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Large diffusivity and rate of convergence of attractors in parabolic systems

Leonardo Pires[a], Rodrigo A. Samprogna[b]

[a] *State University of Ponta Grossa, 84030-900 Ponta Grossa PR, Brazil*
[b] *Federal University of Alfenas, 37715-400 Poços de Caldas MG, Brazil*

Abstract. In this paper, we are concerned with the rate of convergence of parabolic systems with large diffusion. We exhibit the exact moment that spatial homogenization occurs and estimate the continuity of attractors by a rate of convergence. We present an example where our estimate is optimal.

1. Introduction

Many reaction-diffusion equations originated by models of heat diffusion do not display the creation of stable patterns, that is, stable solutions that are not spatially dependent in an adequate limit process involving parameters. Diffusive processes where solutions have this spatial homogenization have been studied in the works [6] and [7].

Here, we consider a system of a parabolic equation with large diffusion in all domain in which the limiting problem is an ordinary differential equation in $\mathbb{R}^n$. More precisely, we will impose conditions in the limiting ODE system to ensure that the PDE has global attractor converging to the limiting attractor with a precise rate of convergence. Hence in this paper, we generalize some results obtained in the above works presenting an estimate of how fast can be spatial homogeneity and exhibiting the exact moment that this phenomenon occurs.

To state our results let Ω be a bounded open set in $\mathbb{R}^N$, $N \leq 3$, with boundary $\Gamma = \partial\Omega$ smooth, and consider the system of reaction-diffusion equations of the form

$$\begin{cases} u_t^\varepsilon - E\Delta u^\varepsilon + u^\varepsilon = F(u^\varepsilon), & t > 0, \ \ x \in \Omega, \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0, & t > 0, \ \ x \in \Gamma, \end{cases} \tag{1}$$

where $u^\varepsilon = (u_1^\varepsilon, ..., u_n^\varepsilon) \in \mathbb{R}^n$, $n \geq 1$, $E = \operatorname{diag}(\varepsilon_1, ..., \varepsilon_n)$, with $\varepsilon_i \geq m_0 > 0$, $i = 1, ..., n$, $\vec{n}$ is the outward normal vector to the Γ and $\frac{\partial u^\varepsilon}{\partial \vec{n}} = (\left\langle \nabla u_1^\varepsilon, \vec{n} \right\rangle, \ldots, \langle \nabla u_n^\varepsilon, \vec{n}\rangle)$. We assume that the nonlinearity $F : \mathbb{R}^n \to \mathbb{R}^n$ is bounded and continuously differentiable and satisfies other hypotheses stated later.

We will see when $d_\varepsilon := \min\limits_{i=1,...,n} \varepsilon_i \to \infty$ the solutions of (1) converge to a solution of the following ordinary differential equation

$$\dot{u}^\infty(t) + u^\infty(t) = F(u^\infty(t)), \tag{2}$$

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Email addresses: lpires@uepg.br (Leonardo Pires), rodrigo.samprogna@unifal-mg.edu.br (Rodrigo A. Samprogna)

where $u^\infty(t) \in \mathbb{R}^n$ and we have assumed with loss of generality $|\Omega| = 1$, where $|\Omega|$ indicates the Lebesgue measure. In fact, if we integrate (1) on Ω and if we use $d_\varepsilon \to \infty$ means that the diffusion is large in Ω then, the spatial variable tends to disappear in the limit process. Thus, we can assume $u^\varepsilon(t,x) \to u^\infty(t)$ in order to obtain $\dot{u}^\infty(t) + u^\infty(t) = |\Omega|^{-1}F(u^\infty(t)) := \tilde{F}(u^\infty(t))$.

We will see in next section that under standard conditions that we affirm later the equations (1) and (2) are globally well-posed in a Hilbert space $X_\varepsilon^{\frac{1}{2}}$ and $\mathbb{R}^n$ respectively. Moreover, the nonlinear semigroup generated by its solutions have a global attractor $\mathcal{A}_\varepsilon \subset X_\varepsilon^{\frac{1}{2}}$ and $\mathcal{A}_\infty \subset \mathbb{R}^n$. We consider $\mathbb{R}^n$ embedding in $X_\varepsilon^{\frac{1}{2}}$ as the constant functions, the main result of this paper states

$$\mathrm{d}_H(\mathcal{A}_\varepsilon, \mathcal{A}_\infty) \leq \frac{C}{\sqrt{d_\varepsilon}}, \tag{3}$$

for d_ε in a appropriate bounded interval, where d_H denotes the Hausdorff distance between sets in $X_\varepsilon^{\frac{1}{2}}$ and C denotes a constant independent of d_ε.

Surprisingly, the spatial homogenization due to large diffusion makes the dynamics of (1) and (2) equal to a finite parameter. In fact, it was showed in [7], for d_ε sufficiently large we have $\mathcal{A}_\varepsilon = \mathcal{A}_\infty$. This equality means that all equilibrium points and connections between them are preserved for d_ε sufficiently large but finite. Our second result will be to calculate the upper limit value μ to bound d_ε. Therefore, (3) ensure the continuity of the family $\{\mathcal{A}_\varepsilon\}_\varepsilon$ as $\varepsilon \to \mu^-$. The $\omega-$limit set of every solution of (1) lies in a bounded set of $\mathbb{R}^n$ and it must be a union of invariant sets of (2) which belongs to this bounded set. But, such invariant sets must belong to $\mathcal{A}_\infty$ and it is clear that for d_ε sufficiently large this bounded set will be $\mathcal{A}_\mu = \mathcal{A}_\infty$.

