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The reducible solution to a system of matrix equations over the Hamilton quaternion algebra

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Abstract. Reducible matrices are closely associated with the connection of directed graph and can be used in stochastic processes, biology and others. In this paper, we investigate the reducible solution to a system of matrix equations over the Hamilton quaternion algebra. We establish the necessary and sufficient conditions for the system to have a reducible solution and derive a formula of the general reducible solution of the system when it is solvable. Finally, we present a numerical example to illustrate the main results of this paper.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real number field and the set of all $m \times n$ matrices over $\mathbb{H}$, where

$$\mathbb{H} = \{u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ u_0, u_1, u_2, u_3 \in \mathbb{R}\}.$$

$\mathbb{H}$ is called the Hamilton quaternion algebra. $r(A)$, I and 0 are denoted by the rank of a given quaternion matrix A, an identity matrix and a zero matrix of appropriate sizes, respectively. The Moore-Penrose inverse of $A \in \mathbb{H}^{l\times k}$ is denoted by $A^{\dagger} = K$, which is defined as $AKA = A,\ KAK = K,\ (AK)^* = AK$ and $(KA)^* = KA$. Further, we define $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$.

In 1843, William Rowan Hamilton discoved quaternions. It is well known that the quaternion algebra is an associative noncommutative division ring, which is widely used in computer science, orbital mechanics, signal and color image processing, and control theory (see, e.g. [4], [28], [29], [35]).

A square quaternion matrix X is said to be reducible, if there exists a permutation matrix K such that

$$X = K\begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}K^{-1},$$

where X_1 and X_3 are square matrices with suitable dimensions. If the order of X_3 is k $(1 \le k < n)$, we call X to be k-reducible concerning the permutation matrix K. For an any but fixed permutation matrix K, we put

$$\mathbb{H}_k^{n\times n} = \left\{ X = K\begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}K^{-1} \middle| 1 \le k < n, X_1 \in \mathbb{H}^{(n-k)\times(n-k)}, X_3 \in \mathbb{H}^{k\times k} \right\}.$$

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We know that linear matrix equations are one of the active research topics in matrix theory and applications (see, e.g., [6], [13], [14], [11], [18], [19], [20], [25], [46], [38], [39], [40], [41]). They have many applications in singular system control [32], system design [33], perturbation theory [21], sensitivity analysis [3] and so on. A large number of papers have presented several approaches to solve some linear matrix equations (see, e.g., [1], [2], [5], [7], [10], [15], [16], [31], [37], [42], [44]). For example, the system of matrix equations

$$AX = C, XB = D, \tag{1}$$

the classic linear matrix equation

$$AZB = C, \tag{2}$$

and the system

$$A_1Z = C_1, A_2ZB_2 = C_2 \tag{3}$$

have been investigated by a crowd of papers for different kinds of solutions. Li, Hu and Zhang [22] gave a generalized reflexive solution of system (1). Qiu and Wang [30] established the least-squares solution of system (1). Zhang [47] investigated the Hermitian and positive solutions of system (1). Nie, Wang and Zhang [26] considered the k-reducible solution of system (1). In 2003, Liao and Bai [15] presented the least-squares solution to (2) over symmetric positive semidefinite matrices. Huang, Yin and Guo [9] provided the skew-symmetric solution and the optimal approximate solution of the matrix equation (2). Peng [27] derived the centrosymmetric solution of matrix equation (2). Xie and Wang [36] studied the reducible solution of equation (2). Wang [43] established some solvability conditions and the general solution to system (3) over von Neumann regular rings. Wang [45] gave the k-reducible solution of the system (3) over $\mathbb{H}$. In 2013, He and Wang [8] considered some necessary and sufficient conditions for the system

$$A_5ZB_5 = C_5, A_6ZB_6 = C_6, A_7ZB_7 = G \tag{4}$$

to have a solution and derive a formula of its general solution when it is solvable. To our best knowledge, so far, there has been little information on the reducible solution to system (4). This paper aims to investigate the reducible solution to system (4). It is well-known that reducible matrices are closely related to the connection of directed graphs and can be used in compartmental analysis, continuous-time positive systems, stochastic processes, biology, and others (see, e.g., [12], [23], [26], [34]).

Motivated by the work mentioned above and the wide applications of reducible matrices, matrix equations and the quaternions. This paper aims to consider the reducible solution to system (4) over $\mathbb{H}$.

The rest of this paper is organized as follows. In Section 2, we make some preliminaries. In Section 3, we give some necessary and sufficient conditions for system (4) to have a solution $Z \in \mathbb{H}_k^{n\times n}$ and present the expression of this solution in terms of Moore-Penrose inverses and rank equalities of the quaternion matrices involved. We also design a numerical example to illustrate the main results of this paper. Finally, we give a brief conclusion to close this paper in Section 4.

2. Preliminaries

In this section, we review some results on quaternion matrices and quaternion matrix equations which are going to used in the next.

Marsaglia (1974) [24] described the following, which is available over $\mathbb{H}$.

Lemma 2.1. *[24] Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$ be given. Then we have the following rank equality:*

$$r\begin{pmatrix} A & BL_D \\ R_EC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

Lemma 2.2. *[43] Let A_1, A_2, B_2, C_1 and C_2 be provided for matrices with adequate shapes, $A_3 = A_2L_{A_1}$. Then the following statements are equivalent:*

(1) *System* (3) *has a solution.*

(2) $R_{A_1}C_1 = 0,\ R_{A_3}(C_2 - A_2A_1^{\dagger}C_1B_2) = 0,\ C_2L_{B_2} = 0.$

(3) $r(A_1,\ C_1) = r(A_1),\ r\begin{pmatrix} A_1 & C_1B_2 \\ A_2 & C_2 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},\ r\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = r(B_2).$

In this case, the general solution of system (3) *can be expressed as*

$$Z = A_1^{\dagger}C_1 + L_{A_1}A_3^{\dagger}(C_2 - A_2A_1^{\dagger}C_1B_2)B_2^{\dagger} + L_{A_1}L_{A_3}Q_1 + L_{A_1}Q_2R_{B_2}.$$

where U_1, U_2 and U_3 are any matrices over $\mathbb{H}$ with appropriate dimensions.

