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Stability analysis for pricing options via time fractional Heston model

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Abstract. In this work, we have studied the time fractional-order derivative of the pricing European options under Heston model. We found some positivity conditions for the solution obtained relative to the numerical methods used. Also, thanks to the properties of the Mittag-Leffler function, we were able to establish a stability result of the solution. Some numerical experiments are carried out to confirm the theoretical results obtained.

1. Introduction and Preliminaries

Pricing options stands for one of the most popular problems in mathematical financial literature. Basically, European options considered among the most in the worldwide financial markets. Over the last few decades, numerous papers addressed the problem of pricing options tackled by different models using multiple methods, for instance [3], [5], [9] and [16]. The most outstanding one is the Black and Scholes model [5], which rests upon the concept that the stock price of the underlying asset is log-normally distributed conditional on the current stock price with constant volatility. Compared to the case of the Black and Scholes model, where the volatility is constant, the Heston model [9] proves to be more important since the volatility is stochastic, owing to the fact that the dynamics of the volatility is fundamental to elaborate strategies for hedging and arbitrage. Indeed, a model based on constant volatility cannot explain the reality of the financial markets. Therefore, pricing option under stochastic volatility model is then extremely significant and highly needed.

The fractional calculus is applied in various fields [1], [4], [6], [7], [15], [19], [21] and [23]. For instance, fractional derivation models have displayed an ability to characterize shape-memory materials better than full derivation models. When a material is purely elastic, it is indicated by an integer derivation of order zero. However, when it is purely viscous it is denoted by an integer derivation of order one. We can therefore immediately describe a viscous-elastic material by a derivation between zero and one. This accounts for the use of fractional derivation for this kind of material. From this perspective, driven by mathematical curiosity and in order to get closer to the reality of the financial market, it is crucial to use models based on fractional derivatives. Recently, they have been integrated in the mathematical finance field [13], [14], [22] and [23]. They have been particularly designed to resolve the pricing option problem, for instance [8],[11], [12], [16] and [24] which are basically devoted for the evaluation of the European option.

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For this reason, using the splitting method, we attempt to elaborate a new resolution for the pricing European option under the fractional Heston model. The aforementioned method allows to solve a mixed problem Parabolic/Hyperbolic by decoupling the parabolic and hyperbolic operators, (for more details see [2]). A nonlinear mixed problem generated by two completely different operators, (Parabolic/Hyperbolic), can generate difficulties in in terms of the numerical simulations. During discretization, the splitting method makes it possible to handle each operator whether Parabolic and Hyperbolic by an adequate numerical scheme. This method preserves the numerical properties (stability, consistency, $\cdots$) of each used scheme for each operator. Additionally, this new method allows to provide relevant numerical results. It is worth noting that the state of arts works reported that the coefficient of correlation ρ, (see equation (1)), lies always between -0.7 and 0.7. With our new numerical method, we can extend the previously stated coefficient ranging between -0.9 and 0.9.

In summary, the novelties of this work are as follows:

- ⋄ Theoretical and numerical study of the time fractional-order derivative of the pricing European options under Heston model.
- ⋄ Obtain a positivity result of the solution relative to the numerical methods used, (see Theorem 5.4 and Lemma 5.6),
- ⋄ Establish a stability result for the solution to the time fractional-order derivative of the pricing European options under Heston model, (see Theorem 5.7),
- ⋄ Prove the effectiveness of the numerical methods used and this by showing the consistency between the theoretical results and the numerical experiments,
- ⋄ Extending the coefficient of correlation between -0.9 and 0.9,
- ⋄ Compered the numerical results between the time fractional-order derivative of the pricing European options under Heston model versus the time integer-order derivative of the pricing European options under Heston model.

In the following definition, we exhibit the modified right Riemann-Liouville derivative and the Caputo time-fractional derivative.

Definition 1.1. ([10]).

1. *The modified right Riemann-Liouville derivative for $0 < \gamma < 1$ is defined by:*
$$\frac{\partial^\gamma f(x,y,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_t^T \frac{f(x,y,s) - f(x,y,T)}{(s-t)^\gamma}ds,$$
where $x, y \in \mathbb{R}$ and $t \in [0, T]$.
2. *The Caputo time fractional derivative of order $\gamma \in (0,1)$, $a, t \in \mathbb{R}$ can be indicated as follows:*
$${}^cD_a^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-\tau)^{-\gamma}\frac{d}{d\tau}f(\tau)d\tau,$$
where $\Gamma(\cdot)$ is the Gamma function.

Definition 1.2. ([25]) *The Mittag-Leffler function of one parameter is determined as:*
$$E_\jmath(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{\Gamma(\jmath n+1)}, \quad Re(\jmath) > 0,\ \sigma \in \mathbb{C}.$$
The Mittag-Leffler function of two parameters is defined as:
$$E_{\jmath,\xi}(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{\Gamma(\jmath n+\xi)}, \quad Re(\jmath) > 0,\ \xi > 0,\ \sigma \in \mathbb{C},$$
where $\Gamma(\cdot)$ is the gamma function.

2. Time-fractional model

In the following section, we introduce the European put option problem under the fractional Heston model:

$$\frac{\partial^\gamma \mathcal{W}}{\partial \tau^\gamma}+\frac{1}{2}s^2 v\frac{\partial^2 \mathcal{W}}{\partial s^2}+\rho\sigma s v\frac{\partial^2 \mathcal{W}}{\partial s\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 \mathcal{W}}{\partial v^2}+rs\frac{\partial \mathcal{W}}{\partial s}+\kappa(\theta-v)\frac{\partial \mathcal{W}}{\partial v}-r\mathcal{W}=0, \tag{1}$$

for all $(s,v,\tau)\in Q^T_\infty=(0,\infty)\times(0,\infty)\times(0,T)$ and where $\frac{\partial^\gamma \mathcal{W}}{\partial \tau^\gamma}$ is the modified right Riemann-Liouville derivative for $0<\gamma<1$ defined in Definition 1.1, $\rho\in(-1,1)$ is the instantaneous correlation, σ is the volatility of the variance, θ is the long-run variance, κ is the mean reversion rate and r is the interest rate.

