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# The Mexican hat wavelet Stieltjes transform

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**Abstract.** In the present article, we define the Mexican hat wavelet Stieltjes transform (MHWST) by applying the concept of Mexican hat wavelet transform [9]. The proposed transform serves as a centralized method to analyze both discrete and continuous time-frequency localization. Besides the formulation of all the fundamental results, a reconstruction formula is also obtained for MHWST. Further, a unified approach is applied to obtain the necessary and sufficient conditions for the same. Moreover, simplified construction for the jump operator is also presented for the Mexican hat wavelet Stieltjes transform.

## 1. Introduction

Using dilation parameter $a \in T_+ = (0, \infty)$ and translation parameter $b \in \mathbb{R}$, the wavelet $\psi_{b,a}(t)$ is given by

$$\psi_{b,a}(t) = (\sqrt{a})^{-1}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R}. \tag{1}$$

Now, for a square integrable function $\phi$, its wavelet transform with respect to the wavelet $\psi_{b,a}(t) \in L^2(\mathbb{R})$, is given by [7]

$$(W\phi)(b,a) = \int_{\mathbb{R}} \phi(t)\overline{\psi_{b,a}}(t)dt \ \text{ for } a \in T_+ \text{ and } t, b \in \mathbb{R}. \tag{2}$$

The inversion formula for (2) is given as follows:

$$\phi(x) = \frac{2}{C_\psi}\int_0^\infty \left[\int_{-\infty}^\infty \frac{1}{\sqrt{a}}(W\phi)(b,a)\psi\left(\frac{x-b}{a}\right)db\right]\frac{da}{a^2}, \quad x \in \mathbb{R}, \tag{3}$$

where

$$\frac{C_\psi}{2} = \int_0^\infty \frac{|\hat{\psi}(v)|^2}{|v|}dv = \int_0^\infty \frac{|\hat{\psi}(-v)|^2}{|v|}dv < \infty \qquad [2, \text{p. } 64].$$

Wavelet transform has been rising as a powerful mathematical tool used in symbolic calculus, approximation theory, Fourier series, and in the solution of boundary-value problems. It acts as a time-frequency

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localization operator that helps to identify the frequency in the temporary or spatial domain. Hence, wavelet transforms have various applications in wave propagation, computer graphics, data compression, image processing, and also in medical image technology. Recently, the theory of wavelets has seen remarkable advancement through the involvement of creative and efficient wavelet families. In [20, 21], Srivastava et al. formulated the operational matrices of integration for the family of Fibonacci wavelets and used them with the wavelet collocation method to solve the dual-phase lag bioheat transfer model and to obtain a solution for the non-linear Hunter–Saxton equation. Subsequently, by employing the approach of Dirac representation, a detailed theory of linear canonical wavelet transform in the framework of quantum mechanics was discussed in [17]. In [19], the authors investigated the characteristics and properties of some continuous and discrete fractional Bessel wavelet transforms. The relationship between the fractional Bessel wavelet transform and the fractional Hankel transform is also discussed in the same. The relation between the wavelet multipliers and localization operators associated with the integral representation is investigated in [18]. Srivastava et al. introduced the application of wavelet transform to efficiently localize any non-transient signal in the time-frequency plane with more degrees of freedom in the linear canonical domain [22–24]. The wavelet transform has also been comprehensively studied in many functions and distribution spaces and inversion formulae for the same are established in the sense of distributions (See, for example, Pathak [6, 7], Pathak and Singh [8–10], and Pandey [5]).

Lately, the family of Mexican hat wavelets has attained considerable attention from researchers working across various disciplines, mainly due to their several distinct characteristics. In addition to this, the Mexican hat wavelets are generated from the second derivative of the Gaussian function therefore, it is symmetrical and satisfies the Gaussian decays in both space and frequency, which helps to extract data in the space-frequency window. Therefore, ample research work has been devoted to studying the properties of Mexican hat wavelet transform in both classical [12] and distribution sense [9]. In [11], Rawat et al. studied the properties of the Mexican hat wavelet transform on generalized functions in $\mathcal{G}'$-space. More recently, Srivastava et al. [22] investigated a certain family of Mexican hat wavelet transforms and an isometry was achieved in the heat equation by implementing the theory of reproducing kernel.

The Mexican hat wavelet is defined by:

$$\psi(t) = exp\left(\frac{-t^2}{2}\right)(1-t^2) = -\frac{d^2}{dt^2}\, exp\left(\frac{-t^2}{2}\right). \tag{4}$$

Therefore,

$$\psi_{b,a}(t) = -a^{3/2} D_t^2 exp\left(-\frac{(b-t)^2}{2a^2}\right), \qquad \left(D_t = \frac{d}{dt}\right). \tag{5}$$

Hence, the wavelet transform with respect to (5) is given by

$$(W\phi)(b,a) = -a^{3/2}\int_{\mathbb{R}} \phi(t)\, D_t^2\, exp\left(-\frac{(b-t)^2}{2a^2}\right)dt, \qquad a \in T_+. \tag{6}$$

Now, under certain conditions on $\phi$, we have

$$(W\phi)(b,a) = -a^{3/2}\int_{\mathbb{R}} \phi^{(2)}(t)\, exp\left(-\frac{(b-t)^2}{2a^2}\right)dt, \qquad b \in \mathbb{R} \text{ and } a \in T_+. \tag{7}$$