Lemma 2.3. *[17] Consider the quaternion matrix equation*

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B \tag{5}$$

where A_i, B_i and B ($i = \overline{1,4}$) are given quaternion matrices and the others are unknown quaternion matrices with appropriate sizes. Put

$$\begin{aligned}
&R_{A_1}A_2 = A_{11},\ R_{A_1}A_3 = A_{22},\ R_{A_1}A_4 = A_{33},\ B_2L_{B_1} = B_{11},\ B_{22}L_{B_{11}} = N_1,\\
&B_3L_{B_1} = B_{22},\ B_4L_{B_1} = B_{33},\ R_{A_{11}}A_{22} = M_1,\ S_1 = A_{22}L_{M_1},\ R_{A_1}BL_{B_1} = T_1,\\
&C = R_{M_1}R_{A_{11}},\ C_1 = CA_{33},\ C_2 = R_{A_{11}}A_{33},\ C_3 = R_{A_{22}}A_{33},\ C_4 = A_{33},\\
&D = L_{B_{11}}L_{N_1},\ D_1 = B_{33},\ D_2 = B_{33}L_{B_{22}},\ D_3 = B_{33}L_{B_{11}},\ D_4 = B_{33}D,\\
&E_1 = CT_1,\ E_2 = R_{A_{11}}T_1L_{B_{22}},\ E_3 = R_{A_{22}}T_1L_{B_{11}},\ E_4 = T_1D,\\
&C_{11} = (L_{C_2},\ L_{C_4}),\ D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\ C_{22} = L_{C_1},\ D_{22} = R_{D_2},\ C_{33} = L_{C_3},\\
&D_{33} = R_{D_4},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}},\\
&M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_2 - F_1,\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M.\\
&F_{11} = C_2L_{C_1},\ G_1 = E_2 - C_2C_1^{\dagger}E_1D_1^{\dagger}D_2,\ F_{22} = C_4L_{C_3},\ G_2 = E_4 - C_4C_3^{\dagger}E_3D_3^{\dagger}D_4,\\
&F_1 = C_1^{\dagger}E_1D_1^{\dagger} + L_{C_1}C_2^{\dagger}E_2D_2^{\dagger},\ F_2 = C_3^{\dagger}E_3D_3^{\dagger} + L_{C_3}C_4^{\dagger}E_4D_4^{\dagger}.
\end{aligned}$$

Then following statements are equivalent:

(1) *Equation* (5) *is consistent.*

(2)

$$R_{C_i}E_i = 0,\ E_iL_{D_i} = 0\ (i = \overline{1,4}),\ R_{E_{22}}EL_{E_{33}} = 0.$$

(3)

$$\begin{aligned}
&r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(A_2,\ A_3,\ A_4,\ A_1),\\
&r\begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_4,\ A_1) + r\begin{pmatrix} B_3 \\ B_1 \end{pmatrix},\\
&r\begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_3,\ A_4,\ A_1) + r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix},\\
&r\begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_1 \end{pmatrix} + r(A_4,\ A_1),
\end{aligned}$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \\ B_1 \end{pmatrix},$$

$$r\begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_2,\ A_1),$$

$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \\ B_1 \end{pmatrix} + r(A_3,\ A_1),$$

$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix} + r(A_1),$$

$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & B_1 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}.$$

In this case, the general solution to equation (5) *can be expressed as*

$$X_1 = A_1^{\dagger}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4) - A_1^{\dagger}U_1B_1 + L_{A_1}U_2,$$
$$X_2 = R_{A_1}(B - A_2Y_1B_2 - A_3Y_2B_3 - A_4Y_3B_4)B_1^{\dagger} + A_1A_1^{\dagger}U_1 + U_3R_{B_1},$$
$$Y_1 = A_{11}^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_{11}^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_{22}B_{11}^{\dagger} - A_{11}^{\dagger}S_1U_4R_{N_1}B_{22}B_{11}^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_{11}},$$
$$Y_2 = M_1^{\dagger}TB_{22}^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_{22}} + L_{M_1}U_4R_{N_1},$$
$$Y_3 = F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\ \text{or}\ Y_3 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},$$

where $T = T_1 - A_{33}Y_3B_{33}$, $U_i (i = \overline{1,8})$ *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$,

$$V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right],$$
$$W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right],$$
$$W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},$$
$$W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},$$

$U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ *and* U_{42} *are arbitrary matrices with appropriate sizes over* $\mathbb{H}$.

3. The reducible solution to system (4) over $\mathbb{H}$

In this section, we give the necessary and sufficient conditions for the system (4) to have a reducible solution and derive an expression of the solution $Z \in \mathbb{H}_k^{n\times n}$ to (4).

Theorem 3.1. *Let* $E_1 \in \mathbb{H}^{m_1\times(n-k)}$, $E_4 \in \mathbb{H}^{m_1\times k}$, $F_4 \in \mathbb{H}^{k\times k}$, $E_2 \in \mathbb{H}^{m_2\times(n-k)}$, $E_3 \in \mathbb{H}^{m_2\times k}$, $F_2 \in \mathbb{H}^{(n-k)\times(n-k)}$, $F_1 \in \mathbb{H}^{k\times(n-k)}$, $F_3 \in \mathbb{H}^{k\times(n-k)}$, $C_4 \in \mathbb{H}^{m_1\times k}$, $C_2 \in \mathbb{H}^{m_2\times(n-k)}$, $C_3 \in \mathbb{H}^{m_2\times k}$, $C_1 \in \mathbb{H}^{m_1\times(n-k)}$ *and* $A \in \mathbb{H}^{(n-k)\times k}$ *be known.* $K \in \mathbb{H}^{n\times n}$ *is a permutation matrix,* $1 \le k < n$. I_1, I_2 *denote the identity matrices of order* $n-k$ *and* k, *respectively. Put*