Moreover, we consider the following boundary conditions:

$$\mathcal{W}(s,v,T)=\max(K-s,0), \tag{2}$$

$$\lim_{s\to 0}\mathcal{W}(s,v,\tau)=K, \qquad \lim_{s\to\infty}\mathcal{W}(s,v,\tau)=0, \tag{3}$$

$$\lim_{v\to 0}\mathcal{W}(s,v,\tau)=\lim_{v\to\infty}\mathcal{W}(s,v,\tau)=\max(K-s,0), \tag{4}$$

Now, let us reformulate the above problem (1)-(4), (as in [24]), with the new variable in time t defined as follows:

$$t=T-\tau, \quad \text{for} \quad 0<\gamma<1.$$

We can deduce, as in [24], that:

$$\begin{aligned}\frac{\partial^\gamma \mathcal{W}(s,v,\tau)}{\partial \tau^\gamma} &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{d\tau}\int_\tau^T \frac{\mathcal{W}(s,v,\xi)-\mathcal{W}(s,v,T)}{(\xi-\tau)^\gamma}d\xi,\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{-d}{dt}\int_{T-t}^T \frac{\mathcal{W}(s,v,\xi)-\mathcal{W}(s,v,T)}{(\xi-(T-t))^\gamma}d\xi,\\ &= -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t \frac{\mathcal{W}(s,v,T-\eta)-\mathcal{W}(s,v,T)}{(t-\eta)^\gamma}d\eta. \end{aligned}\tag{5}$$

By denoting $\zeta(s,v,t)=\mathcal{W}(s,v,T-t)$, we deduce form equation (5) that:

$$\frac{\partial^\gamma \mathcal{W}(s,v,\tau)}{\partial \tau^\gamma}=-\frac{\partial^\gamma \zeta(s,v,t)}{\partial t^\gamma}, \qquad 0<\gamma<1,$$

where the fractional derivative $\frac{\partial^\gamma \zeta}{\partial t^\gamma}$ is given by:

$$\frac{\partial^\gamma \zeta(s,v,t)}{\partial t^\gamma}=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t \frac{\zeta(s,v,\eta)-\zeta(s,v,0)}{(t-\eta)^\gamma}d\eta, \qquad 0<\gamma<1.$$

So, the system (1) can be rewritten as:

$$\frac{\partial^\gamma \zeta}{\partial t^\gamma}=\frac{1}{2}vs^2\frac{\partial^2 \zeta}{\partial s^2}+\rho\sigma v s\frac{\partial^2 \zeta}{\partial s\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 \zeta}{\partial v^2}+rs\frac{\partial \zeta}{\partial s}+\kappa(\theta-v)\frac{\partial \zeta}{\partial v}-r\zeta, \tag{6}$$

with the following new boundary conditions:

$$\zeta(s,v,0)=\max(K-s,0), \tag{7}$$

$$\lim_{s\to 0}\zeta(s,v,t)=K, \qquad \lim_{s\to\infty}\zeta(s,v,t)=0, \tag{8}$$

$$\lim_{v\to 0}\zeta(s,v,t)=\lim_{v\to\infty}\zeta(s,v,t)=\max(K-s,0), \tag{9}$$

If we suppose that $\zeta\in C^1$ about the variable t, $(0<\gamma<1)$. Then, we can prove, as in [24], that:

$$\begin{aligned}\frac{\partial^\gamma \zeta}{\partial t^\gamma}(s,v,t) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t \frac{\zeta(s,v,\eta)-\zeta(s,v,0)}{(t-\eta)^\gamma}d\eta,\\ &= \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{d\zeta(s,v,\xi)}{d\xi}(t-\xi)^{-\gamma}\,d\xi,\\ &= {}^cD_t^\gamma \zeta(s,v,t), \end{aligned}\tag{10}$$

where ${}^cD_t^\gamma\zeta$ is the Caputo fractional derivative of order $0 < \gamma < 1$, (see Definition 1.1). In this case, equation (6) can be rewritten as:

$$ {}^cD_t^\gamma\zeta = \frac{1}{2}vs^2\frac{\partial^2\zeta}{\partial s^2} + \rho\sigma vs\frac{\partial^2\zeta}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2\zeta}{\partial v^2} + rs\frac{\partial\zeta}{\partial s} + \kappa(\theta - v)\frac{\partial\zeta}{\partial v} - r\zeta. \tag{11} $$

In the rest of this article, we will study equation (11) numerically. In fact, we will give some conditions for the stability and the positivity of the numerical solution of equation (11).

3. Splitting method

Equation (11) corresponds to a time-dependent two-dimensional nonlinear Diffusion/Advection equation that includes five types of spatial derivatives. Handling finite-difference methods, the existence of these derivatives together in the same equation can distort the quality of the numerical solution. Furthermore, an implicit finite difference scheme in the presence of five spatial derivatives generates numerous unknowns in the numerical scheme producing considerable difficulties for the numerical implementation and entailing rounding accumulation errors.

In what follows, we suggest using a splitting method [2]. Within the process of discretization, the splitting method allows to handle separately each operator Diffusion and Advection by an adequate numerical scheme. This method keeps the numerical properties (stability, consistency, quality, $\cdots$) of each used scheme for each operator.

We divide the time interval $[0, T]$ into $(N_t + 1)$ equidistant points as follows:

$$ \Delta t = \frac{T}{N_t} \quad \text{where} \quad t^k = k\Delta t, \quad \text{for all} \quad k = 0, \cdots, N_t. $$

Consequently, we have:

$$ [0, T] = \bigcup_{k=0}^{N_t-1}[t^k, t^{k+1}]. $$

Consider the approximation exhibited:

$$ \zeta(s, v, t^k) \approx \zeta^k(s, v), $$

for all $k = 0, \cdots, N_t$ and $(s, v) \in (0, \infty) \times (0, \infty)$.

The splitting method rests on solving the equation (11) on each interval $[t^k, t^{k+1}]$ for all $k = 0, \cdots, N_t - 1$, based upon the three steps:

(i) We solve the equation:

$$ {}^cD_t^\gamma\zeta = \rho\sigma vs\frac{\partial^2\zeta}{\partial s\partial v}, \quad \text{on} \quad [t^k, t^{k+\frac{1}{3}}], \tag{12} $$

with initial condition at t^k : ζ^k.

Therefore, we obtain a solution at time step $t^{k+\frac{1}{3}}$ denoted by $\zeta^{k+\frac{1}{3}}$.

