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On the weighted maximal operators of Marcinkiewicz type Cesàro means of two-dimensional Walsh-Fourier series

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Abstract. In this paper we investigate the behaviour of the weighted maximal operators of Marcinkiewicz type (C,α)-means $\sigma_p^{\alpha,*}(f) := \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\alpha(f)|}{n^{2/p-(2+\alpha)}}$ in the Hardy space $H_p(G^2)$ ($0 < \alpha < 1$ and $p < 2/(2+\alpha)$). It is showed that the maximal operators $\sigma_p^{\alpha,*}(f)$ are bounded from the dyadic Hardy space $H_p(G^2)$ to the Lebesgue space $L^p(G^2)$, and that this is in a sense sharp. It was also proved a strong convergence theorem for the Marcinkiewicz type (C,α) means of Walsh-Fourier series in $H_p(G^2)$.

1. Introduction

In 1987, Simon [26] proved a strong summation theorem for Walsh-Fourier series. Namely, he certified that for any function $f \in H_1(G)$ the following inequality holds

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{\left\|S_n(f)\right\|_1}{n} \leq c\left\|f\right\|_{H_1}.$$

This result has a trigonometric analogue verified by Smith [28]. Analogical theorems with respect to Vilenkin and Vilenkin-like systems were proved by Gát [5] (even in the unbounded case) and Blahota [1]. Later, Simon [27] proved a similar result, he showed that there exists a constant C_p depending only on p, such that the inequality

$$\sum_{k=1}^{\infty}\frac{\left\|S_k(f)\right\|_p^p}{k^{2-p}} \leq C_p\left\|f\right\|_{H_p}^p \tag{1}$$

holds for all $f \in H_p(G)$ $(0 < p < 1)$. Tephnadze in [30] proved that sequence $\left\{k^{2-p} : k \in \mathbb{P}\right\}$ in expression (1) is sharp. The next strong summation theorem for Fejér means was proved also by Tephnadze [29]. There exists a constant $c_p > 0$ which depends only on p, that

$$\frac{1}{\log^{[1/2+p]} n}\sum_{k=1}^{n}\frac{\left\|\sigma_k(f)\right\|_{H_p}^p}{k^{2-p}} \leq c_p\left\|f\right\|_{H_p}^p$$

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holds for all $f \in H_p$ $(0 < p \leq 1/2)$, where $[x]$ denotes the integer part of x. Blahota, Tephnadze and Toledo [2, 3] generalized this result for (C, α) means (see later inequalities (2) and (3)).

Weisz [32] investigated the maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ of (C, α) means of one-dimensional Walsh-Fourier series. Several results were proved with respect to this operator. Weisz proved that $\sigma^{\alpha,*}\colon H_p(G) \to L^p(G)$ is bounded for $p > 1/(1+\alpha)$. Later, Goginava proved that $\sigma^{\alpha,*}$ is not bounded from the dyadic Hardy space $H_{1/(1+\alpha)}(G)$ to the space $L^{1/(1+\alpha)}(G)$ [9]. This means that the endpoint of the boundedness of the maximal operator $\sigma^{\alpha,*}$ is $p_0 := 1/(1+\alpha)$. Weisz and Simon [24] also investigated the properties of the maximal operator $\sigma^{\alpha,*}$ in this endpoint. They showed that the maximal operator is bounded from the dyadic Hardy space $H_{1/(1+\alpha)}(G)$ to the space weak-$L^{1/(1+\alpha)}(G)$. Blahota and Tephnadze [2] continued the investigations of this topic. In 2014, they proved that the exact rate of the deviant behaviour of the nth (C, α) means is $\log^{1+\alpha}(n+1)$. In addition they proved the next strong summation theorem. Let $0 < \alpha < 1$, than there exists a positive constant $c(\alpha)$ depending only on α, such that

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\left\| \sigma_m^{\alpha}(F) \right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{m} \leq c(\alpha) \left\| F \right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} \tag{2}$$

holds for all $F \in H_{1/(1+\alpha)}(G)$. Analogical theorems for $p < p_0 = 1/(1+\alpha)$ are discussed in [3] by the first author, Tephnadze and Toledo. Namely, the following result was proved. There exists a positive constant $c_{\alpha,p}$ which depends only on α and p such that

$$\sum_{m=1}^{\infty} \frac{\left\| \sigma_m^{\alpha}(F) \right\|_{H_p}^{p}}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} \left\| F \right\|_{H_p}^{p} \tag{3}$$

holds for all $F \in H_p$ $(p < p_0 = 1/(1+\alpha))$. The case of Fejér means (setting $\alpha = 1$) was proved by Tephnadze [29]. The properties of the maximal operator of Fejér means were investigated by several authors [4, 12, 13, 23, 25, 31, 35].

Goginava discussed the two-dimensional situation. He investigated the maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ of Marcinkiewicz type (C, α) means [9]. Namely, he proved that the maximal operator

$$\sigma^{\alpha,*}(f) := \sup_{n \in \mathbb{P}} \left| \sigma_n^{\alpha}(f) \right| := \sup_{n \in \mathbb{P}} \frac{1}{A_{n-1}^{\alpha}} \left| \sum_{j=1}^{n} A_{n-j}^{\alpha-1} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space $H_p(G^2)$ to the space $L^p(G^2)$ for $p > 2/(2+\alpha)$. Moreover, he showed that the assumption $p > 2/(2+\alpha)$ is essential for the boundedness of this operator, which means that the endpoint of the boundedness of two-dimensional Marcinkiewicz type maximal operator $\sigma^{\alpha,*}$ is $p_0 := 2/(2+\alpha)$.