$$A_5K = (E_1,\ E_4),\ K^{-1}B_5 = \begin{pmatrix} I_1 & 0 \\ 0 & F_4 \end{pmatrix}, \tag{6}$$

$$A_6K = (E_2,\ E_3),\ K^{-1}B_6 = \begin{pmatrix} F_2 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{7}$$

$$A_7K = (F_1,\ I_2),\ K^{-1}B_7 = \begin{pmatrix} I_1 \\ F_3 \end{pmatrix}, \tag{8}$$

$$\begin{aligned} &E_5 = E_2L_{E_1},\ E_6 = E_4L_{E_3},\ F_1L_{E_1}L_{E_5} = A_1,\ F_1L_{E_1} = A_2,\ C_5 = (C_1,\ C_4 - E_1AF_4),\\ &L_{E_3}L_{E_6} = A_3,\ L_{E_3} = A_4,\ R_{F_2} = B_2,\ F_3 = B_3,\ R_{F_4}F_3 = B_4,\ C_6 = (C_2,\ C_3 - E_2A),\\ &B = G - F_1AF_3 - \left(F_1E_1^{\dagger}C_1 + F_1L_{E_1}E_5^{\dagger}(C_2 - E_2E_1^{\dagger}C_1F_2)F_2^{\dagger} + E_3^{\dagger}(C_3 - E_2A)F_3\right)\\ &- \left(L_{E_3}E_6^{\dagger}(C_4 - E_1AF_4 - E_4E_3^{\dagger}(C_3 - E_2A)F_4)F_4^{\dagger}F_3\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &R_{A_1}A_2 = A_{11},\ R_{A_1}A_3 = A_{22},\ R_{A_1}A_4 = A_{33},\ R_{A_{11}}A_{22} = M_1,\ S_1 = A_{22}L_{M_1},\ R_{A_1}B = T_1,\\ &C = R_{M_1}R_{A_{11}},\ H_1 = CA_{33},\ H_2 = R_{A_{11}}A_{33},\ H_3 = R_{A_{22}}A_{33},\ H_4 = A_{33},\ B_3L_{B_2} = N_1,\\ &D = L_{B_2}L_{N_1},\ D_1 = B_4,\ D_2 = B_4L_{B_3},\ D_3 = B_4L_{B_2},\ D_4 = B_4D,\\ &G_1 = CT_1,\ G_2 = R_{A_{11}}T_1L_{B_3},\ G_3 = R_{A_{22}}T_1L_{B_2},\ G_4 = T_1D,\\ &C_{11} = (L_{H_2},\ L_{H_4}),\ D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix},\ C_{22} = L_{H_1},\ D_{22} = R_{D_2},\ C_{33} = L_{H_3},\\ &D_{33} = R_{D_4},\ E_{11} = R_{C_{11}}C_{22},\ E_{22} = R_{C_{11}}C_{33},\ E_{33} = D_{22}L_{D_{11}},\ E_{44} = D_{33}L_{D_{11}},\\ &M = R_{E_{11}}E_{22},\ N = E_{44}L_{E_{33}},\ F = F_6 - F_5,\ E = R_{C_{11}}FL_{D_{11}},\ S = E_{22}L_M,\\ &F_{11} = H_2L_{H_1},\ L_1 = G_2 - H_2H_1^{\dagger}G_1D_1^{\dagger}D_2,\ F_{22} = H_4L_{H_3},\ L_2 = G_4 - H_4H_3^{\dagger}G_3D_3^{\dagger}D_4,\\ &F_5 = H_1^{\dagger}G_1D_1^{\dagger} + L_{H_1}H_2^{\dagger}G_2D_2^{\dagger},\ F_6 = H_3^{\dagger}G_3D_3^{\dagger} + L_{H_3}H_4^{\dagger}G_4D_4^{\dagger}. \end{aligned} \tag{10}$$

Then the following statements are equivalent:

(i) *System* (4) *has a solution* $Z \in \mathbb{H}_k^{n\times n}$.

(ii)

$$\begin{aligned} &R_{E_1}C_1 = 0,\ R_{E_5}(C_2 - E_2E_1^{\dagger}C_1F_2) = 0,\ C_2L_{F_2} = 0,\ R_{E_3}(C_3 - E_2A) = 0,\\ &R_{E_6}(C_4 - E_1AF_4 - E_4E_3^{\dagger}(C_3 - E_2A)F_4) = 0,\ (C_4 - E_1AF_4)L_{F_4} = 0, \end{aligned} \tag{11}$$

$$R_{H_i}G_i = 0,\ G_iL_{D_i} = 0(i = \overline{1,4}),\ R_{E_{22}}EL_{E_{33}} = 0. \tag{12}$$

(iii)

$$r(E_1,\ C_1) = r(E_1),\ r\begin{pmatrix} E_1 & C_1F_2 \\ F_2 & C_2 \end{pmatrix} = r\begin{pmatrix} E_1 \\ E_2 \end{pmatrix},\ r\begin{pmatrix} C_2 \\ F_2 \end{pmatrix} = r(F_2), \tag{13}$$

$$r(E_3,\ C_3 - E_2A) = r(E_3),\ r\begin{pmatrix} E_3 & (C_3 - E_2A)F_4 \\ F_4 & C_4 - E_1AF_4 \end{pmatrix} = r\begin{pmatrix} E_3 \\ E_4 \end{pmatrix},\ r\begin{pmatrix} C_4 \\ F_4 \end{pmatrix} = r(F_4), \tag{14}$$

$$r\begin{pmatrix} C_1 & E_1 \\ C_3F_3 - E_3(G - F_1AF_3) - E_2AF_3 & -E_3F_1 \end{pmatrix} = r\begin{pmatrix} E_1 \\ -E_3F_1 \end{pmatrix}, \tag{15}$$

$$r\begin{pmatrix} F_3 & 0 \\ C_1 & E_1 \\ E_3G & E_3F_1 \end{pmatrix} = r(F_3) + r\begin{pmatrix} E_1 \\ E_3F_1 \end{pmatrix}, \tag{16}$$