Let for $a \in (0,\infty)$ and $b \in \mathbb{R}$, we define

$$k(b,a) = \frac{1}{\sqrt{2\pi a}} exp\left(\frac{-b^2}{2a}\right). \tag{8}$$

Clearly,

$$k^{(2)}(b-t,a^2) = D_t^2 k(b-t,a^2) = \frac{1}{\sqrt{2\pi}a} D_t^2\left(exp\left(\frac{-(b-t)^2}{2a^2}\right)\right). \tag{9}$$

Therefore by (5), we have

$$\psi_{b,a}(t) = -(2\pi)^{1/2} a^{5/2} k^{(2)}(b-t, a^2), \tag{10}$$

and hence the Mexican hat wavelet transform of $\phi(t)$ is defined by [9]

$$(W\phi)(b,a) = a^{3/2} \int_{\mathbb{R}} \phi^{(2)}(t) exp\left(\frac{-(b-t)^2}{2a^2}\right) dt, \qquad b \in \mathbb{R},\ a > 0. \tag{11}$$

In particular, the Mexican hat wavelet transform can be extended for complex values of the translation parameter, whenever required.

In this paper we concentrate on the family of Mexican hat wavelets associated with the Stieltjes transform. There are numerous results in the literature on Stieltjes transforms. To mention a few, Srivastava et al. [14] proved a Parseval Goldstein-type theorem involving a Stieltjes-type integral transform. The theorem is then shown to yield a number of new identities involving several well-known integral transforms (see also [13], [15] and [16]). Bielecki et al. [1] introduced Wavelet-Stieltjes transforms and compared them with properties of the classical continuous wavelet transform. Motivated essentially by the aforementioned concepts and theories given by Srivastava [14–16], Bielecki [1], Pathak [9], and other authors, our main objective in this article is to introduce and develop the theory of the Mexican hat wavelet Stieltjes transform.

In the next section, we present the definition of the Mexican hat wavelet Stieltjes transform, including the construction of some of its basic properties by using (11). In Section 3, a reconstructional formula is formulated for the proposed transform. The representation theory for functions as MHWST is developed in Section 4. Further, in the last section, we formulate the jump operator as an application of the introduced transform.

## 2. Mexican hat wavelet Stieltjes transform and its basic properties

In this section, we define MHWST of a real-valued function in the form of continuous and discrete time scales, which may serve as a centralized method to analyze both discrete and continuous time-frequency localization. Some basic ideas are known from the theory of wavelet Stieltjes transform [1].

**Definition 2.1.** *Let [1]*

$$BV(1,2) = \left\{ F | F : \mathbb{R} \to \mathbb{R},\ F(\cdot) = \int_{-\infty}^{\cdot} f(t)dt + \sum_{s \leq \cdot} \rho(s), \right.$$
$$\left. f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),\ \rho \in \tilde{l}_1 \cap \tilde{l}_2 \right\},$$

*where*

$$\tilde{l}_1 = \left\{ \rho | \rho : \mathbb{R} \to \mathbb{R}, \rho(s) \neq 0 \text{ at countable points and } \sum_{s \in \mathbb{R}} |\rho(s)| < \infty \right\}$$

*and*

$$\tilde{l}_2 = \left\{ \rho | \rho : \mathbb{R} \to \mathbb{R}, \rho(s) \neq 0 \text{ at countable points and } \sum_{s \in \mathbb{R}} |\rho(s)|^2 < \infty \right\}.$$

Here, $F$ is a function of bounded first variation. Its integral part is absolutely continuous and the sum part is the jump component. Also, without loss of generality, we assume

$$interior\,(supp\ f) \cap supp\ \rho = \emptyset,$$

such that

$$f(t)\rho(t) = 0, \quad -\infty < t < \infty.$$

**Definition 2.2.** *For $F \in BV(1,2)$, we have [1]*

$$\|F\|_1 := \int_{-\infty}^{\infty} |f(t)|dt + \sum_{s\in\mathbb{R}} |\rho(s)| \tag{12}$$

*and*

$$\|F\|_2 := \left[\int_{-\infty}^{\infty} |f(t)|^2 dt + \sum_{s\in\mathbb{R}} |\rho(s)|^2\right]^{1/2} \tag{13}$$

*where $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $BV(1,2)$.*

Here, the bounded variation function $F(t)$ is considered in every finite interval. Let $BV(2)$ denote the completion of $BV(1,2)$ with respect to the $\|\cdot\|_2$ norm given in (13) such that its inner prodduct is defined as follows [1]:

Let $F, G \in BV(2)$ and $F \sim (f,\rho),\ G \sim (g,\gamma)$ then

$$\langle F, G\rangle := \int_{-\infty}^{\infty} f(t)g(t)dt \ + \sum_{s\in(\text{supp }\rho)\cap(\text{supp }\gamma)} \rho(s)\gamma(s). \tag{14}$$

Now to define the Mexican hat wavelet Stieltjes transform let's consider the following class of functions [3]:

$$\Psi := \{\psi|\psi : \mathbb{R} \to \mathbb{C},\ \langle\psi, F\rangle < \infty,\ F, \psi \in BV(2)\}, \tag{15}$$

where

$$\langle F, \psi\rangle = \int_{-\infty}^{\infty} \psi(t)dF(t). \tag{16}$$

The last integral being Lebesgue-Stieltjes integral hence it is finite.