(ii) Hence, we solve the equation:

$$ {}^cD_t^\gamma\zeta = rs\frac{\partial\zeta}{\partial s} + \kappa(\theta - v)\frac{\partial\zeta}{\partial v} - r\zeta, \quad \text{on} \quad [t^{k+\frac{1}{3}}, t^{k+\frac{2}{3}}], \tag{13} $$

with initial condition at $t^{k+\frac{1}{3}}$: $\zeta^{k+\frac{1}{3}}$,

The solution at time step $t^{k+\frac{2}{3}}$ is denoted by $\zeta^{k+\frac{2}{3}}$.

(iii) We solve the equation:

$$\begin{aligned} &{}^{c}D_t^{\gamma}\zeta = \frac{1}{2}vs^2\frac{\partial^2\zeta}{\partial s^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2\zeta}{\partial v^2}, \quad \text{on} \quad [t^{k+\frac{2}{3}}, t^k], \\ &\text{with initial condition at } t^{k+\frac{2}{3}}: \quad \zeta^{k+\frac{2}{3}}. \end{aligned} \tag{14}$$

We proceed in the same way until reaching the final time $T = t^{N_t}$ and we solve simultaneously equations (12), (13) and (14).

Remark 3.1. *It's well known [20] that the mixed derivative term $\frac{\partial^2\zeta}{\partial s\partial v}$ in equation (11) is unstable. Moreover, we know that the diffusion equation (14) is more stable than the equation (12). As a matter of fact, we begin the splitting method by solving equation (12) in the first step and we finish by solving equation (14) to calm the unstable solution arising from the first step. This method allows to obtain relevant numerical results.*

4. Discretization of the Model

To solve the numerical problem (11), we need to select a numerical bounded domain where we can solve (11) by approximations with finite differences. Hence, we consider the following numerical domain:

$$\mathcal{D} = \Big\{(s, v) : s \in [s_l, s_r],\ s_l \neq 0, \quad v \in [v_l, v_r],\ v_l \neq 0\Big\}. \tag{15}$$

We define a uniform grid on the domain $\mathcal{D}$ as follows: let $\Delta s = \frac{s_r - s_l}{N_s}$ and $\Delta v = \frac{v_r - v_l}{N_v}$. Now, we can build the sequences $(s_i)_i$, $(v_j)_j$:

$$\begin{aligned} s_i &= s_l + i\Delta s \quad \text{for all} \quad i = 0, \cdots, N_s, \\ v_j &= v_l + j\Delta v \quad \text{for all} \quad j = 0, \cdots, N_v. \end{aligned} \tag{16}$$

Consider the approximations presented below:

$$\zeta(s_i, v_j, t^k) \approx \zeta_{i,j}^k,$$

for all $i = 0, \cdots, N_s$, $j = 0, \cdots, N_v$ and $k = 0, \cdots, N_t$.

- *Discretization of the Caputo fractional time derivative term* (${}^{c}D_t^{\gamma}\zeta$): we know from equation (10) that the Caputo fractional time derivative term (${}^{c}D_t^{\gamma}\zeta$) is defined by:

$${}^{c}D_t^{\gamma}\zeta(s, v, t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{d\zeta(s, v, \xi)}{d\xi}(t-\xi)^{-\gamma}\, d\xi.$$

  As identified in [24], at the point (s_i, v_j, t^{k+1}), we get the following approximation:

$$\begin{aligned} {}^{c}D_t^{\gamma}\zeta(s_i, v_j, t^{k+1}) &\approx \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{m=0}^{k}(\zeta_{i,j}^{k+1-m} - \zeta_{i,j}^{k-m})b_m, \\ &\approx \alpha_0\sum_{m=0}^{k-1}(b_{m+1} - b_m)\zeta_{i,j}^{k-m} - \alpha_0\zeta_{i,j}^0 b_k + \alpha_0\zeta_{i,j}^{k+1}, \end{aligned}$$

  where $\alpha_0 = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}$ and $b_m = (m+1)^{1-\gamma} - m^{1-\gamma}$.

- *Discretization of the problem* (12): At the point (s_i, v_j, t^{k+1}), we have:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{m=0}^{k}(\zeta_{i,j}^{k+1-m} - \zeta_{i,j}^{k-m})b_m = \frac{\rho\sigma s_i v_j}{\Delta s\Delta v}\Big(\zeta_{i+1,j+1}^{k+1} - \zeta_{i,j+1}^{k+1} - \zeta_{i+1,j}^{k+1} + \zeta_{i,j}^{k+1}\Big). \tag{17}$$

- *Discretization of the problem* (13): At the point (s_i, v_j, t^{k+1}), we have:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{m=0}^{k}(\zeta_{i,j}^{k+1-m} - \zeta_{i,j}^{k-m})b_m = \frac{rs_i}{\Delta s}\Big(\frac{1}{2}(\zeta_{i+1,j+1}^{k+1} + \zeta_{i+1,j-1}^{k+1}) - \zeta_{i,j}^{k+1}\Big) + \frac{\kappa(\theta - v_j)}{\Delta v}\Big(\zeta_{i,j+1}^{k+1} - \zeta_{i,j}^{k+1}\Big) - r\zeta_{i,j}^{k}. \tag{18}$$

- *Discretization of the problem* (14): At the point (s_i, v_j, t^{k+1}), we have:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{m=0}^{k}(\zeta_{i,j}^{k+1-m} - \zeta_{i,j}^{k-m})b_m = \frac{s_i^2 v_j}{2(\Delta s)^2}\Big(\zeta_{i+1,j}^{k+1} - 2\zeta_{i,j}^{k+1} + \frac{1}{2}(\zeta_{i-1,j+1}^{k+1} + \zeta_{i-1,j-1}^{k+1})\Big) + \frac{\sigma^2 v_j}{2(\Delta v)^2}\Big(\zeta_{i,j+1}^{k+1} - 2\zeta_{i,j}^{k+1} + \zeta_{i,j-1}^{k+1}\Big). \tag{19}$$

Remark 4.1. *1) For the discretization of equations* (12), (13) *and* (14) *we have used an implicit time finite difference schemes. The basic merit of these schemes resides in the fact that they are unconditionally stable.*