$$r\begin{pmatrix} E_3F_1 & E_3GF_2 - (C_3 - E_2A)F_3F_2 \\ E_1 & C_1F_2 + E_1AF_3F_2 \\ E_2 & C_2 \end{pmatrix} = r\begin{pmatrix} E_3F_1 \\ E_1 \\ E_2 \end{pmatrix}, \tag{17}$$

$$r\begin{pmatrix} 0 & F_3F_2 \\ E_3F_1 & E_3GF_2 \\ E_1 & C_1F_2 \\ E_2 & C_2 \end{pmatrix} = r\begin{pmatrix} E_3F_1 \\ E_1 \\ E_2 \end{pmatrix} + r(F_3F_2), \tag{18}$$

$$r\begin{pmatrix} F_3 & 0 & F_4 \\ C_1 + E_1AF_3 & E_1 & 0 \\ C_3F_3 - E_2F_3 - E_3G & -E_3F_1 & 0 \\ E_4G & E_4F_1 & C_4 - E_1AF_4 \end{pmatrix} = r(F_3,\ F_4) + r\begin{pmatrix} E_1 \\ -E_3F_1 \\ E_4F_1 \end{pmatrix}, \tag{19}$$

$$r\begin{pmatrix} G & F_1 \\ F_3 & 0 \\ C_1 & E_1 \end{pmatrix} = r\begin{pmatrix} F_1 \\ E_1 \end{pmatrix} + r(F_3), \tag{20}$$

$$r\begin{pmatrix} 0 & 0 & F_4 \\ E_3F_1 & E_3GF_2 - C_3F_3F_2 - (E_3F_1 - E_2)AE_3F_2 & 0 \\ E_4F_1 & E_4GF_3 - F_1AF_3F_2 & E_1AF_4 - C_4 \\ E_1 & C_1F_2 & 0 \\ E_2 & C_2 & 0 \end{pmatrix} = r\begin{pmatrix} E_3F_1 \\ E_4F_1 \\ E_1 \\ E_2 \end{pmatrix} + r(F_3F_2,\ F_4), \tag{21}$$

$$r\begin{pmatrix} F_1 & GF_2 \\ 0 & F_3F_2 \\ E_1 & C_1F_2 \\ E_2 & C_2 \end{pmatrix} = r\left(F_3F_2\right) + r\begin{pmatrix} F_1 \\ E_1 \\ E_2 \end{pmatrix}, \tag{22}$$

$$r\begin{pmatrix} F_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_3F_2 & F_4 \\ C_1 & E_1 & 0 & 0 & 0 \\ E_3G & E_3F_1 & E_3F_1 & E_3GF_2 + E_2AF_3F_2 - C_3F_3F_2 & E_3F_1AF_4 \\ E_4G & E_4F_1 & E_4F_1 & E_4GF_2 & C_4 - E_1AF_4 + E_4F_1AF_4 \\ 0 & 0 & E_1 & C_1F_2 & 0 \\ 0 & 0 & E_2 & C_2 & 0 \\ E_3G & E_3F_1 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_3 & 0 & 0 \\ 0 & F_3F_2 & F_4 \end{pmatrix} + r\begin{pmatrix} E_1 & 0 \\ E_3F_1 & E_3F_1 \\ E_4F_1 & E_4F_1 \\ 0 & E_1 \\ 0 & E_2 \end{pmatrix}. \tag{23}$$

In this case, a reducible solution Z of system (4) *with respect to K can be expressed as*

$$Z = K\begin{pmatrix} X & A \\ 0 & Y \end{pmatrix}K^{-1}, \tag{24}$$

where $X \in \mathbb{H}^{(n-k)\times(n-k)}$, $Y \in \mathbb{H}^{k\times k}$,

$$\begin{aligned}
&X = E_1^{\dagger}C_1 + L_{E_1}E_5^{\dagger}(C_2 - E_2E_1^{\dagger}C_1F_2)F_2^{\dagger} + L_{E_1}L_{E_5}Q_1 + L_{E_1}Q_2R_{F_2},\\
&Y = E_3^{\dagger}(C_3 - E_2A) + L_{E_3}E_6^{\dagger}(C_4 - E_1AF_4 - E_4E_3^{\dagger}(C_3 - E_2A)F_4)F_4^{\dagger} + L_{E_3}L_{E_6}Q_3 + L_{E_3}Q_4R_{F_4},\\
&Q_1 = A_1^{\dagger}(B - A_2Q_2B_2 - A_3Q_3B_3 - A_4Q_4B_4) - A_1^{\dagger}U_1B_1 + L_{A_1}U_2,\\
&Q_2 = A_{11}^{\dagger}TB_2^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_2^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_3B_2^{\dagger} - A_{11}^{\dagger}S_1U_4R_{N_1}B_3B_2^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_2},\\
&Q_3 = M_1^{\dagger}TB_3^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_3} + L_{M_1}U_4R_{N_1},\\
&Q_4 = F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\ \text{or}\ V_2 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4},
\end{aligned} \tag{25}$$

where $T = T_1 - A_{33}Q_4B_{33}$, $U_i (i = 1, ..., 8)$ *are any matrices with the fit dimensions,*

$$\begin{aligned}
&V_1 = (I_m,\ 0)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right],\\
&W_1 = (0,\ I_m)\left[C_{11}^{\dagger}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}^{\dagger}U_{11}D_{11} + L_{C_{11}}U_{12}\right],\\
&W_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} 0\\ I_n\end{pmatrix},\\
&V_2 = \left[R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^{\dagger} + C_{11}C_{11}^{\dagger}U_{11} + U_{21}R_{D_{11}}\right]\begin{pmatrix} I_n\\ 0\end{pmatrix},\\
&V_3 = E_{11}^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}E_{22}M^{\dagger}FE_{33}^{\dagger} - E_{11}^{\dagger}SE_{22}^{\dagger}FN^{\dagger}E_{44}E_{33}^{\dagger} - E_{11}^{\dagger}SU_{31}R_NE_{44}E_{33}^{\dagger} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},\\
&W_3 = M^{\dagger}FE_{44}^{\dagger} + S^{\dagger}SE_{22}^{\dagger}FN^{\dagger} + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\end{aligned}$$

where $U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}$ *and* U_{42} *are any matrices with the suitable dimensions.*