Let $h : \mathbb{R} \to \mathbb{R}$, then for some $a, b \in \mathbb{R}$, we define

$$h_{b,a}(t) = \begin{cases} h\left(\frac{t-b}{a}\right), & a \neq 0, \\ 0, & a = 0. \end{cases}$$

**Definition 2.3.** *The Mexican hat wavelet Stieltjes transform of bounded variation function F in every finite interval with kernel $\psi_{b,a}(t)$ is defined by*

$$\begin{aligned} (WS_\psi F)(b,a) &= \langle\psi_{b,a}, F\rangle \\ &= \sqrt{2\pi}a^{5/2}\int_{\mathbb{R}} k^{(2)}(b-t, a^2)dF(t), \end{aligned} \tag{17}$$

*such that the integral converges for all $b = \sigma + i\omega$. In fact, $(WS_\psi F)(b,a) \in L^2(\mathbb{R}^2, \mu)$, where $\mu(da, db) = |a|^{-3}dadb$, for $\psi \in \Psi$.*

**Theorem 2.4.** *If $f^{(2)}(t)$ is continuous and bounded in $-\infty < t < \infty$, then*

$$(WS_\psi F)(b,a) = \int_{-\infty}^{\infty} k^{(2)}(b-t, a^2)dF(t) = \int_{-\infty}^{\infty} k(b-t, a^2)dF^{(2)}(t).$$

*Proof.* From (9), we have

$$k^{(2)}(b-t,a^2)=\frac{d^2}{dt^2}k(b-t,a^2)=\frac{-1}{\sqrt{2\pi}a}\left\{1-\left(\frac{b-t}{a}\right)^2\right\}\exp\left(-\frac{1}{2}\left(\frac{b-t}{a}\right)^2\right).$$

Now, from (17), we get

$$\begin{aligned}(WS_\psi F)(b,a) &= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\left\{1-\left(\frac{b-t}{a}\right)^2\right\}\exp\left(-\frac{1}{2}\left(\frac{b-t}{a}\right)^2\right)dF(t)\\ &= -\sqrt{2a}\int_{-\infty}^{\infty}\exp(-v^2)\left\{1-2v^2\right\}\left\{dF(b-\sqrt{2}av)\right\} \quad \left(\text{putting } v=\frac{(b-t)}{\sqrt{2}a}\right)\\ &= \sqrt{2a}\int_{-\infty}^{\infty}\exp(-v^2)\left(\frac{1}{\sqrt{2}}\frac{d}{dv}\right)^2\left\{dF(b-\sqrt{2}av)\right\} \quad (\text{by } (4))\\ &= a^{3/2}\int_{-\infty}^{\infty}\exp\left(\frac{-(b-w)^2}{2a^2}\right)\left(\frac{d}{dw}\right)^2\{dF(w)\} \quad \left(\text{putting } b-\sqrt{2}av=w\right)\\ &= \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty}k(b-w,a^2)dF^{(2)}(w). \end{aligned} \tag{18}$$

□

**Theorem 2.5.** *If $k(b,a^2)$ is the function defined by (8) for $b\in\mathbb{R}$ and $a\in\mathbb{R}^+$, then*

*(i)* $k_{bb}(b,a^2)=\frac{1}{a}k_a(b,a^2)=\frac{b^2-a^2}{a^4}k(b,a^2)$

*(ii)* $|k(s,a^2)|\le\frac{A}{a}\exp\left(\frac{\omega^2-\sigma^2}{2a^2}\right),\ \ (s=\sigma+i\omega)$

*(iii)* $|k_{ss}(s,a^2)|\le\frac{A}{a^5}\exp\left(\frac{\omega^2-\sigma^2}{2a^2}\right)(|s|^2+a^2),$

*here $A$ is suitable constant and $s\in\mathbb{C}$.*

*Proof.* From (8), we have

$$k(b,a^2)=\frac{1}{\sqrt{2\pi}a}\exp\left(\frac{-b^2}{2a^2}\right). \tag{19}$$

By differentiating equation (19) with respect to $a$ and twice with respect to $b$, we get

$$k_a(b,a^2)=\frac{b^2-a^2}{a^4\sqrt{2\pi}}\exp\left(\frac{-b^2}{2a^2}\right)=\frac{b^2-a^2}{a^3}k(b,a^2). \tag{20}$$

$$k_{bb}(b,a^2)=\frac{b^2-a^2}{a^5\sqrt{2\pi}}\exp\left(\frac{-b^2}{2a^2}\right)=\frac{b^2-a^2}{a^4}k(b,a^2). \tag{21}$$

Then, condition (*i*) follows from (20) and (21).