2) Notice that in the first step of the splitting method, in equation (17)*, only four unknowns are identified:* $\zeta_{i+1,j+1}^{k+1}$, $\zeta_{i,j+1}^{k+1}$, $\zeta_{i+1,j}^{k+1}$, $\zeta_{i,j}^{k+1}$. *In addition, in the second step of the splitting method, in equation* (18)*, we have three unknowns:* $\zeta_{i,j+1}^{k+1}$, $\zeta_{i+1,j}^{k+1}$, $\zeta_{i,j}^{k+1}$. *In the third step, in equation* (19)*, we have five unknowns:* $\zeta_{i+1,j}^{k+1}$, $\zeta_{i,j+1}^{k+1}$, $\zeta_{i-1,j}^{k+1}$, $\zeta_{i,j-1}^{k+1}$, $\zeta_{i,j}^{k+1}$. *Thus, in each step, the numerical implementation is quite simple and the solution quality(Diffusion, Advection) is ensured.*

5. Study of the stability

The total spatial discretization of equation (11) is expressed as follows: for all $i = 1, \cdots, N_s - 1$ and $j = 1, \cdots, N_v - 1$

$$\begin{aligned} {}^cD_t^\gamma \zeta_{i,j} = {} & \frac{s_i^2 v_j}{2(\Delta s)^2}\Big(\zeta_{i+1,j} - 2\zeta_{i,j} + \frac{1}{2}(\zeta_{i-1,j+1} + \zeta_{i-1,j-1})\Big) + \\ & \frac{\rho\sigma s_i v_j}{4\Delta s\Delta v}\Big(\zeta_{i+1,j+1} - \zeta_{i,j+1} - \zeta_{i+1,j} + \zeta_{i,j}\Big) + \frac{\sigma^2 v_j}{2(\Delta v)^2}\Big(\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}\Big) + \\ & \frac{rs_i}{\Delta s}\Big(\frac{1}{2}(\zeta_{i+1,j+1} + \zeta_{i+1,j-1}) - \zeta_{i,j}\Big) + \frac{k(\theta - v_j)}{\Delta v}\Big(\zeta_{i,j+1} - \zeta_{i,j}\Big) - r\zeta_{i,j}, \end{aligned}$$

where we mean by $\zeta_{i,j} = \zeta_{i,j}(t)$ for all $t \in [0, T]$. Therefore, we obtain the following expression: for all $i = 1, \cdots, N_s - 1$ and $j = 1, \cdots, N_v - 1$

$$\begin{aligned} {}^cD_t^\gamma \zeta_{i,j} &= \tau_{i,j}\zeta_{i+1,j} - \epsilon_{i,j}\zeta_{i,j} + \tilde{\alpha}_{i,j}\zeta_{i-1,j+1} + \alpha_{i,j}\zeta_{i-1,j-1} + \\ &\quad \delta_{i,j}\zeta_{i+1,j+1} + \tilde{\delta}_{i,j}\zeta_{i+1,j-1} + (a_j + f_j)\zeta_{i,j+1} + a_j\zeta_{i,j-1}, \end{aligned} \tag{20}$$

where the coefficients a_j, $\tau_{i,j}$, $\beta_{i,j}$, f_j, γ_i, $\delta_{i,j}$, $\alpha_{i,j}$ and $\epsilon_{i,j}$ are real numbers and are determined by:

$$a_j = \frac{\sigma^2 v_j}{2(\Delta v)^2}, \quad \tau_{i,j} = \frac{s_i^2 v_j}{2(\Delta s)^2}, \quad \beta_{i,j} = \frac{\rho\sigma s_i v_j}{4\Delta s \Delta v}, \quad f_j = \frac{k(\theta - v_j)}{\Delta v}, \quad \gamma_i = \frac{r s_i}{\Delta s}, \tag{21}$$

$$\delta_{i,j} = \frac{1}{2}\gamma_i + \beta_{i,j}, \quad \alpha_{i,j} = \frac{1}{2}\tau_{i,j} + \beta_{i,j}, \quad \epsilon_{i,j} = 2\tau_{i,j} + 2a_j + \gamma_i + f_j + r,$$

where if $q = x + y \in \mathbb{R}$, then $\tilde{q} = x - y \in \mathbb{R}$.

Remark 5.1. *Remark that the coefficients a_j, $\tau_{i,j}$, γ_i, are strictly positive real numbers. On the other side, the sign of each coefficient $\beta_{i,j}$, $\delta_{i,j}$, $\alpha_{i,j}$, $\epsilon_{i,j}$ and f_j depends on the signs of $\rho \in (-1, 1)$ and the parameter θ.*

Let $N_{sv} = (N_s - 1) \times (N_v - 1)$. We define the vector $\boldsymbol{Q} \in \mathbb{R}^{N_{sv}}$ by: $\forall t \in [0, T]$

$$\boldsymbol{Q}(t) = [\zeta_{1,1}, \zeta_{1,2}, \cdots, \zeta_{1,N_v-1}, \zeta_{2,1}, \zeta_{2,2}, \cdots, \zeta_{2,N_v-1}, \cdots, \zeta_{N_s-1,1}, \cdots, \zeta_{N_s-1,N_v-1}] \in \mathbb{R}^{N_{sv}}.$$

Hence, the system (20) can be expressed as follows:

$${}^cD_t^\gamma \boldsymbol{Q}(t) = B\boldsymbol{Q}(t) + \boldsymbol{M}(t), \tag{22}$$

with initial condition:

$$\boldsymbol{Q}(0) = \boldsymbol{Q}_0. \tag{23}$$

Remark 5.2. *1) The matrix $B = (a_{i,j})_{i,j} \in \mathbb{R}^{N_{sv} \times N_{sv}}$ has eight diagonals and is defined in terms of:*

⋄The diagonal elements of B:

$$a_{i,i} \in \{-\epsilon_{k,\ell}, \quad k = 1, \cdots, N_s - 1, \quad \ell = 1, \cdots, N_v - 1\}.$$

⋄The non-diagonal elements of B for $i \neq j$:

$$a_{i,j} \in \{\tau_{k,\ell},\ \delta_{k,\ell},\ \tilde{\delta}_{k,\ell},\ a_\ell,\ a_\ell + f_\ell,\ \alpha_{k,\ell},\ \tilde{\alpha}_{k,\ell},\ k = 1, \cdots, N_s - 1,\ \ell = 1, \cdots, N_v - 1\}.$$