Proof. $(i) \Leftrightarrow (ii)$:

Substituting (24) into the system (4) yields

$$A_5K\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}K^{-1}B_5 = C_5,\ A_6K\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}K^{-1}B_6 = C_6,\ A_7K\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}K^{-1}B_7 = G, \tag{26}$$

where $X \in \mathbb{H}^{(n-k)\times(n-k)}$, $Y \in \mathbb{H}^{k\times k}$ $A \in \mathbb{H}^{(n-k)\times k}$. It follows from (6), (7) and (8) that the system (26) is equivalent to

$$\begin{aligned}
&(E_1,\ E_4)\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}\begin{pmatrix} I_1 & 0\\ 0 & F_4\end{pmatrix} = C_5,\\
&(E_2,\ E_3)\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}\begin{pmatrix} F_2 & 0\\ 0 & I_2\end{pmatrix} = C_6,\\
&(F_1,\ I_2)\begin{pmatrix} X & A\\ 0 & Y\end{pmatrix}\begin{pmatrix} I_1\\ F_3\end{pmatrix} = G,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&E_1X = C_1,\ E_4YF_4 = C_4 - E_1AF_4,\\
&E_3Y = C_3 - E_2A,\ E_2XF_2 = C_2,\\
&\quad F_1X + YF_3 = G - F_1AF_3.
\end{aligned} \tag{27}$$

Thus, system (4) has a solution $Z \in \mathbb{H}_k^{n\times n}$ is equivalent to (27) is consistent for X and Y.

We divided the system (27) into the following:

$$\begin{aligned}
&\qquad E_1X = C_1,\ E_2XF_2 = C_2,\\
&E_3Y = C_3 - E_2A,\ E_4YF_4 = C_4 - E_1AF_4,
\end{aligned} \tag{28}$$

$$F_1X + YF_3 = G - F_1AF_3. \tag{29}$$

We want to show that system (28) and equation (29) have a common solution if and only if (*ii*) holds or (*iii*) holds. The outline of the proof is as follows: We first prove that system (28) and equation (29) have a common solution if and only if (*ii*) holds and the general common solution to (28) and (29) has the form of (25); We then show that (*ii*) ⇔ (*iii*).

We now assume system (28) and (29) have a common solution $(X,\ Y)$. By Lemma 2.2, it follows from (28) that (11) holds and

$$\begin{aligned} X &= E_1^{\dagger}C_1 + L_{E_1}E_5^{\dagger}(C_2 - E_2E_1^{\dagger}C_1F_2)F_2^{\dagger} + L_{E_1}L_{E_5}Q_1 + L_{E_1}Q_2R_{F_2},\\ Y &= E_3^{\dagger}C_3 - E_3^{\dagger}E_2A + L_{E_3}E_6^{\dagger}(C_4 - E_1AF_4 - E_4E_3^{\dagger}(C_3 - E_2A)F_4)F_4^{\dagger} + L_{E_3}L_{E_6}Q_3 + L_{E_3}Q_4R_{F_4}, \end{aligned} \tag{30}$$

where $Q_i(i = \overline{1,4})$ are any matrices with the suitable dimensions over $\mathbb{H}$. Substituting (30) into (29) yields

$$A_1Q_1 + A_2Q_2B_2 + A_3Q_3B_3 + A_4Q_4B_4 = B, \tag{31}$$

where $A_i,\ B_i\ (i = \overline{1,4})$ and B are defined by (9). According to Lemma 2.3, we have from (31) that (12) holds and

$$\begin{aligned} Q_1 &= A_1^{\dagger}(B - A_2Q_2B_2 - A_3Q_3B_3 - A_4Q_4B_4) - A_1^{\dagger}U_1B_1 + L_{A_1}U_2,\\ Q_2 &= A_{11}^{\dagger}TB_2^{\dagger} - A_{11}^{\dagger}A_{22}M_1^{\dagger}TB_2^{\dagger} - A_{11}^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger}B_3B_2^{\dagger} - A_{11}^{\dagger}S_1U_4R_{N_1}B_3B_2^{\dagger} + L_{A_{11}}U_5 + U_6R_{B_2},\\ Q_3 &= M_1^{\dagger}TB_3^{\dagger} + S_1^{\dagger}S_1A_{22}^{\dagger}TN_1^{\dagger} + L_{M_1}L_{S_1}U_7 + U_8R_{B_3} + L_{M_1}U_4R_{N_1},\\ Q_4 &= F_1 + L_{C_2}V_1 + V_2R_{D_1} + L_{C_1}V_3R_{D_2},\ or\ Q_4 = F_2 - L_{C_4}W_1 - W_2R_{D_3} - L_{C_3}W_3R_{D_4}, \end{aligned} \tag{32}$$

where $T = T_1 - A_{33}Q_4B_{33}$, $U_i(i = 1, ..., 8)$ are any matrices with the fit dimensions over $\mathbb{H}$. Hence, we have shown that if (28) and (29) have a common solution, then all equalities of (*ii*) are satisfied and X and Y can be expressed as (25).

Conversely, suppose that (*ii*) holds, for any X, Y of the form (25), it is easy to verify from (11) that X and Y satisfy the system (28). Let $Q_i\ (i = \overline{1,4})$ be expressed as (32). According to (12), we have that $Q_i\ (i = \overline{1,4})$ satisfy (31). Note X and Y can be expressed as (25), we easily get that (29) holds. Hence, X and Y having the form of (25) are a common solution of system (28) and (29) under the hypothesis (*ii*). To sum up, system (28) and equation (29) have a common solution if and only if (*ii*) holds and the general solution to (28) and (29) have the form of (25), i.e., system (4) has a solution $Z \in \mathbb{H}_k^{n\times n}$ if and only if (*ii*) holds.