But we go further, we will show the existence of an invariant manifold $\mathcal{M}_\varepsilon$ for (1) containing the global attractor $\mathcal{A}_\varepsilon$. Notice that trivially $\mathbb{R}^n$ is a invariant manifold for (2) containing $\mathcal{A}_\infty$. We will show that, in some sense, $\mathcal{M}_\varepsilon$ approaches $\mathbb{R}^n$ with the same rate $1/\sqrt{d_\varepsilon}$.

The values of λ such that

$$\begin{cases} E\Delta u^\varepsilon + u^\varepsilon = \lambda u^\varepsilon, & x \in \Omega, \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} = 0, & x \in \Gamma, \end{cases} \tag{4}$$

has non-zero solutions are called eigenvalues. We will see that they are real numbers and can be ordered in the following way $\{1 < \lambda_2^\varepsilon < \lambda_3^\varepsilon, \dots\}$. Moreover we have $\lambda_j^\varepsilon \to \infty$ as $d_\varepsilon \to \infty$ for $j \geq 2$. Thus we can consider the spectral projection Q_ε whose image can be identified with $\mathbb{R}^n$. We will prove that

$$\|Q_\varepsilon - P\|_{\mathcal{L}(L^2(\Omega,\mathbb{R}^n), X_\varepsilon^{\frac{1}{2}})} \leq \frac{C}{\sqrt{d_\varepsilon}},$$

where C is a constant independent of d_ε and P denotes the average projection on Ω.

An interesting question arises when we ask if the exponent $-1/2$ in the above estimate is optimal. We will exhibit an example where the convergence of resolvent operators in exact $Cd_\varepsilon^{-\frac{1}{2}}$. Our third result is to show that this convergence of resolvent operators will imply the convergence of global attractors and invariant manifolds.

This paper is divided as follows: in Section 2, we present the functional phase space to deal with (1) and (2) and we state conditions to ensure the existence of global attractors. In Section 3, we make precise in what sense the spatial homogenization occurs. In Section 4, we deal with the spectral convergence and we obtain a rate of convergence for the resolvent operators. In Section 5, we prove the main result of this work concerning to rate of convergence of attractors.

2. Functional Setting

The phase space to deal with the system of reaction-diffusion equation as (1) is generally the Sobolev space $H^1(\Omega, \mathbb{R}^n)$, but since we have the diffusion coefficient as the parameter ε_i is natural to consider a metric with some weight. We use the fractional power spaces associated with sectorial operators. They play a key role in the theory of the existence of solutions to nonlinear partial differential equations of parabolic type and in the analysis of the asymptotic behavior of its solutions.

Consider the operator $A_\varepsilon = \mathrm{diag}(A_1, ..., A_n)$, where $A_i : D(A_i) \subset L^2(\Omega) \to L^2(\Omega)$, $i = 1, ..., n$, is given by

$$\begin{cases} D(A_i) = \{\varphi \in H^1(\Omega) : \frac{\partial \varphi}{\partial \vec{n}} = 0, \text{ on } \Gamma\}, \\ A_i\varphi = -\varepsilon_i \Delta\varphi + \varphi. \end{cases} \tag{5}$$

Let A_i^α be the fractional power of operator A_i and denote X_i^α its fractional power space endowed with the graph norm. If $N \leq 3$ and $\frac{3}{4} < \alpha < 1$, according to [8], we have $X_i^\alpha \hookrightarrow H^1(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ with continuous inclusion and

$$D(A_\varepsilon^{\frac{1}{2}}) = X_1^{\frac{1}{2}} \times \cdots \times X_n^{\frac{1}{2}} = H^1(\Omega, \mathbb{R}^n). \tag{6}$$

Thus we take as phase space for (1) the space $X_\varepsilon^{\frac{1}{2}} := H^1(\Omega, \mathbb{R}^n)$ with the inner product given by

$$\langle \varphi, \psi \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_\Omega E\nabla\varphi\nabla\psi \, dx + \int_\Omega \varphi\psi \, ds, \quad \varphi, \psi \in X_\varepsilon^{\frac{1}{2}}, \tag{7}$$

and we rewrite (1) in the abstract form

$$\begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon = f(u^\varepsilon), \ t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}, \end{cases} \tag{8}$$

where $f : X_\varepsilon^{\frac{1}{2}} \to L^2(\Omega, \mathbb{R}^n)$ is given by $f(u)(x) = F(u(x))$, for $u \in X_\varepsilon^{\frac{1}{2}}$.

To obtain the well posedness and existence of the global attractor for equation (8), we need to impose some growth and dissipativeness conditions, these conditions are statement in [1, 2] and [9]. Let $F = (f_1, ..., f_n)$, and recall that $\Omega \subset \mathbb{R}^N$, $N \leq 3$.

(i) **Growth condition**. If $N = 2$, for all $\eta > 0$, there is a constant $C_\eta > 0$ such that

$$|f_i(u) - f_i(v)| \leq C_\eta(e^{\eta|u_i|^2} + e^{\eta|v_i|^2})|u - v|, \quad i = 1, ..., n,$$

and if $N = 3$, there is a constant $\tilde{C} > 0$ such that

$$|f_i(u) - f_i(v)| \leq \tilde{C}|u_i - v_i|(|u_i|^{\rho-1} + |v_i|^{\rho-1} + 1), \quad i = 1, ..., n,$$

where $\rho \leq 1 + \frac{4}{N-2}$, $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$.