2) The vector function $\boldsymbol{M}(t) \in \mathbb{R}^{N_{sv}}$ and is defined only by the trace of the solution $\zeta(s, v, t)$ at the boundary $\Gamma_{\mathcal{D}}$ of the domain $\mathcal{D}$ identified in (15) as well as the coefficients $\tau_{i,j}$, $\delta_{i,j}$, $\tilde{\delta}_{i,j}$, a_j, f_j, $\alpha_{i,j}$, $\tilde{\alpha}_{i,j}$. The vector function $\boldsymbol{M}(t)$ is indicated by:

$$\begin{aligned} \boldsymbol{M}(t) = \Big[&\phi(1) + \tilde{\delta}_{1,1}\zeta_{2,0} + a_0\zeta_{1,0}, \phi(2), \cdots, \phi(j), \cdots, \phi(N_v - 2), \phi(N_v - 1) + \\ &\delta_{1,N_v-1}\zeta_{2,N_v} + g_{N_v-1}\zeta_{1,N_v}, \psi(2), \cdots, \psi(i), \cdots, \psi(N_s - 2), \lambda(1) + \\ &\omega_{N_s-1}\zeta_{N_s-2,0} + a_0\zeta_{N_s-1,0}, \lambda(2), \cdots, \lambda(j), \cdots, \lambda(N_v - 2), \lambda(N_v - 1) + \\ &\tilde{\alpha}_{N_s-1}\zeta_{N_s-2,N_v} + g_{N_v-1}\zeta_{N_s-1,N_v}\Big] \in \mathbb{R}^{N_{sv}}, \end{aligned}$$

where $g_j = a_j + f_j$, the functions ϕ, λ and the vector function $\psi \in \mathbb{R}^{(N_s-3)\times(N_v-1)}$ are computed by:

$\triangleright$ $\phi(j) = \tilde{\alpha}_{1,j}\zeta_{0,j+1} + \alpha_{1,j}\zeta_{0,j-1}, \quad j = 1, \cdots, N_v - 1,$

$\triangleright$ $\lambda(j) = \tau_{N_s-1,j}\zeta_{N_s,j} + \delta_{N_s-1,j}\zeta_{N_s,j+1} + \tilde{\delta}_{N_s,j}\zeta_{N_s,j-1}, \quad j = 1, \cdots, N_v - 1,$

$\triangleright$ $\psi(i) = \Big[\alpha_{i,1}\zeta_{i-1,0} + \tilde{\delta}_{i,1}\zeta_{i+1,0} + a_1\zeta_{i,0}, 0, \cdots\cdots, 0,$
$\tilde{\alpha}_{i,N_v-1}\zeta_{i-1,N_v} + \delta_{i,N_v-1}\zeta_{i+1,N_v} + g_{N_v-1}\zeta_{i,N_v}\Big] \in \mathbb{R}^{N_v-1}, \quad i = 2, \cdots, N_s - 2.$

Remark that $\zeta_{N_s,j}$, $\zeta_{0,j}$, ζ_{i,N_v} and $\zeta_{i,0}$ represent the discrete boundary conditions of the problem given by $\zeta(s, v, t)|_{\Gamma_{\mathcal{D}}}$.

Definition 5.3. ([26]) *A matrix $M = (m_{i,j})_{i,j} \in \mathbb{R}^{n\times n}$ is called Metzler, if its off-diagonal elements are positive, i.e.: $m_{i,j} \geq 0$, for all $1 \leq i \neq j \leq n$.*

Theorem 5.4. *If $\rho = 0$, then the matrix B is Metzler.*
Let $\rho \in (-1,1)\backslash\{0\}$ and $d = s_r - s_l$. If θ and the ratio $\frac{\Delta s}{\Delta v}$ of the spatial steps satisfy the identities:

$$\theta \geq v_r \quad and \quad \frac{\Delta s}{\Delta v} \leq \frac{1}{|\rho|}\min\{\frac{s_l}{\sigma}, \frac{2rd}{\sigma v_l N_s}\}, \tag{24}$$

then the matrix B is Metzler.

Proof. It's clear, from Remark 5.1, that we need to study the signs of the coefficients $\delta_{i,j}$, $\tilde{\delta}_{i,j}$, $\alpha_{i,j}$, $\tilde{\alpha}_{i,j}$, f_j with respect to the values of ρ, θ. From this perspective, it is useful to distinguish the following two cases:

- When $\theta \geq v_r$, the coefficients f_j are positive. Indeed, from relations (15) and (16) we have $v_j \in [v_l, v_r]$ for all j. Consequently, we have $\theta \geq v_r \geq v_j$. Hence, the coefficients f_j, (see relation (21)), are positive for all j.

- When $\rho \in (0,1)$: since $\delta_{i,j}$, $\alpha_{i,j}$ are positive, then the terms $\tilde{\delta}_{i,j}$, $\tilde{\alpha}_{i,j}$ must be positive to make sure the matrix B is Metzler:

$$\tilde{\delta}_{i,j} = \frac{1}{2}\gamma_i - \beta_{i,j} = \frac{rs_i}{2\Delta s} - \frac{\rho\sigma s_i v_j}{4\Delta s\Delta v} \geq 0,$$

$$\tilde{\alpha}_{i,j} = \frac{1}{2}\tau_{i,j} - \beta_{i,j} = \frac{s_i^2 v_j}{4(\Delta s)^2} - \frac{\rho\sigma s_i v_j}{4\Delta s\Delta v} \geq 0.$$

  Hence, we obtain:

$$\begin{aligned} \frac{\Delta s}{\Delta v} &\leq \frac{2r\Delta s}{\rho\sigma v_j}, \quad \forall j = 0,\cdots,N_v, \\ \frac{\Delta s}{\Delta v} &\leq \frac{s_i}{\rho\sigma}, \quad \forall i = 0,\cdots,N_s. \end{aligned}$$

  Or

$$\begin{aligned} \frac{\Delta s}{\Delta v} &\leq \frac{2rd}{\rho\sigma v_l N_s}, \quad \text{where}\, d = s_r - s_l, \\ \frac{\Delta s}{\Delta v} &\leq \frac{s_l}{\rho\sigma}. \end{aligned}$$

  Thus, we deduce:

$$\frac{\Delta s}{\Delta v} \leq \frac{1}{\rho}\min\{\frac{2rd}{\sigma v_l N_s}, \frac{s_l}{\sigma}\}. \tag{25}$$

- When $\rho \in (-1,0)$: since $\tilde{\delta}_{i,j}$, $\tilde{\alpha}_{i,j}$ are positive, then the terms $\delta_{i,j}$, $\alpha_{i,j}$ must be positive to make sure the matrix B is Metzler:

$$\delta_{i,j} = \frac{1}{2}\gamma_i + \beta_{i,j} = \frac{rs_i}{2\Delta s} + \frac{\rho\sigma s_i v_j}{4\Delta s\Delta v} \geq 0,$$

$$\alpha_i = \frac{1}{2}\tau_{i,j} + \beta_{i,j} = \frac{s_i^2 v_j}{4(\Delta s)^2} + \frac{\rho\sigma s_i v_j}{4\Delta s\Delta v} \geq 0.$$