The maximal operator $\sigma^{\alpha,*}$ is not bounded from the dyadic Hardy space $H_p(G^2)$ to the space weak-$L^p(G^2)$ for $0 < p < 2/(2+\alpha)$, it can be proved by interpolation. That is why the behaviour of this maximal operator in the endpoint case (when $p_0 = 2/(2+\alpha)$) interesting. Goginava [11] proved the boundedness of the maximal operator $\sigma^{\alpha,*}$ from the dyadic Hardy space $H_{2/(2+\alpha)}(G^2)$ to the space weak-$L^{2/(2+\alpha)}(G^2)$. In the special case, while $\alpha = 1$ we get the so-called Marcinkiewicz means of Walsh-Fourier series. This case was discussed by several authors [12, 14, 18, 20–22, 36]. For recent investigations connected to this topic with respect to almost everywhere convergence see [6], [7] and [15], to strong convergence see [17].

In the paper [19], Nagy and Salim defined the weighted maximal operator $\tilde{\sigma}^{\alpha,*}$ by

$$\tilde{\sigma}^{\alpha,*}(f) := \sup_{n \in \mathbb{P}} \frac{\left| \sigma_n^{\alpha}(f) \right|}{\log^{(2+\alpha)/2}(n+1)}$$

and they showed that the weighted maximal operator is bounded from the dyadic Hardy space $H_{2/(2+\alpha)}(G^2)$ to the space $L^{2/(2+\alpha)}(G^2)$. Moreover, they proved the sharpness of the sequence $\{\log^{(2+\alpha)/2}(n+1) : n \in \mathbb{P}\}$.

That is, they stated that the exact rate of deviant behaviour of nth Marcinkiewicz type (C, α) mean, it is $\log^{(2+\alpha)/2}(n+1)$ in the end point case $p_0 = 2/(2+\alpha)$. They also proved a two-dimensional analogue of the strong summation theorem given in the inequality (2). Namely, there exists a positive constant $c(\alpha)$ depending only on α, such that

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\left\|\sigma_m^\alpha(f)\right\|_{H_{2/(2+\alpha)}}^{2/(2+\alpha)}}{m} \le c(\alpha)\left\|f\right\|_{H_{2/(2+\alpha)}}^{2/(2+\alpha)}$$

holds for all $f \in H_{2/(2+\alpha)}(G^2)$.

In this paper, we deal with the case $0 < p < p_0 = 2/(2+\alpha)$, while $0 < \alpha < 1$. We define the maximal operator

$$\sigma_p^{\alpha,*}(f) := \sup_{n\in\mathbb{P}}\frac{\left|\sigma_n^\alpha(f)\right|}{n^{2/p-(2+\alpha)}}.$$

We prove its boundedness from the martingale Hardy space $H_p(G^2)$ to the Lebesgue space $L^p(G^2)$. We prove the sharpness of the sequence $\{n^{2/p-(2+\alpha)}\}$, as well. At the end of this article, we prove a strong summation theorem for the Marcinkiewicz type (C, α) means of Walsh-Fourier series in the Hardy space H_p $(0 < p < p_0 = 2/(2+\alpha))$. That is, we prove the two-dimensional version of the strong summation theorem given in the inequality (3).

2. Definitions and notation

$L^p\left(G^2\right), 0 < p \le \infty$ with norms or quasi-norms $\|\cdot\|_p$ denotes the Lebesque spaces, as usual. If $1 \le p \le \infty$, for two-dimensional Walsh-Paley system the number

$$\hat{f}(i,j) := \int_{G^2} f(x,y)w_i(x)w_j(y)d\mu(x,y)$$

is called (i, j)th Walsh-Fourier coefficient of function f. For more details see [33, 34]. For two-dimensional Walsh-Fourier series the rectangular partial sums are defined by

$$S_{N,M}(f,x,y) := \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\hat{f}(i,j)w_i(x)w_j(y).$$

Let $0 < \alpha \le 1$ and let

$$A_j^\alpha := \binom{j+\alpha}{j} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+j)}{j!}, \quad (j \in \mathbb{N}; \alpha \ne -1, -2, \dots).$$

It is known that

$$A_j^\alpha \sim j^\alpha, \;\; A_n^\alpha - A_{n-1}^\alpha \sim A_n^{\alpha-1}, \;\; \sum_{k=0}^{n} A_k^{\alpha-1} = A_n^\alpha.$$

(see Zygmund [38], page 42.).

The Marcinkiewicz type (C, α) means of the two-dimensional Walsh-Fourier series are defined as the (C, α) means of the quadratic partial sums. That is,

$$\sigma_n^\alpha(f,x,y) := \frac{1}{A_{n-1}^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_{k,k}(f,x,y).$$

The kernel function is given by

$$K_n^\alpha(x,y) := \frac{1}{A_{n-1}^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k(x)D_k(y).$$

For trigonometric system the behaviour of Marcinkiewicz means ($\alpha = 1$) was investigated by Marcinkiewicz [16] and the properties of Marcinkiewicz type (C,α) means was discussed by Zhizhiashvili [37].

It is well-know that the L^1 norm of the kernels K_n^α $(0 < \alpha \le 1)$ are uniformly bounded [8]. That is, there exists a positive constant $c.$ such that

$$\left\|K_n^\alpha\right\|_1 \le c \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

3. Auxiliary results

Our proofs are based on two lemmas of Goginava [11, page 20, 22] and some results of Weisz [34].