(*ii*) ⇔ (*iii*) : We now show that (*ii*) ⇔ (*iii*). It follows from Lemma 2.2 that (11) are equivalent to (13) and (14). We turn to prove that (12) holds if and only if (15) to (23) hold. By Lemma 2.3, we have that (12) are equivalent to

$$r\begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \end{pmatrix} = r(A_2,\ A_3,\ A_4,\ A_1), \tag{33}$$

$$r\begin{pmatrix} B & A_2 & A_4 & A_1\\ B_3 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_4,\ A_1) + r\begin{pmatrix} B_3 \end{pmatrix}, \tag{34}$$

$$r\begin{pmatrix} B & A_3 & A_4 & A_1\\ B_2 & 0 & 0 & 0 \end{pmatrix} = r(A_3,\ A_4,\ A_1) + r\begin{pmatrix} B_2 \end{pmatrix}, \tag{35}$$

$$r\begin{pmatrix} B & A_4 & A_1\\ B_2 & 0 & 0\\ B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2\\ B_3 \end{pmatrix} + r(A_4,\ A_1), \tag{36}$$

$$r\begin{pmatrix} B & A_2 & A_3 & A_1\\ B_4 & 0 & 0 & 0 \end{pmatrix} = r(A_2,\ A_3,\ A_1) + r\begin{pmatrix} B_4 \end{pmatrix}, \tag{37}$$

$$r\begin{pmatrix} B & A_2 & A_1\\ B_3 & 0 & 0\\ B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3\\ B_4 \end{pmatrix} + r(A_2,\ A_1), \tag{38}$$

$$r\begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_4 \end{pmatrix} + r(A_3,\ A_1), \tag{39}$$

$$r\begin{pmatrix} B & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_3 \\ B_4 \end{pmatrix} + r(A_1), \tag{40}$$

$$r\begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_4 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_3 & 0 \\ 0 & B_2 \\ B_4 & B_4 \end{pmatrix} + r\begin{pmatrix} A_2 & A_1 & 0 & 0 & A_4 \\ 0 & 0 & A_3 & A_1 & A_4 \end{pmatrix}, \tag{41}$$

respectively. Therefore, we need to prove that (15) to (23) hold if and only if (33) to (41) hold. Let that

$$\begin{aligned} X_0 &= E_1^{\dagger}C_1 + L_{E_1}E_5^{\dagger}(C_2 - E_2E_1^{\dagger}C_1F_2)F_2^{\dagger}, \\ Y_0 &= E_3^{\dagger}(C_3 - E_2A) + L_{E_3}E_6^{\dagger}(C_4 - E_1AF_4 - E_4E_3^{\dagger}(C_3 - E_2A)F_4)F_4^{\dagger}. \end{aligned}$$

Then it is easy to check that X_0, Y_0 satisfy

$$\begin{aligned} E_1X_0 = C_1,\ E_2X_0F_2 &= C_2, \\ E_3Y_0 = C_3 - E_2A,\ E_4Y_0F_4 &= C_4 - E_1AF_4. \end{aligned} \tag{42}$$

By (9), we have that $B = G - F_1AF_3 - F_1X_0 - Y_0F_3$. It follows from Lemma 2.1 and (42) that

$$\begin{aligned}
&(33) \Leftrightarrow r\begin{pmatrix} B & F_1L_{E_1} & L_{E_3}L_{E_6} & L_{E_3} & F_1L_{E_1}L_{E_5} \end{pmatrix} \\
&= r\begin{pmatrix} F_1L_{E_1} & L_{E_3}L_{E_6} & L_{E_3} & F_1L_{E_1}L_{E_5} \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} B & F_1 & I \\ 0 & E_1 & 0 \\ 0 & 0 & E_3 \end{pmatrix} = \begin{pmatrix} F_1 & I \\ E_1 & 0 \\ 0 & E_3 \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} C_1 & E_1 \\ C_3F_3 - E_2AF_3 + E_3F_1AF_3 - E_3G & -E_3F_1 \end{pmatrix} = r\begin{pmatrix} E_1 \\ -E_3F_1 \end{pmatrix} \Leftrightarrow (15), \\
&(40) \Leftrightarrow r\begin{pmatrix} B & F_1L_{E_1}L_{E_5} \\ R_{F_2} & 0 \\ F_3 & 0 \\ R_{F_4}F_3 & 0 \end{pmatrix} = r\begin{pmatrix} R_{F_2} \\ F_3 \\ R_{F_4}F_3 \end{pmatrix} + r(F_1L_{E_1}L_{E_5}) \\
&\Leftrightarrow r\begin{pmatrix} B & F_1 & 0 & 0 \\ I & 0 & F_2 & 0 \\ F_3 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & E_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & F_2 & 0 \\ F_3 & 0 & 0 \\ F_3 & 0 & F_4 \end{pmatrix} + r\begin{pmatrix} F_1 \\ E_1 \\ E_2 \end{pmatrix} \\
&\Leftrightarrow r\begin{pmatrix} F_1 & GF_2 \\ 0 & F_3F_2 \\ E_1 & C_1F_2 \\ E_2 & C_2 \end{pmatrix} = r\begin{pmatrix} F_3F_2 \end{pmatrix} + r\begin{pmatrix} F_1 \\ E_1 \\ E_2 \end{pmatrix} \Leftrightarrow (22).
\end{aligned}$$

Similarly, we have that (34) to (39) hold if and only if (16) to (21) hold.