Therefore, proceeding in the same way as above, we obtain:

$$\frac{\Delta s}{\Delta v} \leq -\frac{1}{\rho} \min\{\frac{2rd}{\sigma v_l N_s}, \frac{s_l}{\sigma}\}. \tag{26}$$

Departing from estimations (25) and (26) we deduce the identity (24).
□

Remark 5.5. *Referring to* (7)-(9)*, we deduce that $\zeta(s,v,t)|_{\Gamma_{\mathcal{D}}}$ and $\zeta(S,v,t)|_{t=0}$ are positive. Further more, the coefficients $\tau_{i,j}$, $\delta_{i,j}$, $\tilde{\delta}_{i,j}$, a_j, $a_j + f_j$, $\alpha_{i,j}$, $\tilde{\alpha}_{i,j}$ (in the expression of $\boldsymbol{M}(t)$), are positive under condition* (24)*. Consequently, $\boldsymbol{M}(t)$ and $\boldsymbol{Q}_0$ are positive.*

Based upon [17, 18], we deduce that the system (22)-(23) has an analytic solution defined as follows:

$$\boldsymbol{Q}(t) = E_{\gamma,1}(Bt^{\gamma})\boldsymbol{Q}_0 + \chi_{[0,t]}\Big(t^{\gamma-1}E_{\gamma,\gamma}(Bt^{\gamma})\Big) * \chi_{[0,t]}\boldsymbol{M}(t), \tag{27}$$

where $\chi_{[0,t]}$ is the characteristic function of $[0,t]$, and $E_{\gamma,1}(\cdot)$, $E_{\gamma,\gamma}(\cdot)$ are the Mittag-Leffler functions and the symbol $(*)$ means the convolution product.

Lemma 5.6. *For $\rho = 0$ or for $\rho \in (-1,1)\backslash\{0\}$ and under the condition* (24)*, the numerical solution of the system* (22)-(23) *displayed by the proposed scheme is positive.*

Proof. For $\rho = 0$ or for $\rho \in (-1,1)\backslash\{0\}$ and under the condition (24), the matrix B is Metzler, (see Theorem 5.4). Consequently, we have:

$$E_{\gamma,1}(Bt^{\gamma}) \geq 0, \quad E_{\gamma,\gamma}(Bt^{\gamma}) \geq 0.$$

Knowing from Remark 5.5 that $\boldsymbol{Q}_0 \geq 0$ and $\boldsymbol{M}(t) \geq 0$, we infer that the solution $\boldsymbol{Q}(t)$ is positive. □

Theorem 5.7. *For any $\boldsymbol{Q}_0 \in \mathbb{R}^{N_{sv}}$ such that $\|\boldsymbol{Q}_0\|_\infty \leq \varrho$, ($\varrho > 0$), the solution $\boldsymbol{Q}(t)$ to the problem* (22)-(23) *is stable and satisfies the stability identity:*

$$\|\boldsymbol{Q}(t)\|_\infty \leq \varrho E_{\gamma,1}(\mathcal{M}t^{\gamma}) + \mathcal{E}t^{\gamma}E_{\gamma,\gamma}(\mathcal{M}t^{\gamma}), \quad \forall t \in [0,T], \tag{28}$$

where $\|B\|_\infty \leq \mathcal{M}$, $\mathcal{E} = \mathcal{E}(\frac{1}{\Delta s}, \frac{1}{\Delta v})$ and ϱ are constants independent of t. Moreover, we have:

$$\|\boldsymbol{Q}(t)\|_\infty \leq \mathcal{K}, \quad \forall t \in [0,T], \tag{29}$$

where $\mathcal{K} = \mathcal{K}(\frac{1}{\Delta s}, \frac{1}{\Delta v})$ is a constant independent of t.

Proof. Grounded on the expression of the analytic solution $\boldsymbol{Q}(t)$ exhibited in (27), we deduce that:

$$\begin{aligned}
\|\boldsymbol{Q}(t)\|_\infty &\leq \|\boldsymbol{Q}_0\|_\infty \|E_{\gamma,1}(Bt^{\gamma})\|_\infty + \|\chi_{[0,t]}\Big(t^{\gamma-1}E_{\gamma,\gamma}(Bt^{\gamma})\Big) * \chi_{[0,t]}\boldsymbol{M}(t)\|_\infty,\\
&\leq \varrho\|E_{\gamma,1}(Bt^{\gamma})\|_\infty + \|E_{\gamma,\gamma}(Bt^{\gamma})\|_\infty \|\chi_{[0,t]}\Big(t^{\gamma-1}\Big) * \chi_{[0,t]}\boldsymbol{M}(t)\|_\infty,\\
&\leq \varrho\|E_{\gamma,1}(Bt^{\gamma})\|_\infty + \|E_{\gamma,\gamma}(Bt^{\gamma})\|_\infty \|\boldsymbol{M}(t)\|_\infty \Big(\int_0^t s^{\gamma-1}\, ds\Big),
\end{aligned}$$

where $\chi_{[0,t]}$ is the characteristic function of $[0,t]$. It follows that:

$$\|\boldsymbol{Q}(t)\|_\infty \leq \varrho\|E_{\gamma,1}(Bt^{\gamma})\|_\infty + \frac{t^{\gamma}}{\gamma}\|E_{\gamma,\gamma}(Bt^{\gamma})\|_\infty \|\boldsymbol{M}(t)\|_\infty. \tag{30}$$

We know that:

$$E_{\gamma,1}(Bt^{\gamma}) = \sum_{k\geq 0} \frac{B^k t^{\gamma k}}{\Gamma(\gamma k + 1)}, \tag{31}$$

$$E_{\gamma,\gamma}(Bt^{\gamma}) = \sum_{k\geq 0} \frac{B^k t^{\gamma k}}{\Gamma(\gamma k + \gamma)}. \tag{32}$$

