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On Weil like functors on flag vector bundles with given length

Miroslav Doupovec[a], Jan Kurek[b], Włodzimierz M. Mikulski[c]

[a]*Institute of Mathematics, Brno University of Technology, FSI VUT Brno, Technická 2, Brno, Czech Republic*
[b]*Institute of Mathematics, Maria Curie Sklodowska University, pl. Marii Curie Sklodowskiej 1, Lublin, Poland*
[c]*Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348, Cracow, Poland*

Abstract. Let $\kappa \geq 2$ be a fixed natural number. The complete description is given of the product preserving gauge bundle functors F on the category $\mathcal{F}_\kappa\mathcal{VB}$ of flag vector bundles $K = (K; K_1, \dots, K_\kappa)$ of length κ in terms of the systems $I = (I_1, \dots, I_{\kappa-1})$ of A-module homomorphisms $I_i : V_{i+1} \to V_i$ for Weil algebras A and finite dimensional (over $\mathbf{R}$) A-modules $V_1, \dots, V_\kappa$. The so called iteration problem is investigated. The natural affinors on FK are classified. The gauge-natural operators C lifting κ-flag-linear (i.e. with the flow in $\mathcal{F}_\kappa\mathcal{VB}$) vector fields X on K to vector fields $C(X)$ on FK are completely described. The concept of the complete lift $\mathcal{F}\varphi$ of a κ-flag-linear semi-basic tangent valued p-form φ on K is introduced. That the complete lift $\mathcal{F}\varphi$ preserves the Frölicher-Nijenhuis bracket is deduced. The obtained results are applied to study prolongation and torsion of κ-flag-linear connections.

1. Introduction

We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth (of class C^∞).

Definition 1.1. *A flag vector bundle of length κ is a system $K = (K; K_1, \dots, K_\kappa)$ of a vector bundle $K = (K, \pi, M)$ together with vector sub-bundles $K_i = (K_i, \pi_i, M)$ of K for $i = 1, \dots, \kappa$ such that $K_1 \subset K_2 \subset \cdots \subset K_\kappa = K$. We call M the basis of K. If $K' = (K'; K'_1, \dots, K'_\kappa)$ is another flag vector bundle, a flag vector bundle map $K \to K'$ is a vector bundle map $f : K \to K'$ such that $f(K_i) \subset K'_i$ for $i = 1, \dots, \kappa$.*

We have the trivial flag vector bundle $K = (K; K_1, \dots, K_\kappa)$, where $K = (\mathbf{R}^m \times \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_\kappa}, \pi, \mathbf{R}^m)$ and $K_i = (\mathbf{R}^m \times \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_i} \times \mathbf{R}^0 \times \cdots \times \mathbf{R}^0, \pi_i, \mathbf{R}^m)$ for $i = 1, \dots, \kappa$. We will denote this trivial flag vector bundle by $\mathbf{R}^{m;n_1,\dots,n_\kappa}$.

Any flag vector bundle $K = (K; K_1, \dots, K_\kappa)$ with the basis M is locally trivial. It means that there are integers $m, n_1, \dots, n_\kappa$ such that for any $x \in M$ there is an open neighborhood $\Omega \subset M$ of x such that $K_{|\Omega} = \mathbf{R}^{m;n_1,\dots,n_\kappa}$ modulo flag vector bundle isomorphism.

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Email addresses: doupovec@fme.vutbr.cz (Miroslav Doupovec), kurek@hektor.umcs.lublin.pl (Jan Kurek), Wlodzimierz.Mikulski@im.uj.edu.pl (Włodzimierz M. Mikulski)

Some flag vector bundles appear naturally in differential geometry. For example, if $q : B \to X$ is a bundle, then we have the flag vector bundle $TB = (TB; VB, TB)$ of length 2 with basis B, where TB is the tangent bundle of B and $VB = \ker(Tq) \subset TB$ is the vertical bundle of B. Another example, if $q : E \to M$ is a vector bundle, then we have the flag vector bundle $(J^rE)^* = ((J^rE)^*; E^*, (J^1E)^*, \dots, (J^rE)^*)$ of length $r + 1$ with basis M, where $(J^rE)^*$ is the dual of the r-jet prolongation J^rE of E and the inclusions $E^* \subset (J^1E)^* \subset \cdots \subset (J^rE)^*$ are dual to the jet projections $J^rE \to J^{r-1}E \to \cdots \to J^1E \to E$. Else one example, if $M = (M; \mathcal{F}_1, \dots, \mathcal{F}_n)$ is a manifold with a flag of foliations with $\mathcal{F}_n = \{M\}$, then we have the (obvious) flag vector bundle $TM = (TM; T\mathcal{F}_1, \dots, T\mathcal{F}_n)$ of length n with basis M.

Let $\mathcal{F}_\kappa\mathcal{VB}$ denotes the category of all flag vector bundles of length κ and their flag vector bundle maps and $\mathcal{FM}$ denotes the category of fibred manifolds and fibred maps. The general concept of (gauge) bundle functors can be found in [9]. We need the following particular case of it.

