 \leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{b=0}^{N-1} \sum_{a=0}^{N-1} \frac{1}{m^{1-(2+\alpha)p}} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

The estimate of expression $I_3$ is similar. That is, we have that

$$I_3 \leq c(a, p).$$

At last, we discuss the expression $I_4$. By the decomposition (5) we write

$$I_4 \leq \sum_{m=2^N}^n \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu$$

$$+ \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu =: I_{41} + I_{42}.$$

We discuss $I_{42}$. By decomposition (5) and Lemma 3.2 we get that

$$I_{42} \leq \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

$$+ \frac{c(a, p)2^N}{2Np+a\alpha p} \sum_{m=2^N}^n \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \int_{I_{2N-3}^a} \left( \frac{\|\alpha\|_2}{2^{Np+a(N-3)} N} \right)^p \, d\mu.$$

$$=: I_{4,2}^1 + I_{4,2}^2.$$
and

$$I_{4,2}^1 \leq c(a,p)2^{2N} \sum_{m=2^N}^{n} \sum_{a=0}^{N-1} \sum_{b=0}^{\frac{N-1}{2}} \sum_{r=2^N}^{N} \int_{G_{x}(l_{b+1})} \frac{\left(\frac{2^{\alpha}}{m^{3-2\alpha}} \sum_{j=0}^{B^2} D_2(x + a_3)\right)^p}{m^{3(2\alpha)}p} d\mu(x,y)$$

$$\leq c(a,p)2^{2N} \sum_{m=2^N}^{n} \frac{1}{m^{3-2\alpha}p} \sum_{a=0}^{N-1} \sum_{b=0}^{\frac{N-1}{2}} \sum_{r=2^N}^{N} \frac{2^{b+a+3p}}{2^{2Np}} 2^{-a-b}$$

$$\leq c(a,p)2^{2N} \sum_{m=2^N}^{n} \frac{1}{m^{3-2p}p} \sum_{a=0}^{N-1} \sum_{b=0}^{\frac{N-1}{2}} \frac{2^{b(2\alpha+3p)-2}}{2^{2Np}}$$

$$\leq c(a,p)2^{2N} \frac{1}{2^{2Np}} \frac{2^{b(2\alpha+3p)-2}}{2^{2Np}} \leq c(a,p).$$

Now, we discuss the expression $I_{4,2}^2$. Inequalities (13) and (9) imply

$$I_{4,2}^2 \leq c(a,p)2^{2N} \sum_{m=2^N}^{n} \frac{1}{m^{3-2\alpha}p} \sum_{a=0}^{N-1} \sum_{b=0}^{\frac{N-1}{2}} \sum_{r=2^N}^{N} \sum_{i=0}^{\frac{N-1}{2}} \sum_{i=0}^{\frac{N-1}{2}} \sum_{i=0}^{\frac{N-1}{2}} \frac{2^{b(2\alpha+3p)-2}}{2^{2Np}}$$

$$\leq c(a,p)2^{2N} \frac{1}{2^{2Np}} \frac{2^{b(2\alpha+3p)-2}}{2^{2Np}} \leq c(a,p).$$

We estimate the expression $I_{4,1}$ analogically and write that

$$I_{4,1} \leq c(a,p).$$

Inequality (11) and the properties (6) of the conjugate transform of a martingale yield that

$$\sum_{m=1}^{n} \left\| \sigma_{m}^{a}(f) \right\|_{H_p} \sim \sum_{m=1}^{n} \int_{G} \left\| \left(\sigma_{m}^{a}(f)\right)(t) \right\|_{m^{3-2\alpha}p}^{p} d\mu(t)$$

$$= \int_{G} \sum_{m=1}^{n} \left\| \sigma_{m}^{a}(f)(t) \right\|_{m^{3-2\alpha}p}^{p} d\mu(t)$$

$$\leq c(a,p) \int_{G} \left\| (\tilde{f})(t) \right\|_{H_p}^{p} d\mu(t) \sim \| f \|_{H_p}^{p}.$$
Conjecture 5.2. Let $0 < \alpha < 1$ and $0 < p < 2/(2+\alpha)$. Let $\varphi : \mathbb{P} \to [1, \infty)$ be a non-decreasing function satisfying the condition
\[
\limsup_{k \to \infty} \frac{2^{k(3-2+\alpha)p}}{\varphi(2^k)} = \infty.
\] (14)
Then there exists a martingale $f \in H_p(G^2)$ such that
\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m(f)\|_{\text{weak-}L_p}}{\varphi(m)} = \infty.
\]

For Marcinkiewicz means (setting $\alpha = 1$) the analogue of Conjecture 5.2 is proved in [20].

References