Lemma 3.1 (Goginava [11]). *Let* $(x,y) \in I_N \times (I_b \backslash I_{b+1})$, *where* $0 \le b < N$, $n > 2^N$. *Then*

$$\int_{I_N\times I_N} \left|K_n^\alpha(x+u,y+v)\right| d\mu(u,v) \le \frac{c(\alpha)}{2^{N-\alpha b} n^\alpha}\sum_{j=b}^{N} D_{2^j}(y+e_b)$$

holds.

In order to apply Lemma 3.1, we decompose the set $\overline{I_N}$ in the following way.

$$\overline{I_N} = \bigcup_{b=0}^{N-1} (I_b\backslash I_{b+1}) \quad \text{and} \quad I_b\backslash I_{b+1} = \bigcup_{s=b+1}^{N} I_b^s, \tag{5}$$

where $I_b^s := I_{s+1}(0,\dots,y_b = 1,0,\dots,y_s = 1)$ for $s < N$ and $I_b^N := I_N(0.\dots,y_b = 1,0,\dots,0)$. We use it later.

Lemma 3.2 (Goginava [11]). *Let* $(x,y) \in (I_a\backslash I_{a+1}) \times (I_b\backslash I_{b+1})$, $0 \le a \le b < N$ *and* $n \ge 2^N$. *Then*

$$\int_{I_N\times I_N} |K_n^\alpha(x+u,y+v)| d\mu(u,v) \le \frac{c(\alpha)}{2^{2N} n^\alpha}\Bigg(2^{b+a\alpha}\sum_{j=a}^{b} D_{2^j}(x+e_a)$$
$$+2^{a-b}\sum_{j=b+1}^{N} 2^{j(\alpha-1)} D_{2^j}(y+e_b+x_{b+1,j-1}) \sum_{m=a+1}^{b+1} 2^m D_{2^{b+1}}(x+e_a+e_m)\Bigg),$$

where $x_{i,j} := \sum_{s=i}^{j} x_s e_s$, $x_{i,i-1} := 0$.

For the simplicity we introduce the notation

$$\mathcal{K}_{n,1}^{\alpha,a,b}(x,y) := \frac{c(\alpha)2^{b+a\alpha}}{2^{2N} n^\alpha}\sum_{j=a}^{b} D_{2^j}(x+e_a)$$

and

$$\mathcal{K}_{n,2}^{\alpha,a,b}(x,y) := \frac{c(\alpha)2^{a-b}}{2^{2N} n^\alpha}\sum_{j=b+1}^{N} 2^{j(\alpha-1)} D_{2^j}(y+e_b+x_{b+1,j-1}) \sum_{m=a+1}^{b+1} 2^m D_{2^{b+1}}(x+e_a+e_m).$$

The σ-algebra generated by the dyadic 2-dimensional cubes I_k^2 is denoted by $\mathcal{F}_k$ $(k \in \mathbb{N})$. $f = (f_n : n \in \mathbb{N})$ denotes a one-parameter martingale with respect to the sequence of σ-algebras $(\mathcal{F}_n, n \in \mathbb{N})$. The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} \left|f_n\right|.$$

For $0 < p \leq \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which

$$\left\|f\right\|_{H_p} := \left\|f^*\right\|_p < \infty.$$

For $f \in L^1(G^2)$ the sequence $(S_{2^n,2^n}(f) : n \in \mathbb{N})$ is a martingale. The maximal function can be given in the form

$$f^*(x,y) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x)\times I_n(y))} \left|\int_{I_n(x)\times I_n(y)} f(u,v)d\mu(u,v)\right|.$$

The concept of Walsh-Fourier coefficient of a function f can be extended to martingales in the usual way (see Weisz [33, 34]). The Walsh-Fourier coefficients of $f \in L^1(G^2)$ are the same as the ones of the martingale $(S_{2^n,2^n}(f) : n \in \mathbb{N})$ obtained from f. Consequently, partial sums, Marcinkiewicz means and (C,α) means of quadratic partial sums are defined for martingales, as well.

A useful property of the Hardy spaces $H_p(G^2)$ is the atomic structure. A bounded measurable function a is a p-atom, if there exists a dyadic two-dimensional cube I^2, such that

a) $\int_{I^2} a d\mu = 0$,

b) $\|a\|_\infty \leq \mu(I^2)^{-1/p}$,

c) $\operatorname{supp} a \subset I^2$.

The operator T is said to be p-quasilocal if there exists a positive constant c_p, such that

$$\int_{\overline{I_N\times I_N}} |T(a)|^p d\mu \leq c_p$$

holds for all arbitrary p-atom a with support $I_N \times I_N$.

Lemma 3.3 (Weisz [33]). *Let $0 < p \leq 1$. Suppose that the operator T is σ-sublinear and p-quasilocal. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \leq c_p\|f\|_{H_p} \quad \textit{for all } f \in H_p.$$

For the martingale

$$f = \sum_{n=0}^{\infty} (f_n - f_{n-1})$$

the conjugate transforms are defined as

$$\widetilde{f^{(t)}} = \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in G$ is fixed. We note that $\widetilde{f^{(0)}} = f$. It is well-known (see [33]) that

$$\begin{aligned} \left\|\widetilde{f^{(t)}}\right\|_{H_p} &= \left\|f\right\|_{H_p}, \quad \left\|f\right\|_{H_p}^p \sim \int_G \left\|\widetilde{f^{(t)}}\right\|_p^p d\mu(t), \qquad (6)\\ (\widetilde{\sigma_m^\alpha(f)})^{(t)} &= \sigma_m^\alpha((\widetilde{f})^{(t)}). \end{aligned}$$