(ii) **Dissipativeness condition**

$$\limsup_{|u| \to \infty} \frac{f_i(u)}{u_i} < 0, \ i = 1, ..., n.$$

The theory of well-posedness of abstract parabolic problems that enable us to study (8) is developed in [2] and [9]. Results in local well-posedness in the energy space $X_\varepsilon^{\frac{1}{2}}$ are obtained due to the fact that A_ε generates a strongly continuous semigroup and in addition, f_i is continuously differentiable satisfying the above growth condition (i). To show that all solutions of (8) are globally defined, we need to impose the above dissipativeness condition (ii). Thus, for each initial date u_0^ε in $X_\varepsilon^{\frac{1}{2}}$, the equation (8) has global solution through u_0^ε. This solution is continuously differentiable with respect to the initial data and it is a classical solution for $t > 0$ satisfying the variation of constants formula. Moreover, (8) has global attractor uniformly bounded in ε_i, $i = 1, ..., n$.

Thus, the solution $u^\varepsilon(t,u_0^\varepsilon)$ of (8) through $u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}$ is well defined for positive time $t \geq 0$ and the nonlinear semigroup defined by $T_\varepsilon(t)u_0^\varepsilon = u^\varepsilon(t,u_0^\varepsilon)$ satisfies the variation of constants formula

$$T_\varepsilon(t)u_0^\varepsilon = e^{-A_\varepsilon t}u_0^\varepsilon + \int_0^t e^{-A_\varepsilon(t-s)}f(T_\varepsilon(s)u_0^\varepsilon)\,ds,\ \ t > 0. \tag{9}$$

Moreover, there is the global attractor $\mathcal{A}_\varepsilon \subset X_\varepsilon^{\frac{1}{2}}$ such that

$$\sup_{u\in\mathcal{A}_\varepsilon} \|u\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq K,$$

for some constant K independent of ε (see [2] and [9]). Here $e^{-A_\varepsilon t}$ is the strongly linear semigroup whose infinitesimal generator is $-A_\varepsilon$.

The equation (2) is well-posed in $\mathbb{R}^n$ since F is continuous with Lipschitz continuous first derivative. Moreover if we assume that F satisfy the above dissipativeness conditions (ii) then (2) has solutions defined for all time and a global attractor $\mathcal{A}_\infty \subset \mathbb{R}^n$.

3. Asymptotic Behavior

Once the problem is well posed in the energy space $X_\varepsilon^{\frac{1}{2}}$, we will prove that the ordinary differential equation (2) will describe the asymptotic behavior of (1). For this, we take $\delta > 0$ sufficiently small and define the spectral projection $Q_\varepsilon : L^2(\Omega,\mathbb{R}^n) \to L^2(\Omega,\mathbb{R}^n)$, given by

$$Q_\varepsilon = \frac{1}{2\pi i}\int_{|\xi+1|=\delta} (\xi + A_\varepsilon)^{-1}\,d\xi. \tag{10}$$

Thus, the eigenspace $Q_\varepsilon X_\varepsilon^{\frac{1}{2}}$ is isomorphic to $\mathbb{R}^n$. In fact, the operator A_ε has compact resolvent and $1 \in \sigma(A_\varepsilon)$ is its first eigenvalue, thus Q_ε is well defined projection with finit rank since $Q_\varepsilon X_\varepsilon^{\frac{1}{2}} = \mathrm{span}[\varphi_1^\varepsilon]$, where $\varphi_1^\varepsilon = (\tilde{\varphi}_1^\varepsilon, ..., \tilde{\varphi}_n^\varepsilon)$ is the first eigenfunction of A_ε associated with the eigenvalue $\lambda = 1$.

With the aid of the projection Q_ε, we can decompose the phase space $X_\varepsilon^{\frac{1}{2}}$ in a finite-dimensional subspace and its complement. This decomposition will allow us to decompose the operator A_ε in order to obtain estimates for the linear semigroup $e^{-A_\varepsilon t}$ restricted to these spaces in the decomposition. Let us show how to obtain the explained above.

In what follows we denote $L^2 = L^2(\Omega,\mathbb{R}^n)$.

Lemma 3.1. *Let Q_ε be the spectral projection defined in* (10). *If we denote $Y_\varepsilon = Q_\varepsilon X_\varepsilon^{\frac{1}{2}}$ and $Z_\varepsilon = (I - Q_\varepsilon)X_\varepsilon^{\frac{1}{2}}$ and define the projected operators*

$$A_\varepsilon^+ = A_\varepsilon|_{Y_\varepsilon} \quad \text{and} \quad A_\varepsilon^- = A_\varepsilon|_{Z_\varepsilon},$$

then the following estimates are valid,

(i) $\|e^{-A_\varepsilon^- t}z\|_{X_\varepsilon^{\frac{1}{2}}} \leq Me^{-(d_\varepsilon\lambda_1+1)t}\|z\|_{X_\varepsilon^{\frac{1}{2}}}, \quad t > 0, \quad z \in Z_\varepsilon,$

(ii) $\|e^{-A_\varepsilon^- t}z\|_{X_\varepsilon^{\frac{1}{2}}} \leq Me^{-(d_\varepsilon\lambda_1+1)t}t^{-\frac{1}{2}}\|z\|_{L^2}, \quad t > 0, \quad z \in Z_\varepsilon,$

where $-\lambda_1$ is the first nonzero eigenvalue of the Laplacian with homogeneous Neumann boundary conditions on Ω and M is a constant independent of d_ε.