Condition (*ii*), follows directly from the definition of $k(b,a^2)$. For $s$ being complex number, $s=\sigma+i\omega$

$$\begin{aligned}|k(s,a^2)| &= \left|\frac{1}{\sqrt{2\pi}a}\exp\left(\frac{-(\sigma+i\omega)^2}{2a^2}\right)\right|\\ &\le \frac{A}{a}\exp\left(\frac{\omega^2-\sigma^2}{2a^2}\right).\end{aligned}$$

Next, we consider

$$\begin{aligned}|k_{ss}(s,a^2)| &\leq \left|\frac{(s^2-a^2)}{a^4}\right||k(s,a^2)| && \text{[by (i)]}\\ &\leq \frac{A}{\sqrt{2\pi}a^5}\exp\left(\frac{(\omega^2-\sigma^2)}{2a^2}\right)(|s|^2+a^2) && \text{[by (ii)].}\end{aligned}$$

Thus, condition (*iii*) follows from (*i*) and (*ii*). □

## 3. A real inversion formula for the Mexican hat wavelet Stieltjes transform

In this section, the operator $\exp(-a^2D^2)$ is used to invert the MHWST to obtain the continuous bounded variation function.

**Theorem 3.1.** - *If*

*(I)* $\int_{-\infty}^{\infty} k(b_0-t,1)\phi^{(2)}(t)dt$ converges for some $b_0$

*(II)* $\int_x^t \left[\phi^{(2)}(v)-\phi^{(2)}(x)\right]dv = o\left(|t-x|\right) \qquad$ as $t\to x$,

*then*

$$\lim_{a^2\to 0^+}\int_{-\infty}^{\infty} k(x-t,a^2)\phi^{(2)}(t)dt = \phi^{(2)}(x). \tag{22}$$

*Proof.* Let us rewrite the integral (22) as the sum of two others with respect to $(-\infty,x)$ and $(x,\infty)$, i.e.,

$$\begin{aligned}\lim_{a^2\to 0^+} &\int_{-\infty}^{\infty} k(x-t,a^2)\phi^{(2)}(t)dt\\ &= \lim_{a^2\to 0^+}\left\{\int_{-\infty}^{x} k(x-t,a^2)\phi^{(2)}(t)dt + \int_{x}^{\infty} k(x-t,a^2)\phi^{(2)}(t)dt\right\}\\ &= \lim_{a^2\to 0^+}\left\{\int_{-x}^{\infty} k(x+t,a^2)\phi^{(2)}(-t)dt + \int_{x}^{\infty} k(x-t,a^2)\phi^{(2)}(t)dt\right\}\\ &= I_1(a)+I_2(a).\end{aligned}$$

Choose an arbitrary $\delta>0$, such that the integral $I_1(a)$ can be written as the sum of the two others, $I_3(a)$ and $I_4(a)$, corresponding to the intervals $(-x,-x+\delta)$ and $(-x+\delta,\infty)$. Set

$$\begin{aligned}\beta_1(v) &= \exp\left(\frac{-(v+x)^2}{2a^2}\right)\exp\left(\frac{(v+b_0)^2}{2}\right)\\ \alpha_1(v) &= \int_{-x+\delta}^{v}\exp\left(\frac{-(t+b_0)^2}{2}\right)\phi^{(2)}(-t)dt.\end{aligned}$$

Then $\alpha_1(-x+\delta)=0$ and $\alpha(+\infty)$ exists by hypothesis. Moreover, $\beta_1(v)$ is positive, continuous, non-increasing and $\beta_1(+\infty)=0$.
Then by Theorem 2.1 of [4], we have

$$\frac{1}{\sqrt{2\pi}a}\int_{-x+\delta}^{\infty}\beta_1(v)d\alpha_1(v) = \int_{-x+\delta}^{\infty}\phi^{(2)}(-t)k(x+t,a^2)dt. \tag{23}$$

Hence by (23), for small value of $a$, we have

$$\begin{aligned} I_4(a) &= \lim_{a^2\to 0^+} \frac{1}{\sqrt{2\pi}a} \int_{-x+\delta}^{\infty} \beta_1(v) d\alpha_1(v) \\ &= \lim_{a^2\to 0^+} \frac{-1}{\sqrt{2\pi}a} \int_{-x+\delta}^{\infty} \alpha_1(v) d\beta_1(v). \end{aligned}$$

Let $M$ be a constant not larger than $|\alpha_1(a)|$, then

$$\begin{aligned} |I_4(a)| &\leq \lim_{a^2\to 0^+} \frac{1}{\sqrt{2\pi}a} M \int_{-x+\delta}^{\infty} d[-\beta_1(v)] \\ &= \lim_{a^2\to 0^+} \frac{M}{\sqrt{2\pi}a} \left(-\beta_1(+\infty) + \beta_1(-x+\delta)\right) \\ &= \lim_{a^2\to 0^+} \frac{M}{\sqrt{2\pi}a} \beta_1(-x+\delta) \\ &\leq \lim_{a^2\to 0^+} \frac{M}{\sqrt{2\pi}a} \exp\left(\frac{-\delta^2}{2a^2}\right) \exp\left(\frac{(-x+\delta+b_0)^2}{2}\right) \\ &= o(1). \end{aligned}$$

Next, we consider

$$\begin{aligned} I_3(a) &= \lim_{a^2\to 0^+} \int_{-x}^{-x+\delta} \phi^{(2)}(-t) k(x+t, a^2) dt \\ &= \lim_{a^2\to 0^+} \frac{1}{\sqrt{2\pi}a} \int_{-x}^{-x+\delta} \exp\left(\frac{-(x+t)^2}{2a^2}\right) \phi^{(2)}(-t) dt. \end{aligned}$$