Definition 1.2. *A gauge bundle functor on $\mathcal{F}_\kappa\mathcal{VB}$ is a covariant functor $F : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{FM}$ sending any $\mathcal{F}_\kappa\mathcal{VB}$-object K with the basis M into fibred manifold $p_K : FK \to M$ over M and any $\mathcal{F}_\kappa\mathcal{VB}$-map $f : K \to K'$ with the base map $\underline{f} : M \to M'$ into fibred map $Ff : FK \to FK'$ over $\underline{f} : M \to M'$ and satisfying the following conditions:*

(i) (Localization condition) For every $\mathcal{F}_\kappa\mathcal{VB}$-object K with the basis M and any open subset $U \subset M$ the inclusion map $i_{K|U} : K|U \to K$ induces diffeomorphism $Fi_{K|U} : F(K|U) \to p_K^{-1}(U)$, and

(ii) (Regularity condition) F transforms smoothly parametrized families of $\mathcal{F}_\kappa\mathcal{VB}$-maps into smoothly parametrized families of $\mathcal{FM}$-maps.

A gauge bundle functor F on $\mathcal{F}_\kappa\mathcal{VB}$ is product preserving (ppgb-functor) if $F(K \times K') = F(K) \times F(K')$ for any $\mathcal{F}_\kappa\mathcal{VB}$-objects K and K'. (If $K = (K; K_1, \dots, K_\kappa)$ and $K' = (K'; K'_1, \dots, K'_\kappa)$ then (of course) $K \times K' = (K \times K'; K_1 \times K'_1, \dots, K_\kappa \times K'_\kappa)$.)

A simple example of a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$ is the tangent functor T sending any $\mathcal{F}_\kappa\mathcal{VB}$-object K into the tangent bundle TK (over M) and any $\mathcal{F}_\kappa\mathcal{VB}$-map $f : K \to K'$ into the tangent map $Tf : TK \to TK'$.

Given gauge bundle functors F_1, F_2 on $\mathcal{F}_\kappa\mathcal{VB}$, a natural transformation $\eta : F_1 \to F_2$ is a system of base preserving fibred maps $\eta_K : F_1K \to F_2K$ for every $\mathcal{F}_\kappa\mathcal{VB}$-object K satisfying $F_2f \circ \eta_K = \eta_{K'} \circ F_1f$ for every $\mathcal{F}_\kappa\mathcal{VB}$-map $f : K \to K'$.

In the present note, if $\kappa \geq 2$, we describe the ppgb-functors F on $\mathcal{F}_\kappa\mathcal{VB}$ in terms of the systems $I = (I_1, \dots, I_{\kappa-1})$ consisting of A-module homomorphisms $I_i : V_{i+1} \to V_i$ for $i = 1, \dots, \kappa - 1$, where A is a Weil algebra (i.e. a finite dimensional real associative commutative algebra with unity of the form $A = \mathbf{R} \oplus n_A$ with nilpotent ideal n_A) and $V_1, \dots, V_\kappa$ are finite dimensional (over $\mathbf{R}$) A-modules (over commutative ring A with unity). We study the so called iteration problem, too. Then we classify all natural affinors on ppgb-functors and all natural liftings of the so called κ-flag-linear vector fields (i.e. with the flows being $\mathcal{F}_\kappa\mathcal{VB}$-local isomorphisms) to ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$. We also define the complete lifting of κ-flag-linear semi-basic tangent valued p-forms to ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$ and observe that this complete lifting preserves the Frölicher-Nijenuis bracket. Finally, we apply the obtained results to study curvature and torsion of κ-flag-linear connections.

Clearly, $\mathcal{F}_1\mathcal{VB}$ is equivalent to the category of vector bundles $\mathcal{VB}$. In [15], we described the ppgb-functors F on $\mathcal{VB}$ in terms of finite dimensional (over $\mathbf{R}$) A-modules V.

Product preserving (gauge) bundle functors are studied in many papers, see [1, 8, 9, 12, 14–16, 19–21]. Natural operators lifting vector fields to product preserving (gauge) bundle functors are studied in [7, 13]. Complete lifts of semi-basic tangent valued p-forms are studied in [3, 4, 17]. Natural affinors are classified in many papers, see e.g. [2, 5, 10, 11]. Natural affinors are used to study torsion of connections, see e.g. [2, 10].

2. The ppgb-functors $F^{[I]}$

Suppose we have a system $I = (I_1, \dots, I_{\kappa-1})$ consisting of A-module homomorphisms $I_i : V_{i+1} \to V_i$ for $i = 1, \dots, \kappa - 1$, where A is a Weil algebra and $V_1, \dots, V_\kappa$ are finite dimensional (over $\mathbf{R}$) A-modules. (A rather simple but non-trivial example of such a system I is presented in the end of this section). We are going to construct ppgb-functor $F^{[I]} : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{FM}$.