4. The properties of the weighted maximal function $\sigma_p^{\alpha,*}(f)$

Theorem 4.1. *Let* $0 < \alpha < 1$ *and* $0 < p < 2/(2+\alpha)$.

a) Then the maximal operator

$$\sigma_p^{\alpha,*}(f) := \sup_{n\in\mathbb{P}} \frac{\left|\sigma_n^\alpha(f)\right|}{n^{2/p-(2+\alpha)}}$$

is bounded from the dyadic Hardy space $H_p(G^2)$ *to the space* $L^p(G^2)$.

b) Let $\varphi : \mathbb{P} \to [1,\infty)$ *be a non-decreasing function satisfying the condition*

$$\limsup_{n\to\infty} \frac{n^{2/p-(2+\alpha)}}{\varphi(n)} = \infty. \tag{7}$$

Then the weighted maximal operator

$$\sigma_\varphi^{\alpha,*}(f) := \sup_{n\in\mathbb{P}} \frac{\left|\sigma_n^\alpha(f)\right|}{\varphi(n)}$$

is not bounded from the Hardy space $H_p(G^2)$ *to the space weak-*$L^p(G^2)$.

Proof. First, we prove part a).

Inequality (4) yields the boundedness of the operator $\sigma_p^{\alpha,*}$ from the space L^∞ to the space L^∞. Applying Lemma 3.3, we have to show that the maximal operator $\sigma_p^{\alpha,*}$ is p-quasilocal. That is, there exists a constant $c(\alpha) > 0$ such that

$$\int_{\overline{I^2}} \left|\sigma_p^{\alpha,*}(a)\right|^p d\mu \le c(\alpha) < \infty$$

for every p-atom a, where the dyadic cube I^2 is the support of the p-atom a.

Let a be an arbitrary p-atom with support I^2 and $\mu(I^2) = 2^{-2N}$. Without loss of generality, we may assume that $I^2 := I_N \times I_N$. It is easily seen that $\sigma_n^\alpha(a) = 0$ if $n \le 2^N$. Therefore, we set $n > 2^N$. We know that $\|a\|_\infty \le 2^{2N/p}$. Thus,

$$\begin{aligned} \left|\sigma_n^\alpha(a;x,y)\right| &\le \int_{I_N\times I_N} |a(u,v)| \left|K_n^\alpha(x+u,y+v)\right| d\mu(u,v) \\ &\le c(\alpha)2^{2N/p} \int_{I_N\times I_N} \left|K_n^\alpha(x+u,y+v)\right| d\mu(u,v) \end{aligned}$$

and

$$\left|\sigma_p^{\alpha,*}(a)\right| \le c(\alpha)2^{2N/p} \sup_{n>2^N} \int_{I_N\times I_N} \frac{\left|K_n^\alpha(x+u,y+v)\right|}{n^{2/p-(2+\alpha)}} d\mu(u,v). \tag{8}$$

We decompose the set $\overline{I_N \times I_N}$ as

$$\overline{I_N \times I_N} = \left(I_N \times \overline{I_N}\right) \cup \left(\overline{I_N} \times I_N\right) \cup \left(\overline{I_N} \times \overline{I_N}\right).$$

This yields that

$$\begin{aligned} \int_{\overline{I_N\times I_N}} \left|\sigma_p^{\alpha,*}(a)\right|^p d\mu &= \int_{I_N\times\overline{I_N}} \left|\sigma_p^{\alpha,*}(a)\right|^p d\mu + \int_{\overline{I_N}\times I_N} \left|\sigma_p^{\alpha,*}(a)\right|^p d\mu \\ &+ \int_{\overline{I_N}\times\overline{I_N}} \left|\sigma_p^{\alpha,*}(a)\right|^p d\mu =: L_1 + L_2 + L_3. \end{aligned}$$

First, we discuss the expression L_1 (the expression L_2 is discussed analogously). Lemma 3.1 and decomposition (5) imply that

$$\begin{aligned} L_1 &\leq \sum_{b=0}^{N-1}\sum_{s=b+1}^{N}\int_{I_N\times I_b^s}\left(\frac{c(\alpha)2^{2N/p}}{2^{N(2/p-2-\alpha)}2^{N+\alpha(N-b)}}\sum_{j=b}^{s}D_{2^j}(y+e_b)\right)^p d\mu(x,y) \\ &\leq c(\alpha)\sum_{b=0}^{N-1}\sum_{s=b+1}^{N}\frac{2^{2N}2^{sp}2^{-N-s}}{2^{N(2-2p-p\alpha)+Np+p\alpha(N-b)}} \\ &= c(\alpha)\sum_{b=0}^{N-1}\sum_{s=b+1}^{N}\frac{2^{p\alpha b}2^{s(p-1)}}{2^{N(1-p)}} \\ &\leq c(\alpha)\sum_{b=0}^{N-1}\frac{2^{b(p(\alpha+1)-1)}}{2^{N(1-p)}}. \end{aligned}$$

There are three cases. $0 < p < 1/(1+\alpha)$, $p = 1/(1+\alpha)$ and $1/(1+\alpha) < p < 2/(2+\alpha)$.