Thus, we deduce that:

$$\|E_{\gamma,1}(Bt^{\gamma})\|_{\infty} \leq \sum_{k\geq 0} \frac{t^{\gamma k}}{\Gamma(\gamma k+1)}\|B\|_{\infty}^{k}, \tag{33}$$

$$\|E_{\gamma,\gamma}(Bt^{\gamma})\|_{\infty} \leq \sum_{k\geq 0} \frac{t^{\gamma k}}{\Gamma(\gamma k+\gamma)}\|B\|_{\infty}^{k}. \tag{34}$$

Recall that:

$$\|B\|_{\infty} = \max_{k}\Big\{\sum_{\ell=1}^{N_{sv}} |a_{k,\ell}|\Big\}$$

Departing from the definition of the matrix B given in Remark 5.2 and under the condition (24), all the non-diagonal elements of the matrix B are positive and the diagonal elements are negative. Additionally, the maximum number of non-zero elements at all the rows of B is equal to eight elements. These elements are expressed as follows: $\tau_{i,j}$, $\delta_{i,j}$, $\tilde{\delta}_{i,j}$, a_j, $a_j + f_j$, $\alpha_{i,j}$, $\tilde{\alpha}_{i,j}$ and $-\epsilon_{i,i}$, (where $\epsilon_{i,i} > 0$). Hence, we can establish that:

$$\begin{aligned}\|B\|_{\infty} &= \epsilon_{i,i} + \Big(\tau_{i,j} + \delta_{i,j} + \tilde{\delta}_{i,j} + a_j + a_j + f_j + \alpha_{i,j} + \tilde{\alpha}_{i,j}\Big)_{i\neq j},\\ &= 4\tau_{i,j} + 2\gamma_i + 4a_j + 2f_j + r, \qquad \forall i = 0,\cdots,N_s,\ j = 0,\cdots,N_v.\end{aligned}$$

Referring to the definition of the coefficients $\tau_{i,j}$, γ_i, a_j, f_j, we can prove easily that there exists a constant $\mathcal{M} > 0$ such that:

$$\|B\|_{\infty} \leq \mathcal{M}. \tag{35}$$

Consequently, using the identities (33), (34) and (35), we get:

$$\|E_{\gamma,1}(Bt^{\gamma})\|_{\infty} \leq E_{\gamma,1}(\mathcal{M}t^{\gamma}) \quad \text{and} \quad \|E_{\gamma,\gamma}(Bt^{\gamma})\|_{\infty} \leq E_{\gamma,\gamma}(\mathcal{M}t^{\gamma}). \tag{36}$$

Based on the definition of the vector function $\boldsymbol{M}(t)$ as well as Remark 5.2, there exists a constant $\mathcal{E} = \mathcal{E}(\frac{1}{\Delta s}, \frac{1}{\Delta v})$ that is independent of t such that:

$$\|\boldsymbol{M}(t)\|_{\infty} \leq \mathcal{E}, \quad \forall t \in [0,T]. \tag{37}$$

In fact, the constant $\mathcal{E} = \mathcal{E}(\frac{1}{\Delta s}, \frac{1}{\Delta v})$ comes from the estimates made on each coefficient $\tau_{i,j}$, $\delta_{i,j}$, $\tilde{\delta}_{i,j}$, a_j, $a_j + f_j$, $\alpha_{i,j}$, $\tilde{\alpha}_{i,j}$ where each one depends on $\frac{1}{\Delta s}$ and $\frac{1}{\Delta v}$, (see Remark 5.2). Finally, the identity (28) can be obtained from (30), (36) and (37). Since the Mittag-Leffler functions $E_{\gamma,1}(\mathcal{M}t^{\gamma})$, $E_{\gamma,\gamma}(\mathcal{M}t^{\gamma})$ are bounded for all $t \in [0,T]$, then we can establish the identity (29). □

6. Numerical simulations and interpretations

In this section, we consider the time fractional-order derivative Heston model of order $\gamma \in (0,1)$ and the time integer-order derivative Heston model of order 1 indicated by:

$${}^{c}D_t^{\gamma}\zeta = \frac{1}{2}vs^2\frac{\partial^2\zeta}{\partial s^2} + \rho\sigma vs\frac{\partial^2\zeta}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2\zeta}{\partial v^2} + rs\frac{\partial\zeta}{\partial s} + \kappa(\theta - v)\frac{\partial\zeta}{\partial v} - r\zeta, \tag{38}$$

$$\frac{\partial\zeta}{\partial t} = \frac{1}{2}vs^2\frac{\partial^2\zeta}{\partial s^2} + \rho\sigma vs\frac{\partial^2\zeta}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2\zeta}{\partial v^2} + rs\frac{\partial\zeta}{\partial s} + \kappa(\theta - v)\frac{\partial\zeta}{\partial v} - r\zeta. \tag{39}$$

We consider the same boundary conditions as presented in (7)-(9).

Example:1– The parameters considered in this example are determined as:

$$\begin{aligned}&T = 0.25, \quad [s_l, s_r] = [0.25, 40], \quad [v_l, v_r] = [0.002, 1.2], \quad N_s = N_v = 30, \quad N_t = 100\\ &\gamma = 0.8, \quad K = 10, \quad \sigma = 0.2, \quad \rho = 0.001, \quad r = 0.1, \quad \kappa = 5.\end{aligned}$$

In this example, we investigate the numerical solution with respect to the stability condition (24) given in Theorem 5.4 and verify the positivity of the solution given in Lemma 5.6. Remark that the numerical solutions are in coherence with the theoretical results. Indeed, when the condition (24) is not verified $\theta = 0.1 < v_r = 1.2$ the solution is not positive and unstable see Figure 2. On the other hand, when (24) is verified $\theta = 2 > v_r = 1.2$ the solution is positive and stable see Figure 1.