Proof. The operator A_ε is positive and self-adjoint. If we denote its ordered spectrum $\sigma(A_\varepsilon) = \{1 < \lambda_2^\varepsilon < \dots\}$ and $\{\varphi_1^\varepsilon, \varphi_2^\varepsilon, \dots\}$ the associated eigenfunctions, for $z \in Z_\varepsilon$ we have

$$e^{-A_\varepsilon^- t}z = e^{-A_\varepsilon t}(I - Q_\varepsilon)z = \sum_{i=2}^{\infty} e^{-\lambda_i^\varepsilon t}\left\langle z, \varphi_i^\varepsilon\right\rangle_{L^2}\varphi_i^\varepsilon, \quad t > 0,$$

but $\lambda_2^\varepsilon < \lambda_i^\varepsilon$ implies $e^{-\lambda_i^\varepsilon t} < e^{-\lambda_2^\varepsilon t}$ for $t > 0$. Thus

$$\|e^{-A_\varepsilon^- t}z\|_{X_\varepsilon^{\frac{1}{2}}} \leq \Big(e^{-2\lambda_2^\varepsilon t}\sum_{i=2}^{\infty}\big\langle z,\varphi_i^\varepsilon\big\rangle_{L^2}^2\,\lambda_i^\varepsilon\Big)^{\frac{1}{2}} \leq Me^{-\lambda_2^\varepsilon t}\|z\|_{X_\varepsilon^{\frac{1}{2}}},\quad t>0.$$

The function $f(\eta) = e^{-2\eta t}\eta$ attains its maximum at $\eta = 1/2t$, $t > 0$. Then,

$$\|e^{-A_\varepsilon^- t}z\|_{X_\varepsilon^{\frac{1}{2}}} \leq \begin{cases} e^{-\lambda_2^\varepsilon t}(\lambda_2^\varepsilon)^{\frac{1}{2}}\|z\|_{L^2}, & 1/2t < \lambda_2^\varepsilon,\\ e^{-\lambda_2^\varepsilon t}2^{-\frac{1}{2}}t^{-\frac{1}{2}}\|z\|_{L^2}, & 1/2t > \lambda_2^\varepsilon.\end{cases}$$

The result follows by noticing that $\lambda_2^\varepsilon = d_\varepsilon\lambda_1 + 1$. □

Now, we consider the decomposition $X_\varepsilon^{\frac{1}{2}} = Y_\varepsilon \oplus Z_\varepsilon$ we have $Y_\varepsilon \approx \mathbb{R}^n$ and $Z_\varepsilon = \{\varphi \in X_\varepsilon^{\frac{1}{2}} : \langle\psi,\varphi\rangle_{L^2} = 0,\ \psi \in Y_\varepsilon\}$, with

$$\langle\varphi,\psi\rangle_{L^2} = \int_\Omega \varphi(x)\psi(x)\,dx,\quad \varphi\in Y_\varepsilon,\ \psi\in Z_\varepsilon,$$

where φ is a constant map. Thus, $\psi \in L^\infty(\Omega,\mathbb{R}^n)$ and the above integral is well defined for all $\psi \in X_\varepsilon^{\frac{1}{2}}$. Hence, if $u(t,\cdot) \in X_\varepsilon^{\frac{1}{2}}$ is a solution of (1), it can be written as $u(t,x) = v(t) + w(t,x)$, where $v \in Y_\varepsilon$ and $w \in Z_\varepsilon$ satisfy

$$v(t) = \int_\Omega u(t,x)\,dx \quad\text{and}\quad \int_\Omega w(t,x)\,dx = 0,\ \ t>0.$$

Moreover,

$$\begin{aligned}\dot v(t) &= \int_\Omega u_t(t,x)\,dx = \int_\Omega E\Delta u(t,x) - u(t,x)\,dx + \int_\Omega F(u(t,x))\,dx\\ &= -v(t) + \int_\Omega F(v(t)+w(t,x))\,dx\end{aligned}$$

and

$$\begin{aligned}w_t(t,x) &= u_t(t,x) - \dot v(t)\\ &= E\Delta u(t,x) - u(t,x) + F(u(t,x)) - \int_\Omega F(v(t)+w(t,x))\,dx + v(t)\\ &= E\Delta w(t,x) - w(t,x) + F(v(t)+w(t,x)) - \int_\Omega F(v(t)+w(t,x))\,dx.\end{aligned}$$

Therefore, we can write every solution of (1) as a solution to the problem

$$\begin{cases}\dot v + v = S(v,w),\ t>0,\\ w_t - E\Delta w + w = Q(v,w),\ \ t>0,\ x\in\Omega,\\ E\frac{\partial w}{\partial \vec n} = 0,\ t>0,\ x\in\Gamma,\\ w(0) = w_0 \in Z_\varepsilon,\end{cases} \tag{11}$$

where

$$\begin{cases} S(v,w) = \int_\Omega F(v+w)\,dx, & v\in Y_\varepsilon,\ w\in Z_\varepsilon,\\ Q(v,w) = F(v+w) - \int_\Omega F(v+w)\,dx, & v\in Y_\varepsilon,\ w\in Z_\varepsilon.\end{cases} \tag{12}$$

It is expected that for d_ε sufficiently large the part $w(t,x)$ in (11) will not play an important role in the asymptotic behavior and, in that case, the limiting equation should be

$$\dot u^\infty(t) + u^\infty(t) = F(u^\infty(t)), \tag{13}$$

where we have used the notation $u^\infty = v$ and we have taken $w = 0$ in (12). In fact, the next Theorem inspired by the Theorem 1.1 in [7] shows that $w(t,x)$ and $g(t, v+w) = F(v+w) - S(v,w) = Q(v,w)$ goes to zero exponentially as t goes to infinity in the energy space $X_\varepsilon^{\frac{1}{2}}$ when d_ε is sufficiently large.

Theorem 3.2. *Let Q be as in the definition* (12). *Then there is a positive constant C independent of d_ε such that*

$$\|Q(v(t), w(t))\|_{L^2} \leq Ce^{-(d_\varepsilon\lambda_1+1-\mu)t} \quad \text{and} \quad \|w(t)\|_{Z_\varepsilon} \leq Ce^{-(d_\varepsilon\lambda_1+1-\mu)t},$$

where $\mu = (2M\Gamma(\frac{1}{2}))^2$ and M is given by the Lemma 3.1.