Let $k = a^{-2}$ and $h(t) = \dfrac{-(x+t)^2}{2}$, such that $h(-x) = 0,\ h'(-x) = 0$ and $h''(-x) = -1$. Then by Theorem 2b of [25, p. 278], we have

$$\begin{aligned} I_3(a) &= \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{2\pi}} \phi^{(2)}(x) \exp\left(kh(-x)\right) \left(\frac{-\pi}{2kh''(-x)}\right)^{1/2} \\ &= \frac{\phi^{(2)}(x)}{2}. \end{aligned} \tag{24}$$

Similarly, for $I_2(a)$ we choose positive $\delta$, such that it can be written as sum of other two integrals $I_5(a)$ and $I_6(a)$ with the intervals $(x, x+\delta)$ and $(x+\delta, \infty)$. Set

$$\begin{aligned} \beta_2(v) &= \exp\left(\frac{-(v-x)^2}{2a^2}\right) \exp\left(\frac{(v-b)^2}{2}\right) \\ \alpha_2(v) &= \int_{x+\delta}^{v} \exp\left(\frac{-(t-b_0)^2}{2}\right) \phi^{(2)}(t) dt, \end{aligned}$$

with $\alpha_2(x+\delta) = 0$, $\beta_2(+\infty) = 0$ and $\alpha_2(+\infty)$ exists. Then by simple computations, we get

$$I_6(a) = o(1).$$

For $I_5(a)$, consider $k = a^{-2}$ and $h(t) = \dfrac{-(x-t)^2}{2}$ such that it satisfies conditions of Theorem 2b of [25, p. 278]. Therefore

$$I_5(a) = \frac{\phi^{(2)}(x)}{2}. \tag{25}$$

Thus by (24) and (25), we have the desired result. □

The last theorem acts as a foundation for the inversion of the MHWST in the most general case. However, to implement it we require the next result.

**Theorem 3.2.** *If the Mexican hat wavelet Stieltjes transform*

$$(WS_{\psi}F)(b,1) = \int_{-\infty}^{\infty} k(b-t,1)dF^{(2)}(t) \tag{26}$$

*converges in the interval* $m < b < n$*, then for* $m < d < n$, $0 < a^2 < 1$, $-\infty < b < \infty$,

$$\begin{aligned} \exp(-a^2D^2)(WS_{\psi}F)(b,1) &= \int_{-\infty}^{\infty} k(t+ib,a^2)WS(it,1)dt \\ &= \int_{-\infty}^{\infty} k(b-u,1-a^2)dF^{(2)}(u). \end{aligned} \tag{27}$$

*Proof.* Choose two constants $\xi$ and $\eta$ such that $m < \xi < d < \eta < n$ and let

$$\begin{aligned} F^{(2)}(t) &= o\left(\exp\left(\frac{(t-\eta)^2}{2}\right)\right), \qquad t \to +\infty \\ &= o\left(\exp\left(\frac{(t-\xi)^2}{2}\right)\right), \qquad t \to -\infty. \end{aligned} \tag{28}$$

Now integrate (26) by parts and using (28), we get

$$(WS_{\psi}F)(b,1) = \int_{-\infty}^{\infty} k_1(b-t,1)F^{(2)}(t)dt,$$

where

$$k_1(b-t,a^2) = \frac{d}{db}k(b-t,a^2) = \frac{-(b-t)}{a^2}k(b-t,a^2).$$

Therefore,

$$\begin{aligned} \exp(-a^2D^2)(WS_{\psi}F)(b,1) &= \int_{-\infty}^{\infty} k(t+ib,a^2)dt \int_{-\infty}^{\infty} k_1(t-iu,1)F^{(2)}(u)du \\ &= \int_{-\infty}^{\infty} F^{(2)}(u)du \int_{-\infty}^{\infty} k(t+ib,a^2)k_1(t-iu,1)dt. \end{aligned} \tag{29}$$

By Theorem 2.5 of [4], for $-\infty < b < \infty$, $-\infty < u < \infty$, we have

$$\int_{-\infty}^{\infty} k(t+ib,a^2)k_1(t-iu,1)dt = k_1(b-u,1-a^2).$$

Therefore, (29) becomes

$$\exp(-a^2D^2)(WS_{\psi}F)(b,1) = \int_{-\infty}^{\infty} k_1(b-u,1-a^2)F^{(2)}(u)du. \tag{30}$$

Now integrating (30) by parts and using (28), we get

$$\exp(-a^2D^2)(WS_{\psi}F)(b,1) = \int_{-\infty}^{\infty} k(b-u,1-a^2)dF^{(2)}(u).$$

□

**Theorem 3.3.** *If $F(u)$ is normalized function of bounded variation in every finite interval and if*

$$(WS_\psi F)(b,1) = \int_{-\infty}^{\infty} k(b-u,1)dF^{(2)}(u),$$

*the integral converging for $m < b < n$, then for $m < d < n$ and any two real numbers $b_1,\ b_2$*

$$\begin{aligned}\int_{b_1}^{b_2} exp(-D^2)(WS_\psi F)(b,1)db &= \lim_{a^2\to 1-} \frac{1}{2\pi i}\int_{b_1}^{b_2} db \int_{d-i\infty}^{d+i\infty} k(s+b,a^2)WS(s,1)ds \\ &= F^{(2)}(b_2) - F^{(2)}(b_1).\end{aligned}$$