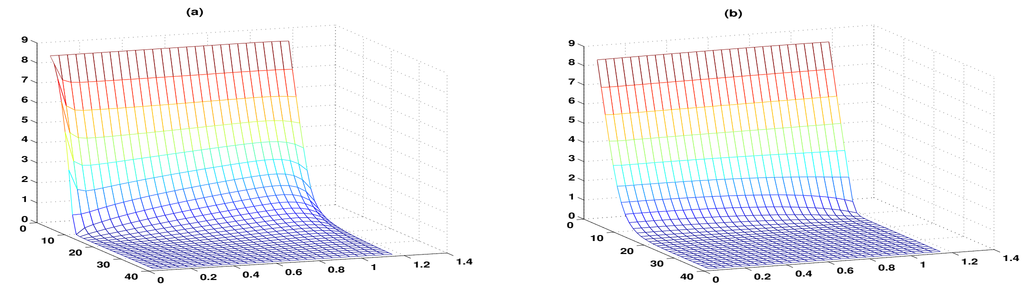

Figure 1: Solution with fractional derivative figure (a) versus solution with classical derivative figure (b): $\theta = 2 > v_r = 1.2$

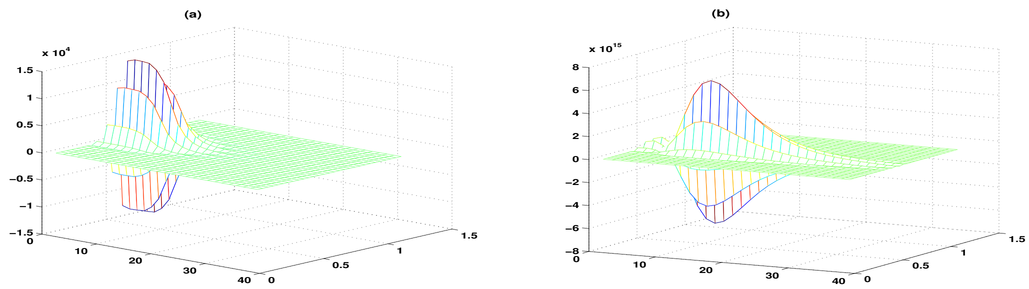

Figure 2: Solution with fractional derivative figure (a) versus solution with classical derivative figure (b): $\theta = 0.1 < v_r = 1.2$

Example:2– Notice that when $|\rho|$ is close to zero, the spatial mixed derivative term $\frac{\partial^2 \zeta}{\partial s \partial v}$ of equation (11) has a very weak influence on the numerical solution, (stable solution), see Figure 1 in Example:1. In fact, the problem arises for large values of $|\rho|$, (when $|\rho|$ is close to 1), which we do not find in literature [20], where the maximum value of ρ was equal to 0.7. In this paper, thanks to our numerical method, we managed to reach the value of $\rho = -0.9, 0.9$, see Figure 3.

$$T = 0.25, \quad [s_l, s_r] = [0.25, 40], \quad [v_l, v_r] = [0.002, 1.2], \quad N_s = N_v = 18, \quad N_t = 100$$
$$K = 10, \quad \sigma = 0.9, \quad \rho = -0.9, 0.9, \quad \theta = 2.0, \quad r = 0.1, \quad \kappa = 5.$$

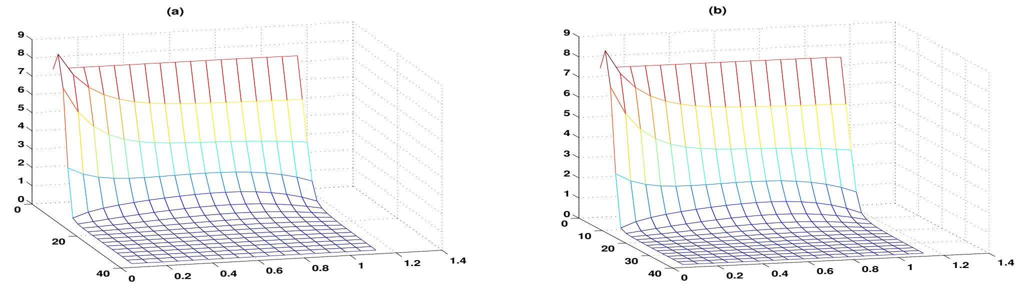

Figure 3: Solution with$\gamma = 0.9$: (a): $\rho = 0.9$, (b): $\rho = -0.9$.

Example:3– In this example, we examine the properties of the numerical solution by exploring the behavior of the first order partial derivatives $\Delta = \frac{\partial \zeta}{\partial s}$ and $\nu = \frac{\partial \zeta}{\partial v}$.

$$T = 0.25, \quad [s_l, s_r] = [0.25, 40], \quad [v_l, v_r] = [0.002, 1.2], \quad N_s = N_v = 30, \quad N_t = 100$$
$$\gamma = 0.8, \quad K = 10, \quad \sigma = 0.2, \quad \rho = 0.1, \quad \theta = 2, \quad r = 0.1, \quad \kappa = 5.$$

Notice that when S tends to zero, Δ is decreasing fast up to -1. On the other side, when $S \approx K$, Δ is increasing fast up to 0, see Figure 4. As expected, the put option price tends to zero for large asset price.

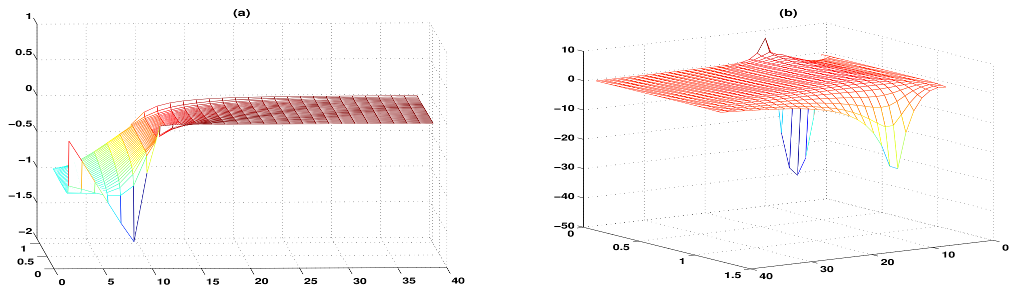

Figure 4: $\Delta = \frac{\partial \zeta}{\partial s}$ of the option (a) and $\nu = \frac{\partial \zeta}{\partial v}$ of the option (b).

7. Conclusion

In this research paper, we have studied the time fractional-order derivative of the pricing European options under Heston model. We found some positivity conditions for the solution obtained relative to the numerical methods used. Also, we were able to establish a stability result of the solution. Moreover, we elaborate a new resolution for the pricing European option under the fractional Heston model based on the splitting method. we have invested implicit time finite difference schemes. The intrinsic merit of these schemes lies in the fact that they are unconditionally stable. This new method allowed us to provide relevant numerical results in addition to what we found in literature confirming that the coefficient of correlation lies always between -0.7 and 0.7. With our new numerical method, we are able to extend the absolute value of the aforementioned coefficient to 0.9.
To corroborate the reliability of our results, we compared obtained results related to the fractional time derivative Heston model to the classical Hesston model and we deduced that all results are correlated with the options theory.

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