Proof. Note that $S(v,0) = F(v)$, $Q(v,0) = 0$ and S, Q are continuously differentiable with $Q_v(0,0) = 0 = S_v(0,0)$, thus there is $\rho > 0$ such that for $v, \tilde{v} \in Y_\varepsilon$ and $w, \tilde{w} \in Z_\varepsilon$,

$$\|Q(v,w)\|_{L^2} \leq \rho,$$

$$\|Q(v,w) - Q(\tilde{v},\tilde{w})\|_{L^2} \leq \rho(\|v-\tilde{v}\|_{Y_\varepsilon} + \|w-\tilde{w}\|_{Z_\varepsilon}).$$

Thus

$$\|Q(v,w)\|_{L^2} \leq \rho\|w\|_{Z_\varepsilon}, \quad v \in Y_\varepsilon,\ w \in Z_\varepsilon.$$

Hence, we need to estimate $\|w\|_{Z_\varepsilon}$.

We use the variation of constants formula to write

$$w(t) = e^{-A_\varepsilon^- t} w_0 + \int_0^t e^{-A_\varepsilon^-(t-s)} Q(v(s), w(s))\, ds.$$

Using the estimates from the Lemma 3.1, we have

$$e^{(d_\varepsilon\lambda_1+1)t}\|w(t)\|_{Z_\varepsilon} \leq M\|w_0\|_{Z_\varepsilon} + M\int_0^t (t-s)^{-\frac{1}{2}} e^{(d_\varepsilon\lambda_1+1)s}\|w(s)\|_{Z_\varepsilon}\, ds,$$

and by Gronwall's inequality (see [4] pag 168), we obtain for $\mu = (2M\Gamma(\frac{1}{2}))^2$,

$$\|w(t)\|_{Z_\varepsilon} \leq 2M\|w_0\|_{Z_\varepsilon} e^{-(d_\varepsilon\lambda_1+1-\mu)t}.$$

□

Now, we rewrite the ordinary differential equation in (11) as $\dot{v} + v = F(v) + [S(v,w) - F(v)]$. It follows from Theorem 3.2 that for d_ε sufficiently large, the asymptotic behavior of (1) is determined by the ordinary differential equation (13). That is, if $d_\varepsilon\lambda_1 > \mu - 1$ then the solution $u(t, u_0)$ of the problem (1) through $u_0 \in X_\varepsilon^{\frac{1}{2}}$ at $t = 0$ satisfies

$$\|u(t,u_0) - v(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq Ke^{-(d_\varepsilon\lambda_1+1-\mu)t} \overset{t\to\infty}{\longrightarrow} 0, \tag{14}$$

where $v(t)$ is the average of $u(t,u_0)$ in Ω. Since the equation (13) has global attractor $\mathcal{A}_\infty \subset \mathbb{R}^n$ and we are understanding $\mathbb{R}^n$ as the subspace of constant functions in $X_\varepsilon^{\frac{1}{2}}$, we have A_∞ a compact subset in $X_\varepsilon^{\frac{1}{2}}$ invariant under $T_\varepsilon(\cdot)$ and it follows from (14) that $\mathcal{A}_\infty$ attracts under $T_\varepsilon(\cdot)$ bounded set in $X_\varepsilon^{\frac{1}{2}}$, hence $\mathcal{A}_\infty = \mathcal{A}_{\bar{\mu}}$, when $d_\varepsilon\lambda_1 > \mu - 1$, where $\mu = (2M\Gamma(\frac{1}{2}))^2$ and $\bar{\mu} = (\mu-1)\lambda_1^{-1}$.

4. Spectral Convergence

In what follows we prove the convergence of the resolvent operators and we obtain estimates in the convergence of the spectral projections Q_ε. We establish that the rate for these convergences is $d_\varepsilon^{-\frac{1}{2}}$.

We saw that the operators A_ε and A_∞ work in different spaces. The operator A_∞ is the identity in $\mathbb{R}^n$ that can be understood as the space of constant functions in $X_\varepsilon^{\frac{1}{2}}$. Thus, we need to find a way to compare functions between these spaces. The abstract theory that can be used to compare linear problems in different spaces is developed in [3] and named E-convergence. In this context, we consider the inclusion operator $i : \mathbb{R}^n \to X_\varepsilon^{\frac{1}{2}}$ and the projection $P : X_\varepsilon^{\frac{1}{2}} \to \mathbb{R}^n$ given by the average in Ω,

$$Pu = \frac{1}{|\Omega|}\int_\Omega u\, dx, \quad u \in X_\varepsilon^{\frac{1}{2}}.$$

Notice that P can also be considered as an orthogonal projection acting on L^2 onto $\mathbb{R}^n$.

We have seen in the previous section that A_ε is an invertible operator with compact resolvent. The next result shows that the resolvent operator approaches the projection P uniformly in the operator norm.

Lemma 4.1. *For $g \in L^2(\Omega, \mathbb{R}^n)$ such that $\|g\|_{L^2(\Omega,\mathbb{R}^n)} \leq 1$, let u^ε be the weak solution of the elliptic problem $A_\varepsilon u^\varepsilon = g$. Then there is a positive constant C independent of d_ε such that*

$$\|u^\varepsilon - u^\infty\|_{X_\varepsilon^{\frac{1}{2}}} \leq C d_\varepsilon^{-\frac{1}{2}}, \tag{15}$$

where $u^\infty = Pg$.