*Proof.* By Theorem 3.2, for $0 < a^2 < 1$ and $-\infty < b < \infty$

$$\begin{aligned}\exp(-a^2D^2)(WS_\psi F)(b,1) &= \int_{-\infty}^{\infty} k(b-u,1-a^2)dF^{(2)}(u) \\ &= \int_{-\infty}^{\infty} k_1(b-u,1-a^2)F^{(2)}(u)du. \qquad (31)\end{aligned}$$

By Theorem 2.5, the integral (31) converges uniformly for $b_1 \le b \le b_2$. Hence we have

$$\int_{b_1}^{b_2} \exp(-a^2D^2)(WS_\psi F)(b,1)db = \int_{b_1}^{b_2} db \int_{-\infty}^{\infty} k_1(b-u,1-a^2)F^{(2)}(u)du$$

$$= \int_{-\infty}^{\infty} k(b_2-u,1-a^2)F^{(2)}(u)du - \int_{-\infty}^{\infty} k(b_1-u,1-a^2)F^{(2)}(u)du.$$

Now by Theorem 3.1 (replacing $a^2$ by $1-a^2$) to each of these integrals, we get

$$\int_{b_1}^{b_2} \exp(-a^2D^2)(WS_\psi F)(b,1)db = F^{(2)}(b_2) - F^{(2)}(b_1).$$

□

## 4. Necessary and sufficient conditions for representing functions as Mexican hat wavelet Stieltjes transform

In this section, necessary and sufficient conditions on a function $(WS_\psi F)(b,a)$ are derived by applying temperature functions. Following are a few notations, defined for a class of functions to which MHWST will belong.

**Definition 4.1.** *A function $u(b,a)$ is said to be a solution of heat equation in a domain $D$ if and only if $u(b,a) \in C^2$ and $u_{bb}(b,a) = \frac{1}{a}u_a(b,a)$. Then $u(b,a) \in H$ in a closed region $R$, if $R$ can be enclosed in a domain in which $u(b,a) \in H$.*

**Definition 4.2.** *A function $(WS_\psi F)(b,a)$ in an interval $m < b < n$ belongs to a class $A$ if and only if it can be extended analytically into the complex plane as*

1. *$(WS_\psi F)(b+i\omega,a)$ is analytic in the strip $m < b < n$*
2. *$(WS_\psi F)(b+i\omega,a) = o\left(|\omega|e^{\frac{\omega^2}{2}}\right),\quad |\omega| \to \infty$, uniformly in every closed subinterval of $m < b < n$.*

**Theorem 4.3.** *If*

$$u(b,a) = \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} k(b-t,a^2)dF^{(2)}(y), \qquad (32)$$

*the integral converging in $0 < a^2 < c$, then $u(b,a) \in H$.*

*Proof.* Let for a fixed $a$ the integral (32) is the product of the entire function $\exp\left(\frac{-b^2}{2a^2}\right)$ by a Laplace transform which converges for all $b$. Hence $u(b,a)$ is an entire function of $b$, therefore, differentiation under integral sign is valid. Further, differentiating (32) twice with respect to $b$ gives

$$u_{bb}(b,a) = \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} k_{bb}(b-t,a^2)dF^{(2)}(t). \tag{33}$$

Again differentiating (32) with respect to $a$, we get

$$u_a(b,a) = \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} k_a(b-t,a^2)dF^{(2)}(t). \tag{34}$$

Simple application of Theorem 2.5, demonstrates that (33) and (34) are equal. This completes the proof. □

**Theorem 4.4.** *The conditions*

*(I)* $u(b,a) \in H, \qquad -\infty < b < \infty,\ 0 < a < c$

*(II)* $\|u(b,a)\|_1 < M, \qquad 0 < a < c$

*are necessary and sufficient that*

$$u(b,a) = \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} k(b-t,a^2)dF^{(2)}(t), \tag{35}$$

*where*

$$\int_{-\infty}^{\infty} |dF^{(2)}(t)| < M. \tag{36}$$

*Proof.* Let us first prove that ($I$) and ($II$) are necessary and sufficient conditions. Under assumption (36) the integral (35) converges absolutely in the half plane $a > 0$. Indeed

$$\begin{aligned} |u(b,a)| &= \left|\sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} k(b-t,a^2)dF^{(2)}(t)\right| \\ &\le a^{3/2}\int_{-\infty}^{\infty}\left|dF^{(2)}(t)\right| \\ &\le a^{3/2}M. \end{aligned} \tag{37}$$

Therefore by Theorem 4.3, $u(b,a) \in H$. Hence condition ($I$) is satisfied. Further, condition ($II$) is also satisfied since

$$\begin{aligned} \|u(b,a)\|_1 &= \int_{-\infty}^{\infty} |u(b,a)|db \\ &\le \sqrt{2\pi}a^{5/2}\int_{-\infty}^{\infty} db\int_{-\infty}^{\infty} k(b-t,a^2)|dF^{(2)}(t)| \\ &\le a^{3/2}\int_{-\infty}^{\infty}\exp\left(\frac{-(b-t)^2}{2a^2}\right)db\int_{-\infty}^{\infty} dF^{(2)}(t) \\ &\le \sqrt{2\pi}a^{5/2}M \le M, \qquad 0 < a < 1. \end{aligned} \tag{38}$$