$$(41) \Leftrightarrow r\begin{pmatrix} B & F_1L_{E_1} & F_1L_{E_1}L_{E_5} & 0 & 0 & 0 & L_{E_3} \\ F_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & L_{E_3}L_{E_6} & F_1L_{E_1}L_{E_5} & L_{E_3} \\ 0 & 0 & 0 & R_{F_2} & 0 & 0 & 0 \\ R_{F_4}F_3 & 0 & 0 & R_{F_4}F_3 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_3 & 0 \\ 0 & R_{F_2} \\ R_{F_4}F_3 & R_{F_4}F_3 \end{pmatrix} + r\begin{pmatrix} F_1L_{E_1} & F_1L_{E_1}L_{E_5} & 0 & 0 & L_{E_3} \\ 0 & 0 & L_{E_3}L_{E_6} & F_1L_{E_1}L_{E_5} & L_{E_3} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} F_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_3F_2 & F_4 \\ C_1 & E_1 & 0 & 0 & 0 \\ E_3G & E_3F_1 & E_3F_1 & E_3GF_2 + E_2AF_3F_2 - C_3F_3F_2 & E_3F_1AF_4 \\ E_4G & E_4F_1 & E_4F_1 & E_4GF_2 & C_4 - E_1AF_4 + E_4F_1AF_4 \\ 0 & 0 & E_1 & C_1F_2 & 0 \\ 0 & 0 & E_2 & C_2 & 0 \\ E_3G & E_3F_1 & 0 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} F_3 & 0 & 0 \\ 0 & F_3F_2 & F_4 \end{pmatrix} + r\begin{pmatrix} E_1 & 0 \\ E_3F_1 & E_3F_1 \\ E_4F_1 & E_4F_1 \\ 0 & E_1 \\ 0 & E_2 \end{pmatrix} \Leftrightarrow (23).$$

□

Now, we give an example to verify the main results of this paper.
Example 3.2 For system (4), we consider case of $n = 4$ and $k = 2$. Let

$$A_5 = (\mathbf{i},\ 1,\ \mathbf{j},\ \mathbf{k}),\ A_5 = (1,\ \mathbf{i},\ \mathbf{i},\ \mathbf{j}),\ A_7 = \begin{pmatrix} \mathbf{j} & \mathbf{k} & 0 & 1 \\ 0 & \mathbf{i} & 1 & 0 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{i} & \mathbf{j} \\ 0 & 0 & \mathbf{k} & 1 \end{pmatrix},\ B_6 = \begin{pmatrix} \mathbf{i} & \mathbf{j} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ B_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \mathbf{i} & \mathbf{k} \\ 0 & \mathbf{j} \end{pmatrix},\ K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

According to (6)-(8), we get, noting $K^{-1} = K$, that

$$E_1 = \begin{pmatrix} \mathbf{i} & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} \mathbf{i} & 1 \end{pmatrix},\ E_3 = \begin{pmatrix} \mathbf{j} & \mathbf{i} \end{pmatrix},\ E_4 = \begin{pmatrix} \mathbf{j} & \mathbf{i} \end{pmatrix},$$

$$F_1 = \begin{pmatrix} \mathbf{k} & \mathbf{j} \\ \mathbf{i} & 0 \end{pmatrix},\ F_2 = \begin{pmatrix} 1 & 0 \\ \mathbf{i} & \mathbf{j} \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{i} & \mathbf{k} \end{pmatrix},\ F_4 = \begin{pmatrix} \mathbf{k} & 1 \\ \mathbf{i} & \mathbf{j} \end{pmatrix},\ G = \begin{pmatrix} \mathbf{j} - 2 & -\mathbf{i} - 2\mathbf{j} \\ -1 - \mathbf{j} & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -1 & 0 \end{pmatrix},\ C_2 = \begin{pmatrix} -1 & 0 \end{pmatrix},\ C_3 = \begin{pmatrix} \mathbf{i} & 0 \end{pmatrix},\ C_4 = \begin{pmatrix} -\mathbf{j} & -\mathbf{i} \end{pmatrix},\ A = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{i} & 0 \end{pmatrix}.$$

It is easy to compute that (13)-(23) are satisfied and the 2-reducible solution

$$Z = K\begin{pmatrix} X & A \\ 0 & Y \end{pmatrix}K^{-1}$$
$$= \begin{pmatrix} 0 & 0 & \mathbf{i} & 0 \\ \mathbf{i} & 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \end{pmatrix},$$

where $X = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & \mathbf{i} \\ 0 & 0 \end{pmatrix}$.

4. Conclusion

We have established the necessary and sufficient conditions for the system (4) to have a solution $Z \in \mathbb{H}_k^{n\times n}$ and give an expression of this solution of the system. We also have designed a numerical example to illustrate the main result of this paper. It is worthy to see that the results in this paper are also available for both the real number filed and the complex number field. Moreover, the results of this paper can be generalized to the corresponding system of quaternion tensor equations.

References

[1] Z.Z. Bai, On preconditioned iteration methods for complex linear systems, J. Engrg. Math. 93 (2015) 41-60.
[2] J.K. Baksalary, R. Kala, The matrix equations $AX - YB = C$, Linear Algebra Appl. 25 (1979) 41-43.
[3] A. Barraud, S. Lesecq, N. Christov, From sensitivity analysis to random floating point arithmetics-application to Sylvester equations, Numerical Analysis and its Applications Lecture Notes in Computer Science (1998) 351.
[4] N.L. Bihan, J. Mars, Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing, Signal Process. 84(7) (2004) 1177-1199.
[5] H.X. Chang, Q.W. Wang, G.J. Song, (R, S)-conjugate solution to a pair of linear matrix equations, Appl. Math. Comput. 217 (2010) 73-82.
[6] D.S. Cvetković-Ilić, J.N. Radenković, Q.W. Wang, Algebraic conditions for the solvability to some systems of matrix equations, Linear Multilinear Algebra 69 (2021) 1579-1609.
[7] M. Dehghan, M. Hajarian, On the generalized reflexive and anti-reflexive solutions to a system of matrix equations, Linear Algebra Appl. 437 (2012) 2793-2812.
[8] Z.H. He, Q.W. Wang, The general solutions to some systems of matrix equations, Linear Multilinear Algebra 63(10) (2015) 2017-2032.
[9] G.X. Huang, F. Yin, K. Guo, An iterative method for the skew-symmetric solution and the optimal approximate solution of the matrix equation $AXB = C$, J. Comput Appl. Math, 212(2) (2008) 231-244.
[10] Z.H. He, Q.W. Wang, A system of periodic discrete-time coupled Sylvester quaternion matrix equations, Algebra Colloq. 24(1) (2017) 169-180.
[11] Z.H. He, Q.W. Wang, Y. Zhang, A system of quaternary coupled Sylvester-type real quaternion matrix equations, Automatica 87 (2018) 25-31.
[12] S.J. Kirkland, M. Neumann, J. H. Xu, Transition matrices for well-conditioned Markov chains, Linear Algebra Appl. 424(1) (2007) 118-131.
[13] I. Kyrchei, Cramers rules for Sylvester quaternion matrix equation and its special cases, Adv. Appl. Clifford Algebras 28 (2018) 1-26.
[14] I. Kyrchei, Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations, Linear Algebra Appl. 438 (2018) 136-152.
[15] A.P. Liao, Z.Z. Bai, The constrained solutions of two matrix equations, Acta Math. Sinica (English Series) 18(4) (2002) 671-678.
[16] A.P. Liao, Z.Z. Bai, Least-squares solution of $AXB = D$ over symmetric positive semidefinite matrices X, J. Comput. Math. 21 (2003) 175-182.
[17] L.S. Liu, Q.W. Wang, M.S. Mehany, A Sylvester-type Hamilton quaternion matrix equation with an application, Mathematics 10 (2022) 1758.
[18] L. S. Liu, Q.W. Wang, J.F. Chen, Y.Z. Xie, An exact solution to a quaternion matrix equation with an application, Symmetry 14(2) (2022) 375.
[19] X. Liu, The η-anti-Hermitian solution to some classic matrix equations, Appl. Math. Comput. 320 (2018) 264-270.