Proof. We denote $u^\varepsilon = (u_1^\varepsilon, ..., u_n^\varepsilon)$, $u^\infty = (u_1^\infty, ..., u_n^\infty)$ and $g = (g_1, ..., g_n)$, for $i = 1, \dots, n$. Since (6) holds we can only consider one component u_i^ε. Then

$$\int_\Omega \varepsilon_i \nabla u_i^\varepsilon \nabla\varphi\, dx + \int_\Omega u_i^\varepsilon \varphi\, dx = \int_\Omega g_i \varphi\, dx, \quad \varphi \in H^1(\Omega); \tag{16}$$

$$\int_\Omega u_i^\infty \psi\, dx = \int_\Omega P g_i \psi\, dx, \quad \psi \in \mathbb{R}.$$

Thus

$$\int_\Omega \varepsilon_i |\nabla u_i^\varepsilon|^2\, dx + \int_\Omega u_i^\varepsilon (u_i^\varepsilon - u_i^\infty)\, dx = \int_\Omega g_i (u_i^\varepsilon - u_i^\infty)\, dx;$$

$$\int_\Omega u_i^\infty (P u_i^\varepsilon - u_i^\infty)\, dx = \int_\Omega P g_i (P u_i^\varepsilon - u_i^\infty)\, dx,$$

which implies

$$\int_\Omega g_i (u_i^\varepsilon - u_i^\infty)\, dx - \int_\Omega P g_i (P u_i^\varepsilon - u_i^\infty)\, dx = \int_\Omega g_i (I - P) u_i^\varepsilon\, dx$$

and

$$\int_\Omega \varepsilon_i |\nabla u_i^\varepsilon|^2\, dx + \int_\Omega u_i^\varepsilon (u_i^\varepsilon - u_i^\infty)\, dx - \int_\Omega u_i^\infty (P u_i^\varepsilon - u_i^\infty)\, dx = \|u_i^\varepsilon - u_i^\infty\|^2_{X_i^{\frac{1}{2}}}.$$

Therefore

$$\|u_i^\varepsilon - u_i^\infty\|^2_{X_i^{\frac{1}{2}}} \leq \int_\Omega |g_i (I - P) u_i^\varepsilon|\, dx.$$

By Poincaré's inequality for average, we have

$$\int_\Omega |g_i (I - P) u_i^\varepsilon|\, dx \leq \|g_i\|_{L^2} \Big(\int_\Omega |\nabla u_i^\varepsilon|^2\, dx \Big)^{\frac{1}{2}},$$

but

$$d_\varepsilon \int_\Omega |\nabla u_i^\varepsilon|^2 \, dx \leq \|u_i^\varepsilon - u_i^\infty\|^2_{X_i^{\frac{1}{2}}}.$$

Put these estimates together we obtain (15). □

Remark 4.2. *When we work with large diffusion the norm in $X_\varepsilon^{\frac{1}{2}}$, in general, is equivalent to the norm of H^1 but this equivalence is not uniform, indeed it follows from* (7) *the following inequalities*

$$m_0\|u\|^2_{H^1} \leq \|u\|^2_{X_\varepsilon^{\frac{1}{2}}} \leq \max_{i=1,\dots,n}\{\varepsilon_i\}\|u\|^2_{H^1}.$$

Hence estimates in the Sobolev spaces H^1 does not give suitable estimates in the half fractional power space $X_\varepsilon^{\frac{1}{2}}$, since $d_\varepsilon \leq \max_{i=1,\dots,n}\{\varepsilon_i\} \to \infty$ as $d_\varepsilon \to \infty$.

Notice that by Poincare's inequality we can obtain a better estimate if we work in H^1, that is, $\|u^\varepsilon - u^\infty\|_{H^1} \leq C d_\varepsilon^{-1}$, for some constant C independent of d_ε.

Hence, it is clear that due to the non-uniformity in the norms we have some loss when we consider $X_\varepsilon^{\frac{1}{2}}$-norm than H^1-norm. This can be seen in the following example.

Consider the one-dimensional elliptic problem

$$\begin{cases} -\varepsilon u_{xx} = \cos(2\pi x), \ x \in (0,1), \\ u_x(0) = 0 = u_x(1). \end{cases}$$

We have $u^\varepsilon(x) = \frac{1}{\varepsilon}\frac{\cos(2\pi x)}{4\pi^2}$ and $u^\infty = 0$. Thus

$$\|u^\varepsilon - u^\infty\|^2_{X_\varepsilon^{\frac{1}{2}}} = \int_0^1 \varepsilon\left|\frac{1}{\varepsilon}\frac{\sin(2\pi x)}{4\pi^2}\right|^2 dx = C\varepsilon^{-1},$$

where C is a constant independent of ε.

The Lemma 4.1 determines the natural quantity that will be used to study the convergence of the dynamic of the problem (1) when ε is approaches $\bar{\mu} = (\mu - 1)\lambda_1^{-1}$. The rate of convergence is given by $d_\varepsilon^{-\frac{1}{2}}$ that goes to zero as d_ε goes to infinity. In fact, if we denote $u^\varepsilon = A_\varepsilon^{-1} g$ then u^ε is the weak solution of the elliptic problem $A_\varepsilon u^\varepsilon = g$ and since g is an arbitrary map in L^2, we obtain by Lemma 4.1,

$$\|A_\varepsilon^{-1} - P\|_{\mathcal{L}(L^2, X_\varepsilon^{\frac{1}{2}})} \leq C d_\varepsilon^{-\frac{1}{2}}. \tag{17}$$

This estimate imply with the compact convergence in [3] and [5], that is the operator A_ε^{-1} converges compactly to $A_\infty^{-1}P = P$.

Notice that, if we take $\varphi = 1$ as a test function in (16), we have $u^\infty = Pu^\varepsilon$, hence (15) shows that u^ε converge for its average in $X_\varepsilon^{\frac{1}{2}}$ and this rate of convergence is $d_\varepsilon^{-\frac{1}{2}}$.