Let us set $0 < p < 1/(1+\alpha)$. Then

$$L_1 \leq \frac{c(\alpha)}{2^{N(1-p)}} \leq c(\alpha).$$

Now, we set $p = 1/(1+\alpha)$. We get that

$$L_1 \leq \frac{c(\alpha)N}{2^{N(1-p)}} \leq c(\alpha).$$

Let $1/(1+\alpha) < p < 2/(2+\alpha)$. In this case, we immediately write that

$$L_1 \leq c(\alpha)\frac{2^{N(p\alpha+p-1)}}{2^{N(1-p)}} \leq c(\alpha)\frac{2^{N(2-2p+p-1)}}{2^{N(1-p)}} = c(\alpha)$$

Now, we discuss the expression L_3. We introduce the notation $J_a := I_a \backslash I_{a+1}$. We write that

$$\begin{aligned} L_3 &= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1}\int_{J_a\times J_b}\left|\sigma_p^{\alpha,*}(a)\right|^p d\mu \\ &= \sum_{a=0}^{N-1}\sum_{b=0}^{a-1}\int_{J_a\times J_b}\left|\sigma_p^{\alpha,*}(a)\right|^p d\mu + \sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{J_a\times J_b}\left|\sigma_p^{\alpha,*}(a)\right|^p d\mu \\ &=: L_{3,1} + L_{3,2}. \end{aligned}$$

We discuss $L_{3,2}$ (by symmetry the discussion of $L_{3,1}$ is analogous). Inequality (8) and Lemma 3.2 yield that

$$\begin{aligned} L_{3,2} &\leq c(\alpha)2^{2N}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{J_a\times J_b}\left(\sup_{n>2^N}\frac{\mathcal{K}_{n,1}^{\alpha,a,b}}{n^{2/p-(2+\alpha)}}\right)^p d\mu \\ &\quad + c(\alpha)2^{2N}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{J_a\times J_b}\left(\sup_{n>2^N}\frac{\mathcal{K}_{n,2}^{\alpha,a,b}}{n^{2/p-(2+\alpha)}}\right)^p d\mu \\ &=: L_{3,2}^1 + L_{3,2}^2. \end{aligned}$$

Decomposition (5), Lemma 3.2 and $p < 2/(2+\alpha)$ give that

$$\begin{aligned}
L_{3,2}^1 &\leq c(\alpha)2^{2N}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{s=a+1}^{N}\int_{I_a^s\times J_b}\left(\frac{2^{b+a\alpha}}{2^{2N+\alpha N}2^{N(2/p-(2+\alpha))}}\sum_{j=a}^{s}D_{2^j}(x+e_a)\right)^p d\mu(x,y)\\
&\leq c(\alpha)2^{2N}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{s=a+1}^{N}\frac{2^{(b+a\alpha+s)p-s-b}}{2^{(2N+\alpha N)p}2^{N(2-p(2+\alpha))}}\\
&= c(\alpha)\sum_{a=0}^{N-1}2^{a\alpha p}\sum_{b=a}^{N-1}2^{b(p-1)}\sum_{s=a+1}^{N}2^{s(p-1)}\\
&\leq c(\alpha)\sum_{a=0}^{N-1}2^{a\alpha p+2a(p-1)} = c(\alpha)\sum_{a=0}^{N-1}2^{a(p(\alpha+2)-2)} = c(\alpha).
\end{aligned}$$

At last, we discuss $L_{3,2}^2$. $\mathcal{K}_{n,2}^{\alpha,a,b}(x,y)\neq 0$ implies that

$$x\in I_N(0,\ldots,0,x_a=1,0,\ldots,0,x_r=1,0,\ldots,0,x_{b+1},\ldots,x_{N-1}) =: I_N^{a,r}(x)$$

and

$$y\in I_N(0,\ldots,y_b=1,x_{b+1},\ldots,x_{q-1},1-x_q,y_{q+1},\ldots,y_{N-1}) =: I_{q+1}^b(\tilde{x}_{b+1,q})$$

for some q and r, for which $a\leq r\leq b<q<N$ (see [11]). Consequently, we have

$$\mathcal{K}_{n,2}^{\alpha,a,b}(x,y)\leq\frac{c(\alpha)}{2^{2N}n^{\alpha}}2^{a+r+q\alpha}. \tag{9}$$

$$\begin{aligned}
L_{3,2}^2 &\leq c(\alpha)2^{2N}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}\sum_{\substack{x_i=0,\\ i\in\{b+1,\ldots,N-1\}}}^{1}\sum_{q=b+1}^{N-1}\int_{I_N^{a,r}(x)\times I_{q+1}^b(\tilde{x}_{b+1,q})}\left(\frac{2^{a+r+q\alpha}}{2^{N(2+\alpha)}2^{N(2/p-2-\alpha)}}\right)^p d\mu\\
&= c(\alpha)\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}\sum_{\substack{x_i=0,\\ i\in\{b+1,\ldots,N-1\}}}^{1}\sum_{q=b+1}^{N-1}\int_{I_N^{a,r}(x)\times I_{q+1}^b(\tilde{x}_{b+1,q})}2^{p(a+r+q\alpha)}d\mu\\
&\leq c(\alpha)\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}\sum_{\substack{x_i=0,\\ i\in\{b+1,\ldots,N-1\}}}^{1}\sum_{q=b+1}^{N-1}2^{p(a+r+q\alpha)}2^{-N-q}\\
&\leq c(\alpha)\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}2^{p(a+r+b\alpha)}2^{-N-b}2^{N-b}\\
&\leq c(\alpha)\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}2^{p(a+b)}2^{b(p\alpha-2)}\\
&\leq c(\alpha)\sum_{a=0}^{N-1}2^{a(2p+p\alpha-2)}\leq c(\alpha).
\end{aligned}$$

Let us discuss part b) of Theorem 4.1. Let $B\in\mathbb{P}$ and

$$f_B(x_1,x_2) := (D_{2^{2B+1}}(x_1)-D_{2^{2B}}(x_1))(D_{2^{2B+1}}(x_2)-D_{2^{2B}}(x_2)).$$