Hence, both conditions are necessary. To show the converse part we consider

$$u_l(b,a) = \frac{1}{2l}\int_{b-l}^{b+l} u(y,a)dy, \qquad 0 < l \tag{39}$$

and since by direct calculation of the partial derivatives of $u_l$, we have

$$\begin{aligned}\frac{\partial^2}{\partial b^2}u_l(b,a) &= \frac{1}{a}\frac{\partial}{\partial a}u_l(b,a)\\ &= \frac{1}{2l}\left[u_b(b+l,a)-u_b(b-l,a)\right]. \qquad (40)\end{aligned}$$

Therefore, the function $u_l(b,a)$ belongs to class $H$ where $u(b,a)$ does. Moreover, by Holder's inequality

$$|u_l(b,a)| \leq \frac{1}{2l}\int_{-\infty}^{\infty}|u(y,a)|dy \leq \frac{M}{2l}$$

for $-\infty < b < \infty$, $0 < a < c$. Hence by Lemma 6.2 of [4],

$$u(b,a+\delta) = \int_{-\infty}^{\infty}k(b-y,a^2)u_l(y,\delta)dy \qquad (41)$$

for $0 < \delta < c$, $0 < a < c-\delta$, $-\infty < b < \infty$. Then by weak convergence theorem as $l \to 0^+$, we have

$$\lim_{l\to 0^+}u_l(b,a+\delta) \to u(b,a+\delta).$$

Therefore,

$$u(b,a) = \lim_{\delta\to 0}\int_{-\infty}^{\infty}k(b-y,a^2)u(y,\delta)dy.$$

This completes the proof of the theorem. □

The previous theorem leads to the following representation theorem for Mexican hat wavelet Stieltjes transform.

**Theorem 4.5.** *The conditions*

*(I)* $(WS_\psi F)(b,a) \in A, \qquad -\infty < b < \infty$

*(II)* $\|\exp(-a^2D^2)(WS_\psi F)(b,a)\|_1 \leq M, \qquad 0 < a < 1$

*are necessary and sufficient that*

$$(WS_\psi F)(b,a) = \int_{-\infty}^{\infty}k(b-t,1)dF^{(2)}(t), \qquad (42)$$

*where*

$$\int_{-\infty}^{\infty}|dF^{(2)}(t)| < M. \qquad (43)$$

*Proof.* If (43) true, then the integral (42) converges absolutely for all $b$ at $a = 1$. Hence, $(WS_\psi F)(b,a) \in A$ in $-\infty < b < \infty$. Moreover, by Theorem 3.2, we have

$$\exp(-a^2D^2)(WS_\psi F)(b,a) = \int_{-\infty}^{\infty}k(b-t,1-a^2)dF^{(2)}(t). \qquad (44)$$

Now, applying the conditions of Theorem 4.4 to (44), we obtain condition $(II)$

$$\begin{aligned}\|\exp(-a^2D^2)(WS_\psi F)(b,a)\| &\leq \int_{-\infty}^{\infty}db\int_{-\infty}^{\infty}|k(b-t,1-a^2)||dF^{(2)}(t)|\\ &\leq M. \qquad (45)\end{aligned}$$

Conversely, let

$$u(b,a) = \exp(-(1-a^2)D^2)(WS_\psi F)(b,a) \qquad \text{(by Theorem 3.2)} \tag{46}$$

such that $u(b,a) \in H$ in $-\infty < b < \infty$, $0 < a < 1$. Then by Theorem 4.4, the equation (46) becomes

$$u(b,a) = \int_{-\infty}^{\infty} k(b-t, a^2) dF^{(2)}(t),$$

where $F^{(2)}(t)$ satisfies (43). As stated before the integral is absolutely convergent for all $a > 0$, so by continuity

$$u(b, 1-) = \int_{-\infty}^{\infty} k(b-t, 1) dF^{(2)}(t).$$

Hence $u(b,a) \to (WS_\psi F)(b,a)$ as $a \to 1-$. Therefore, (42) holds and this completes the proof of the theorem. □

## 5. Jump operator for Mexican hat wavelet Stieltjes transform

Jump operator provides a unified approach to obtain a relation between the determining function at a point of discontinuity or at a given point in terms of the transform. It acts as an operator which gives $F^{(2)}(b+) - F^{(2)}(b-)$ in terms of $(WS_\psi F)(b,a)$ where $(WS_\psi F)(b,a)$ and $F^{(2)}(b)$ are related by (17). The following theorem gives representation of jump operator for MHWST.