Now we will see how the convergence of the resolvent operators implies the convergence of the eigenvalues and spectral projections defined in (10). We have by Lemma 4.1,

$$\|Q_\varepsilon - P\|_{\mathcal{L}(L^2, X_\varepsilon^{\frac{1}{2}})} \leq \frac{1}{2\pi}\int_{|\xi+1|=\delta} \|(\xi + A_\varepsilon)^{-1} - (\xi + I)^{-1}P\|_{\mathcal{L}(L^2, X_\varepsilon^{\frac{1}{2}})} \, d\xi \leq C d_\varepsilon^{-\frac{1}{2}}. \tag{18}$$

Since $A_\infty = I$ in $\mathbb{R}^n$ we can denote $Q_\infty = I$, in other words, Q_ε converges to $Q_\infty^{-1}P = P$. Note that since the operator A_ε has compact resolvent, the spectral projection Q_ε is a compact operator. Thus, for d_ε sufficiently large, the eigenspace $W_\varepsilon = Q_\varepsilon X_\varepsilon^{\frac{1}{2}}$ has dimension $\dim(W_\varepsilon) = \dim(\mathbb{R}^n) = n$. Moreover, the eigenvalues λ_i^2, $i \geq 2$ goes to infinity as d_ε goes to infinity. The last property was used implicitly in the last section when we guessed the limiting ordinary differential equation.

Lemma 4.3. *Let A_ε the operator defined in* (5) *and let $\sigma(A_\varepsilon) = \{1 < \lambda_2^\varepsilon < \lambda_3^\varepsilon, \dots\}$ its ordered spectrum. Then $\lambda_j^\varepsilon \to \infty$ as $d_\varepsilon \to \infty$ and $j \geq 2$.*

Proof. Assume that there is $R > 0$ and there are sequences $\varepsilon_k \to \infty$ as $k \to \infty$ and $\{\lambda_j^{\varepsilon_k}\}_k$, $j \geq 2$, such that, $\lambda_j^{\varepsilon_k} \in \sigma(A_{\varepsilon_k})$ and $|\lambda_j^{\varepsilon_k}| \leq R$. We can assume $\lambda_j^{\varepsilon_k} \to \lambda$. Let $u_j^{\varepsilon_k}$ be the corresponding eigenfunction to $\lambda_j^{\varepsilon_k}$ with $\|u_j^{\varepsilon_k}\|_{X_{\varepsilon_k}^{\frac{1}{2}}} = 1$. Then $u_j^{\varepsilon_k} = \lambda_j^{\varepsilon_k} A_{\varepsilon_k}^{-1} u_j^{\varepsilon_k}$. Since A_{ε_k} converges compactly to $A_\infty^{-1}P$, we can assume $u_j^{\varepsilon_k} \to u$ as $\varepsilon_k \to \infty$ for some $u \in \mathbb{R}^n$. Thus

$$u_j^{\varepsilon_k} = \lambda_j^{\varepsilon_k} A_{\varepsilon_k}^{-1} u_j^{\varepsilon_k} \to \lambda A_\infty^{-1} u,$$

as $\varepsilon_k \to \infty$. Since $u_j^{\varepsilon_k} \to u$, we get $u = \lambda A_\infty^{-1} u$, which implies $\lambda \in \sigma(A_\infty)$, thus $\lambda = 1$ and $\lambda_j^{\varepsilon_k} \to 1$ as $\varepsilon_k \to \infty$, $j \geq 2$, which is an absurd. □

5. Converge of Attractors

In what follows we will consider $d_\varepsilon \in [m_0, \bar{\mu}]$, where $\bar{\mu} = (\mu - 1)\lambda_1^{-1}$. We have seen that $\mathcal{A}_\varepsilon = \mathcal{A}_\infty = \mathcal{A}_{\bar{\mu}}$ for $d_\varepsilon \geq \bar{\mu}$. Thus, we are concerned about what happens when ε approaches $\bar{\mu}$ to the left. We will see that the family of attractors $\{\mathcal{A}_\varepsilon\}$ with $\varepsilon \in [m_0, \bar{\mu}]$ is continuous as $\varepsilon \to \bar{\mu}$ and this continuity can be estimated by a rate of convergence given by $d_\varepsilon^{-\frac{1}{2}}$ that goes to zero when d_ε goes to infinity. Since Y_ε is isomorphic to $\mathbb{R}^n$ and their norms are uniformly equivalent (by (7)) we will consider $Y_\varepsilon = \mathbb{R}^n$.

In order to obtain estimate for the convergence of the attractor $\mathcal{A}_\varepsilon$ of the equation (8) to the attractor $\mathcal{A}_\infty$ of the (13) as $d_\varepsilon \to \bar{\mu}$ following the results of the [5], we assume the nonlinear semigroup $T_\infty(\cdot)$ generated by solutions of the (13) is a Morse-Smale semigroup in $\mathbb{R}^n$. More precisely,

$$T_\infty(t)u_0^\infty = e^{-A_\infty t}u_0^\infty + \int_0^t e^{-A_\infty(t-s)}F(T_\infty(s)u_0^\infty)\,ds,\ t > 0,\ \ u_0^\infty \in \mathbb{R}^n, \tag{19}$$

where $A_\infty = I$ denotes the identity in $\mathbb{R}^n$ and if we denote $\mathcal{E}_\infty$ the set of its equilibrium points, then it is composed of p hyperbolic points, that is,

$$\mathcal{E}_\infty = \{\varphi \in \mathbb{R}^n : A_\infty\varphi - F(\varphi) = 0\} = \{u_1^{\infty,*}, \dots, u_p^{\infty,*}\}, \tag{20}$$