In this case, we have

$$\hat{f_B}(i,j) = \begin{cases} 1 & \text{if } i,\ j \in \{2^{2B}+1,\dots,2^{2B+1}-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$S_{i,j}(f_B;x_1,x_2) = \begin{cases} (D_i(x_1) - D_{2^{2B}}(x_1)) & \text{if } i,\ j \in \{2^{2B}+1,\dots,2^{2B+1}\}, \\ \quad \cdot (D_j(x_2) - D_{2^{2B}}(x_2)) & \\ f_B(x_1,x_2) & \text{if } i,\ j \ge 2^{2B+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Using

$$f_B^*(x_1,x_2) = \sup_{n\in\mathbb{N}} \left| S_{M_n,M_n}(f_B;x_1,x_2) \right| = \left| f_B(x_1,x_2) \right|,$$

we get that

$$\begin{aligned} \|f_B\|_{H_p} &= \|f_B^*\|_p = \|D_{2^{2B+1}} - D_{2^{2B}}\|_p^2 \\ &= \left(\left(\int_{I_{2B}\setminus I_{2B+1}} 2^{2Bp} + \int_{I_{2B+1}} (2^{2B+1} - 2^{2B})^p \right)^{1/p} \right)^2 \\ &= \left(\left(2 \cdot \frac{2^{2Bp}}{2^{2B+1}} \right)^{1/p} \right)^2 = 2^{2B(2-2/p)}. \end{aligned}$$

By equality (10) we can write

$$\begin{aligned} |\sigma^{\alpha}_{2^{2B}+1}(f_B;x_1,x_2)| &= \frac{1}{A^{\alpha}_{2^{2B}}} \left| \sum_{j=1}^{2^{2B}+1} A^{\alpha-1}_{2^{2B}+1-j} S_{j,j}(f_B;x_1,x_2) \right| \\ &= \frac{1}{A^{\alpha}_{2^{2B}}} \left| A_0^{\alpha-1} (D_{2^{2B}+1}(x_1) - D_{2^{2B}}(x_1))(D_{2^{2B}+1}(x_2) - D_{2^{2B}}(x_2)) \right| \\ &= \frac{1}{A^{\alpha}_{2^{2B}}} |\omega_{2^{2B}}(x_1)\omega_{2^{2B}}(x_2)| \ge \frac{c}{2^{2B\alpha}} \quad \text{for all } (x_1,x_2) \in G^2. \end{aligned}$$

Using this fact, we get

$$\begin{aligned} \frac{\frac{c}{\varphi(2^{2B})2^{2B\alpha}} \mu \left\{ (x_1,x_2) : \left| \sigma^{\alpha,*}_{\varphi}(f_B) \right| \ge \frac{c}{\varphi(2^{2B})2^{2B\alpha}} \right\}^{1/p}}{\|f_B\|_{H_p}} &\ge \frac{c}{\varphi(2^{2B})2^{2B\alpha}2^{2B(2-2/p)}} \\ &= c\frac{(2^{2B})^{2/p-2-\alpha}}{\varphi(2^{2B})}. \end{aligned}$$

At last, under condition (7), there exists a sequence of positive integers $\{n_k, k \in \mathbb{N}\}$, such that

$$\lim_{k\to\infty} \frac{(2^{2n_k})^{2/p-2-\alpha}}{\varphi(2^{2n_k})} = \infty.$$

This completes the proof of Theorem 4.1. □

5. Strong summation theorem

Now, we prove a strong summation theorem for the Marcinkiewicz type (C,α) means of Walsh-Fourier series in the Hardy space H_p $(0<\alpha<1, 0<p<2/(2+\alpha))$.

Theorem 5.1. *Let $0<\alpha<1$ and $0<p<2/(2+\alpha)$. There exists a positive constant $c(\alpha)$ depending only on α and p, such that*

$$\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}(f)\right\|_{H_p}^{p}}{m^{3-(2+\alpha)p}}\leq c(\alpha,p)\left\|f\right\|_{H_p}^{p}$$

holds for all $f\in H_p(G^2)$.

Proof. In the sequel, we show that there exists a positive constant $c(\alpha,p)$ which depends only on α and p such that the following inequality holds

$$\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}(f)\right\|_{p}^{p}}{m^{3-(2+\alpha)p}}\leq c(\alpha,p)\left\|f\right\|_{H_p}^{p}\quad (f\in H_p(G^2)).\tag{11}$$

Inequality (4) gives that σ_n^{α} is bounded from the space L_∞ to the space L_∞. By Lemma 3.3 it is enough to prove that inequality (11) holds for every arbitrary p-atom a. Taking into account that $\|a\|_{H_p}\leq 1$ we show that there exists a positive constant $c(\alpha)$ such that

$$\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}(a)\right\|_{p}^{p}}{m^{3-(2+\alpha)p}}<c(\alpha,p)\tag{12}$$

holds.

Let a be an arbitrary p-atom with support I^2 and $\mu(I^2)=2^{-2N}$. Without loss of generality, we may assume that $I^2:=I_N\times I_N$. We know that $\|a\|_\infty\leq 2^{2N/p}$. By a simple consideration $\sigma_n^{\alpha}(a)=0$ if $n\leq 2^N$. Therefore, we set $n>2^N$.