**Theorem 5.1.** *Let $F^{(2)}(t)$ be of bounded variation in any finite interval and let Mexican hat wavelet Stieltjes transform defined by (17) is related to $F^{(2)}(y)$ and converges in $m < b < n$, then for $d$ satisfying $m < d < b$ and $-\infty < b < \infty$,*

$$\lim_{a^2 \to 1-} -i\sqrt{1-a^2} \int_{d-i\infty}^{d+i\infty} \exp\left(\frac{(s-b)^2}{2a^2}\right) WS(s,1) ds = F^{(2)}(b+) - F^{(2)}(b-).$$

*Proof.* By Theorem 3.2, we have

$$\begin{aligned}
\exp(-a^2D^2)(WS_\psi F)(b,1) &= \int_{-\infty}^{\infty} k(t+ib, a^2) WS(it,1) dt \\
&= \int_{-\infty}^{\infty} k_1(b-u, 1-a^2) F^{(2)}(u) du.
\end{aligned}$$

Consider,

$$\begin{aligned}
\int_{-\infty}^{\infty} k(t+ib, a^2) WS(it,1) dt &= \int_{-\infty}^{\infty} k(i^2 t - ib, a^2) WS(it,1) dt \\
&= -i^2 \int_{-\infty}^{\infty} k(i(it-b), a^2) WS(it,1) dt.
\end{aligned}$$

Let $t = -i(s-d)$, then

$$\begin{aligned}
& -i \int_{d-i\infty}^{d+i\infty} k(i(s-b), a^2) WS(s,1) ds \\
= \quad & \frac{-i}{\sqrt{2\pi}a} \int_{d-i\infty}^{d+i\infty} \exp\left(\frac{(s-b)^2}{2a^2}\right) WS(s,1) ds.
\end{aligned}$$

Therefore,

$$-i\sqrt{(1-a^2)}\int_{d-i\infty}^{d+i\infty}\exp\left(\frac{(s-b)^2}{2a^2}\right)WS(s,1)ds$$
$$= \sqrt{2\pi}a\sqrt{(1-a^2)}\int_{-\infty}^{\infty}k_1(b-u,1-a^2)F^{(2)}(u)du, \quad (47)$$

where

$$k_1(b-u,a^2)=\frac{\partial}{\partial b}k(b-u,a^2)=\frac{-(b-u)}{(2\pi)^{1/2}a^3}\exp\left(\frac{-(b-u)^2}{2a^2}\right).$$

Hence for some positive $\delta$, we have

$$\begin{aligned}&\lim_{a^2\to 1-}a(2\pi)^{1/2}(1-a^2)^{1/2}\int_{-\infty}^{\infty}k_1(b-u,1-a^2)F^{(2)}(u)du\\ =&\lim_{a^2\to 1-}a(2\pi)^{1/2}(1-a^2)^{1/2}\left\{\int_{-\infty}^{b-\delta}+\int_{b-\delta}^{b}+\int_{b}^{b+\delta}+\int_{b+\delta}^{\infty}\right\}k_1(b-u,1-a^2)F^{(2)}(u)du\\ =&\ I_1(a)+I_2(a)+I_3(a)+I_4(a).\end{aligned}$$

For $I_2(a)$, we can choose a $\delta>0$ so that $|F^{(2)}(u)-F^{(2)}(b-)|<\epsilon$ for $b-\delta<u<b$ and therefore,

$$\begin{aligned}|I_2(a)+F^{(2)}(b-)| &= \lim_{a^2\to 1-}\left|(2\pi)^{1/2}(1-a^2)^{1/2}\int_{b-\delta}^{b}k_1(b-u,1-a^2)F^{(2)}(u)du\right|\\ &\le \epsilon+\mathrm{o}(1).\end{aligned}$$

For $\epsilon$ being arbitrary, we have $I_2(a)\approx -F^{(2)}(b-)$.
Similarly $|I_3(a)-F^{(2)}(b+)|\le\epsilon+\mathrm{o}(1)$ as $a^2\to 1-$. Therefore, $I_3(a)\approx F^{(2)}(b+)$.

For $I_1(a)$ and $I_4(a)$ by Lemma 2.1c of [4], for some $\xi$ and $\eta$ such that $m<\xi<\eta<n$, at $a=1$

$$\begin{aligned}F^{(2)}(u) &= o\left[exp\left(\frac{(u-\eta)^2}{2}\right)\right], \quad u\to\infty,\\ F^{(2)}(u) &= o\left[exp\left(\frac{(u-\xi)^2}{2}\right)\right], \quad u\to\infty,\end{aligned}$$

and since $F^{(2)}(u)$ is locally of bounded variation

$$|F^{(2)}(u)|\le\begin{cases}M\exp\left(\dfrac{(u-\eta)^2}{2}\right), & u>x,\\ M\exp\left(\dfrac{(u-\xi)^2}{2}\right), & u<x.\end{cases}$$

Therefore,

$$\begin{aligned}|I_1(a)| &= \lim_{a^2\to 1-}\left|(2\pi)^{1/2}(1-a^2)^{1/2}\int_{-\infty}^{b-\delta}k_1(b-u,1-a^2)F^{(2)}(u)du\right|\\ &\le \lim_{a^2\to 1-}(1-a^2)^{-5/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)|F^{(2)}(u)|du\\ &\le \lim_{a^2\to 1-}M(1-a^2)^{-5/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)\exp\left(\frac{-(u-\xi)^2}{2}\right)du\\ &= \mathrm{o}(1).\end{aligned}$$

Hence, $I_1(a)=\mathrm{o}(1)$ and similarly $I_4(a)=\mathrm{o}(1)$ as $a^2\to 1-$, which concludes the proof of the theorem. □

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