where the spectrum set $\sigma(A_\infty - F'(u_i^{\infty,*})) \cap \{\varphi \in \mathbb{R}^n : \|\varphi\|_{\mathbb{R}} = 1\} = \emptyset$, $i = 1, \dots, p$. Moreover, $T_\infty(\cdot)$ is dynamically gradient (see [4]),

$$\mathcal{A}_\infty = \bigcup_{i=1}^{p} W^u(u_i^{\infty,*}), \tag{21}$$

where $W^u(u_i^{\infty,*})$ is the unstable manifold associated to the equilibrium point in $\mathcal{E}_\infty$ and for $i \neq j$ the local unstable manifold $W^u_{\text{loc}}(u_i^{\infty,*})$ and the stable manifold $W^s(u_j^{\infty,*})$ has transversal intersection. We notice that the Kupka-Smale theorem for ODEs ensures that this situation is generic, in the sense that this must occurs in the most interesting cases. Thus our assumptions about hyperbolicity and transversality are not restrictive.

We will study the problem (1) as a small perturbation of (13) and the continuity of attractors will be considered the assumptions above enable us to obtain the geometric equivalence of phase diagrams when ε approaches $\bar{\mu}$. This property is known as geometric structural stability and it is the main feature of Morse-Smale problems. In this way, we are under the conditions described in [5], where results about the rate of convergence of attractor for Morse-Smale problems were obtained. More precisely, it is valid the following result.

Theorem 5.1. *Let K_ε, $\varepsilon \geq 0$ be a family of separable Hilbert spaces such that $K_0 \hookrightarrow K_\varepsilon$ and $dim(K_0) = n$. Suppose B_ε is a self-adjoint positive and invertible operator and consider the following evolution equation*

$$\begin{cases} u_t^\varepsilon + B_\varepsilon u^\varepsilon = h(u^\varepsilon),\ t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \in K_\varepsilon^{\frac{1}{2}}, \end{cases} \tag{22}$$

where $K_\varepsilon^{\frac{1}{2}}$ is the fractional power space associated with B_ε ($Y_0^{\frac{1}{2}} = \mathbb{R}^n$) and h is a bounded Lipschitz function. Assume that there is a increasing function $\tau(\varepsilon)$ such that $\tau(0) = 0$ and

$$\|B_\varepsilon^{-1} - E_\varepsilon B_0^{-1} M_\varepsilon\|_{\mathcal{L}(Y_\varepsilon, Y_\varepsilon^{\frac{1}{2}})} \leq C\tau(\varepsilon), \tag{23}$$

where $E_\varepsilon : K_0 \to K_\varepsilon^{\frac{1}{2}}$ and $M_\varepsilon : K_\varepsilon^{\frac{1}{2}} \to K_0$ are bounded linear operators and C is a constant independent of ε. Then there is an invariant manifold for (22) *given by a graph of a Lipschitz function k_*^ε such that $\sup_{u^\varepsilon \in Y_0} \|k_*^\varepsilon(u^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} \leq C\tau(\varepsilon)$. Moreover if* (22) *with $\varepsilon = 0$ generates a Morse-Smale semigroup and if there is the global attractor $\mathcal{B}_\varepsilon$, for* (22) *with $\varepsilon \geq 0$, then*

$$\mathrm{d}_H(\mathcal{B}_\varepsilon, \mathcal{B}_0) \leq C\tau(\varepsilon).$$

Now, we can state the main result of this paper.

Theorem 5.2. *For $d_\varepsilon \in [m_0, \bar{\mu}]$ there is an invariant manifold $\mathcal{M}_\varepsilon$ for* (8)*, which is given by graph of a certain Lipschitz continuous map $s_*^\varepsilon : \mathbb{R}^n \to Z_\varepsilon$ as*

$$\mathcal{M}_\varepsilon = \{u^\varepsilon \in X_\varepsilon^{\frac{1}{2}} \,;\, u^\varepsilon = Q_\varepsilon u^\varepsilon + s_*^\varepsilon(Q_\varepsilon u^\varepsilon)\}.$$

The map $s_*^\varepsilon : \mathbb{R}^n \to Z_\varepsilon$ satisfies the condition

$$|\!|\!| s_*^\varepsilon |\!|\!| = \sup_{v^\varepsilon \in \mathbb{R}^n} \|s_*^\varepsilon(v^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} \leq C d_\varepsilon^{-\frac{1}{2}}, \tag{24}$$

for some positive constant C independent of d_ε. The invariant manifold $\mathcal{M}_\varepsilon$ is exponentially attracting and the global attractor $\mathcal{A}_\varepsilon$ of the problem (8) *lying in $\mathcal{M}_\varepsilon$. Moreover, the continuity of the attractors can be estimated by*

$$\mathrm{d}_H(\mathcal{A}_\varepsilon, \mathcal{A}_\infty) \leq \frac{C}{\sqrt{d_\varepsilon}}.$$

Proof. If we define $\tau(\varepsilon) = 1/\sqrt{d_\varepsilon}$, then $\tau(\varepsilon)$ is a increasing function such that

$$\tau(0) = \lim_{d_\varepsilon \to \infty} 1/\sqrt{d_\varepsilon} = 0.$$

We take A_0 as identity in $\mathbb{R}^n$, E_ε as the inclusion $\mathbb{R}^n \hookrightarrow X_\varepsilon^{\frac{1}{2}}$ and $M_\varepsilon = P : L^2 \to \mathbb{R}^n$ the average in Ω, then by (17) we have

$$\|A_\varepsilon^{-1} - P\|_{\mathcal{L}(L^2, X_\varepsilon^{\frac{1}{2}})} \leq C\tau(\varepsilon).$$

Thus all conditions of the Theorem (5.2) are satisfied. □

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