[20] X. Liu, G.J. Song, Y. Zhang, Determinantal representations of the solutions to systems of generalized sylvester equations, Adv. Appl. Clifford Algebras 12 (2019) 1-19.
[21] R.C. Li, A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory, SIAM J. Matrix Anal. Appl. 21(2) (1999) 44-45.
[22] F.L. Li, X.Y. Hu, L. Zhang, The generalized reflexive solution for a class of matrix equations ($AX = C; XB = D$), Acta Math. (Series B) 28(1) (2008) 185-193.
[23] J. Z. Lei, C. Y. Wang, On the reducibility of compartmental matrices, Comput Biol Med 38(8) (2008) 881-885.
[24] G. Marsaglia, G.P. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269-292.
[25] M.S. Mehany, Q.W. Wang, Three symmetrical systems of coupled sylvester-like quaternion matrix equations, Symmetry 14 (2022) 550.
[26] X.R. Nie, Q.W. Wang, Y. Zhang, A system of matrix equations over the quaternion algebra with applications, Algebra Colloq. 24(2) (2017) 233-253.
[27] Z.Y. Peng, The centro-symmetric solutions of linear matrix equation $AXB = C$ and its optimal approximation, Chinese J. Eng. Math. 20(6) (2003) 60-64.
[28] S.C. Pei, C.M. Cheng, Color image processing by using binary quaternion-moment-preserving thresholding techniqe, IEEE Trans. Image Process 8 (1999) 614-628.
[29] L. Qi, Z.Y. Luo, Q.W. Wang, X.Z. Zhang, Quaternion matrix optimization: motivation and analysis, J. Optim. Theory Appl. https://doi.org/10.1007/s10957-021-01906-y.
[30] Y.Y. Qiu, A.D. Wang, Least squares solutions to the equations $AX = B, XC = D$ with some constraints, Appl. Math. Comput. 204 (2008) 872-880.
[31] W.E. Roth, The equations $AX - YB = C$ and $AX - XB = C$ in matrices, Proc. Amer. Math. Soc. 3 (1952) 392-396.
[32] A. Shahzad, B.L. Jones, E.C. Kerrigan, G.A. Constantinides, An efficient algorithm for the solution of a coupled sylvester equation appearing in descriptor systems, Automatica 47 (2011) 24-48.
[33] V.L. Syrmos, F.L. Lewis, Coupled and constrained Sylvester equations in system design, Circ. Syst. Signal Pr. 13(6) (1994) 66-94.
[34] P. Santesso, M. E. Valcher, On the zero pattern properties and asymptotic behavior of continuous-time positive system trajectories, Linear Algebra Appl. 425 (2007) 283-302.
[35] C.C. Took, D.P. Mandic, Augmented second-order statistics of quaternion random signals, Signal Process. 91 (2011) 214-224.
[36] M.Y. Xie, Q.W. Wang, The reducible solution to a quaternion tensor equation, Front. Math. China 15(5) (2020) 1047-1070.
[37] L. Wang, Q.W. Wang, Z.H. He, The common solution of some matrix equations, Algebra Colloq. 23 (2016) 71-81.
[38] Q.W.Wang, Z.H. He, Y. Zhang, Constrained two-sided coupled Sylvester-type quaternion matrix equations, Automatica 101 (2019) 207-213.
[39] Q.W. Wang, A. Rehman, Z.H. He, Y. Zhang, Constrained generalized Sylvester matrix equations, Automatica 69 (2016) 60-64.
[40] Q.W. Wang, Z.H. He, Solvability conditions and general solution for mixed Sylvester equations, Automatica 49 (2013) 2713-2719.
[41] Q.W. Wang, Z.H. He, Systems of coupled generalized Sylvester matrix equations, Automatica 50 (2014) 2840-2844.
[42] Q.W. Wang, J.H. Sun, S.Z. Li, Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra, Linear Algebra Appl. 353 (2002) 169-182.
[43] Q.W. Wang. A system of four matrix equations over von neumann regular rings and its applications, Acta Math. Sinica, English Series 21 (2015) 323-334.
[44] Q.W. Wang, Z.H. He, Y. Zhang, Constrained two-sided coupled Sylvester-type quaternion matrix equations, Automatica 101 (2019) 207-213.
[45] H.H. Wang, The solutions of a quaternion tensor equations and its applications, Master, University of Shanghai, Shang, H. July (2019).
[46] Y. Zhang, R.H. Wang, The exact solution of a system of quaternion matrix equations involving η-Hermicity, Appl. Math. Comput. 222 (2013) 201-209.
[47] Q. Zhang, Common Hermitian and positive solutions to the adjointable operator equations $AX = C, XB = D$, Linear Algebra Appl. 429 (2008) 1-11.