To prove inequality (12), we apply the next decomposition.

$$\begin{aligned}
\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}(a)\right\|_{p}^{p}}{m^{3-(2+\alpha)p}} &= \sum_{m=2^N}^{n}\frac{\left\|\sigma_m^{\alpha}(a)\right\|_{p}^{p}}{m^{3-(2+\alpha)p}}\\
&\leq \sum_{m=2^N}^{n}\int_{I_N\times I_N}\frac{\left|\sigma_m^{\alpha}(a)\right|^{p}}{m^{3-(2+\alpha)p}}d\mu+\sum_{m=2^N}^{n}\int_{I_N\times\overline{I_N}}\frac{\left|\sigma_m^{\alpha}(a)\right|^{p}}{m^{3-(2+\alpha)p}}d\mu\\
&+\sum_{m=2^N}^{n}\int_{\overline{I_N}\times I_N}\frac{\left|\sigma_m^{\alpha}(a)\right|^{p}}{m^{3-(2+\alpha)p}}d\mu+\sum_{m=2^N}^{n}\int_{\overline{I_N}\times\overline{I_N}}\frac{\left|\sigma_m^{\alpha}(a)\right|^{p}}{m^{3-(2+\alpha)p}}d\mu\\
&=: I_1+I_2+I_3+I_4.
\end{aligned}$$

First, we apply inequality (4) and we write that

$$\begin{aligned}
I_1 &\leq \sum_{m=2^N}^{n}\int_{I_N\times I_N}\frac{\left|\sigma_m^{\alpha}(a)\right|^{p}}{m^{3-(2+\alpha)p}}d\mu\\
&\leq c(\alpha,p)\sum_{m=2^N}^{n}\frac{1}{m^{3-(2+\alpha)p}}\left\|a\right\|_{\infty}^{p}2^{-2N}\leq c(\alpha,p).
\end{aligned}$$

Second, we discuss expression I_2. Decomposition (5) and Lemma 3.1 yield that

$$\begin{aligned} I_2 &\leq c(\alpha,p)\sum_{m=2^N}^{n}\sum_{b=0}^{N-1}\sum_{s=b+1}^{N-1}\frac{1}{m^{3-(2+\alpha)p}}\int_{I_N\times I_b^s}\left(\frac{\|a\|_\infty}{2^{N+\alpha(N-b)}}2^s\right)^p d\mu \\ &\leq c(\alpha,p)\sum_{m=2^N}^{n}\sum_{b=0}^{N-1}\sum_{s=b+1}^{N-1}\frac{1}{m^{3-(2+\alpha)p}}\frac{2^{2N}}{2^{Np+\alpha(N-b)p}}2^{s(p-1)}2^{-N} \\ &\leq c(\alpha,p)2^N\sum_{m=2^N}^{n}\frac{1}{m^{3-(2+\alpha)p}2^{Np+\alpha Np}}\sum_{b=0}^{N-1}2^{\alpha bp}2^{b(p-1)} \\ &= \frac{c(\alpha,p)2^N}{2^{Np+\alpha Np}}\sum_{m=2^N}^{n}\frac{1}{m^{3-(2+\alpha)p}}\sum_{b=0}^{N-1}2^{b(\alpha p+p-1)}. \end{aligned}$$

There are 3 cases. $0 < p < 1/(1+\alpha)$, $p = 1/(1+\alpha)$ and $1/(1+\alpha) < p < 2/(2+\alpha)$.

Let us set $0 < p < 1/(1+\alpha)$. Then

$$I_2 \leq \frac{c(\alpha,p)2^N}{2^{Np+\alpha Np}}\frac{1}{2^{(2-(2+\alpha)p)N}} \leq \frac{c(\alpha,p)2^{Np}}{2^N} \leq c(\alpha,p).$$

Now, we set $p = 1/(1+\alpha)$.

$$I_2 \leq \frac{c(\alpha,p)2^N}{2^{Np(1+\alpha)}}\frac{N}{2^{(2-(2+\alpha)p)N}} \leq c(\alpha,p)\frac{N2^{(2+\alpha)pN}}{2^{2N}} = c(\alpha,p)\frac{N}{2^{N(1-p)}} \leq c(\alpha,p).$$

Let us set $1/(1+\alpha) < p < 2/(2+\alpha)$.

$$I_2 \leq \frac{c(\alpha,p)2^N}{2^{Np+\alpha Np}}\frac{2^{N(\alpha p+p-1)}}{2^{(2-(2+\alpha)p)N}} \leq \frac{c(\alpha,p)}{2^{(2-(2+\alpha)p)N}} \leq c(\alpha,p).$$

The estimate of expression I_3 is similar. That is, we have that

$$I_3 \leq c(\alpha,p).$$

At last, we discuss the expression I_4. By the decomposition (5) we write

$$\begin{aligned} I_4 &\leq \sum_{m=2^N}^{n}\sum_{a=0}^{N-1}\sum_{b=0}^{a-1}\int_{(I_a\setminus I_{a+1})\times(I_b\setminus I_{b+1})}\frac{\left|\sigma_m^\alpha(a)\right|^p}{m^{3-(2+\alpha)p}}d\mu \\ &+ \sum_{m=2^N}^{n}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{(I_a\setminus I_{a+1})\times(I_b\setminus I_{b+1})}\frac{\left|\sigma_m^\alpha(a)\right|^p}{m^{3-(2+\alpha)p}}d\mu =: I_{4,1}+I_{4,2}. \end{aligned}$$

We discuss $I_{4,2}$. By decomposition (5) and Lemma 3.2 we get that

$$\begin{aligned} I_{4,2} &\leq c(\alpha,p)\sum_{m=2^N}^{n}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{(I_a\setminus I_{a+1})\times(I_b\setminus I_{b+1})}\frac{\left(\|a\|_\infty\mathcal{K}_{m,1}^{\alpha,a,b}\right)^p}{m^{3-(2+\alpha)p}}d\mu \\ &+ c(\alpha,p)\sum_{m=2^N}^{n}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\int_{(I_a\setminus I_{a+1})\times(I_b\setminus I_{b+1})}\frac{\left(\|a\|_\infty\mathcal{K}_{m,2}^{\alpha,a,b}\right)^p}{m^{3-(2+\alpha)p}}d\mu \qquad (13) \\ &=: I_{4,2}^1+I_{4,2}^2 \end{aligned}$$

and

$$\begin{aligned}
I_{4,2}^1 &\leq c(\alpha,p)2^{2N}\sum_{m=2^N}^{n}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{s=a+1}^{N}\int_{I_a^s\times(I_b\setminus I_{b+1})}\frac{\left(\frac{2^{b+a\alpha}}{2^{2N}m^{\alpha}}\sum_{j=a}^{s}D_{2^j}(x+e_a)\right)^p}{m^{3-(2+\alpha)p}}d\mu(x,y)\\
&\leq c(\alpha,p)2^{2N}\sum_{m=2^N}^{n}\frac{1}{m^{3-(2+\alpha)p}m^{\alpha p}}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{s=a+1}^{N}\frac{2^{(b+a\alpha+s)p}}{2^{2Np}}2^{-s-b}\\
&\leq c(\alpha,p)2^{2N}\sum_{m=2^N}^{n}\frac{1}{m^{3-2p}}\sum_{a=0}^{N-1}\frac{2^{a((2+\alpha)p-2)}}{2^{2Np}}\\
&\leq \frac{c(\alpha,p)2^{2N}}{2^{2Np}}\frac{1}{2^{N(2-2p)}}=c(\alpha,p).
\end{aligned}$$

Now, we discuss the expression $I_{4,2}^2$. Inequalities (13) and (9) imply

$$\begin{aligned}
I_{4,2}^2 &\leq c(\alpha,p)2^{2N}\sum_{m=2^N}^{n}\frac{1}{m^{3-(2+\alpha)p}}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}\sum_{\substack{x_i=0,\\ i\in\{b+1,\ldots,N-1\}}}^{1}\sum_{q=b+1}^{N-1}\int_{I_N^{a,r}(x)\times I_{q+1}^b(\tilde{x}_{b+1,q})}\left(\frac{2^{a+r+q\alpha}}{2^{2N}m^{\alpha}}\right)^p d\mu\\
&\leq \frac{c(\alpha,p)2^{2N}}{2^{2Np}}\sum_{m=2^N}^{n}\frac{1}{m^{3-2p}}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}\sum_{r=a}^{b}\sum_{\substack{x_i=0,\\ i\in\{b+1,\ldots,N-1\}}}^{1}\sum_{q=b+1}^{N-1}2^{(a+r+q\alpha)p}2^{-N-q}\\
&\leq \frac{c(\alpha,p)2^{2N}}{2^{2Np}}\sum_{m=2^N}^{n}\frac{1}{m^{3-2p}}\sum_{a=0}^{N-1}\sum_{b=a}^{N-1}2^{ap}2^{bp}2^{b(\alpha p-1)}2^{N-b}2^{-N}\\
&\leq \frac{c(\alpha,p)2^{2N}}{2^{2Np}}\frac{1}{2^{N(2-2p)}}\sum_{a=0}^{N-1}2^{a((2+\alpha)p-2)}\leq c(\alpha,p).
\end{aligned}$$

We estimate the expression $I_{4,1}$ analogically and write that

$$I_{4,1}\leq c(\alpha,p).$$

Inequality (11) and the properties (6) of the conjugate transform of a martingale yield that

$$\begin{aligned}
\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}(f)\right\|_{H_p}^p}{m^{3-(2+\alpha)p}} &\sim \sum_{m=1}^{n}\int_G\frac{\left\|\left(\widetilde{\sigma_m^{\alpha}(f)}\right)^{(t)}\right\|_p^p}{m^{3-(2+\alpha)p}}d\mu(t)\\
&= \int_G\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}((\widetilde{f})^{(t)})\right\|_p^p}{m^{3-(2+\alpha)p}}d\mu(t)\\
&\leq c(\alpha,p)\int_G\left\|\left((\widetilde{f})^{(t)}\right)\right\|_{H_p}^p d\mu(t)\sim\left\|f\right\|_{H_p}^p.
\end{aligned}$$

For more details see [20]. This completes the proof of Theorem 5.1. □

From the proof of Theorem 5.1 (mainly taking into account the discussion of I_2, while $p=1/(\alpha+1)$), we conclude that the sequence $(m^{3-(2+\alpha)p}: m\in\mathbb{P})$ should be sharp. It is formalized in the next Conjecture. Unfortunately, we were not able to prove it.

Conjecture 5.2. *Let* $0 < \alpha < 1$ *and* $0 < p < 2/(2+\alpha)$. *Let* $\varphi : \mathbb{P} \to [1,\infty)$ *be a non-decreasing function satisfying the condition*

$$\limsup_{k\to\infty} \frac{2^{k(3-(2+\alpha)p)}}{\varphi(2^k)} = \infty. \tag{14}$$

Then there exists a martingale $f \in H_p(G^2)$ *such that*

$$\sum_{m=1}^{\infty} \frac{\left\|\sigma_m^{\alpha}(f)\right\|_{weak-L_p}^{p}}{\varphi(m)} = \infty.$$

For Marcinkiewicz means (setting $\alpha = 1$) the analogue of Conjecture 5.2 is proved in [20].

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