We consider a $\mathcal{F}_\kappa\mathcal{VB}$-object $K = (K; K_1, \ldots, K_\kappa)$ with basis M and a point $x \in M$. Let

$$\mathcal{G}_x(K, \mathbf{R}^{m;n_1,\ldots,n_\kappa}) := \text{the space of germs at } x \text{ of } \mathcal{F}_\kappa\mathcal{VB}\text{-maps } K \to \mathbf{R}^{m;n_1,\ldots,n_\kappa} .$$

The sum map $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ can be treated as the $\mathcal{F}_\kappa\mathcal{VB}$-maps:

$$+ : \mathbf{R}^{1;0_1,\ldots,0_\kappa} \times \mathbf{R}^{1;0_1,\ldots,0_\kappa} \to \mathbf{R}^{1;0_1,\ldots,0_\kappa} \text{ and}$$

$$+ : \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa} \times \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa} \to \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa}$$

for $i = 1, \ldots, \kappa$, where $0_j = 0$ and $1_j = 1$ for $j = 1, \ldots, \kappa$.

The multiplication map $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ can be treated as the $\mathcal{F}_\kappa\mathcal{VB}$-maps

$$\cdot : \mathbf{R}^{1;0_1,\ldots,0_\kappa} \times \mathbf{R}^{1;0_1,\ldots,0_\kappa} \to \mathbf{R}^{1;0_1,\ldots,0_\kappa} \text{ and}$$

$$\cdot : \mathbf{R}^{1;0_1,\ldots,0_\kappa} \times \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa} \to \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa}$$

for $i = 1, \ldots, \kappa$.

The constant map $1 : \mathbf{R} \to \mathbf{R}$ can be treated as the $\mathcal{F}_\kappa\mathcal{VB}$-map

$$1 : \mathbf{R}^{1;0_1,\ldots,0_\kappa} \to \mathbf{R}^{1;0_1,\ldots,0_\kappa}$$

and the constant map $0 : \mathbf{R} \to \mathbf{R}$ can be treated as the $\mathcal{F}_\kappa\mathcal{VB}$-maps:

$$0 : \mathbf{R}^{1;0_1,\ldots,0_\kappa} \to \mathbf{R}^{1;0_1,\ldots,0_\kappa} \text{ and}$$

$$0 : \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa} \to \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa}$$

for $i = 1, \ldots, \kappa$.

Hence $\mathcal{G}_x(K, \mathbf{R}^{1;0_1,\ldots,0_\kappa})$ is (in obvious way) an algebra and $\mathcal{G}_x(K, \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa})$ is (in obvious way) a $\mathcal{G}_x(K, \mathbf{R}^{1;0_1,\ldots,0_\kappa})$-module for $i = 1, \ldots, \kappa$.

The identity map $\mathrm{id}_\mathbf{R} : \mathbf{R} \to \mathbf{R}$ can be treated as $\mathcal{F}_\kappa\mathcal{VB}$-map

$$\iota_{(i)} : \mathbf{R}^{0;0_1,\ldots,0_i,1_{i+1},0_{i+2},\ldots,0_\kappa} \to \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa}$$

for $i = 1, \ldots, \kappa - 1$.

Example 2.1. *We define $F^{[I]}_x K$ to be the space of tuples $(\varphi, \psi_1, \ldots, \psi_\kappa)$ consisting of algebra maps $\varphi : \mathcal{G}_x(K, \mathbf{R}^{1;0_1,\ldots,0_\kappa}) \to A$ and module maps $\psi_i : \mathcal{G}_x(K, \mathbf{R}^{0;0_1,\ldots,0_{i-1},1_i,0_{i+1},\ldots,0_\kappa}) \to V_i$ over φ for $i = 1, \ldots, \kappa$ satisfying*

$$\psi_i(\iota_{(i)} \circ g) = I_i \circ \psi_{i+1}(g) \tag{1}$$

for all $g \in \mathcal{G}_x(K, \mathbf{R}^{0;0_1,\ldots,0_i,1_{i+1},0_{i+2},\ldots,0_\kappa})$ if $i = 1, \ldots, \kappa - 1$. Let $F^{[I]}K := \bigcup_{x\in M} F^{[I]}_x K$. We can see that $F^{[I]}K$ is a fibred manifold over M. Indeed, a $\mathcal{F}_\kappa\mathcal{VB}$- trivialization

$$((x^j), (y^{j_1}_1), \ldots, (y^{j_\kappa}_\kappa)) : K_{|\Omega} \tilde{=} \mathbf{R}^{m;n_1,\ldots,n_\kappa}$$

of K induces an $\mathcal{FM}$-trivialization

$$((\tilde{x}^j), (\tilde{y}^{j_1}_1), \ldots, (\tilde{y}^{j_\kappa}_\kappa)) : F^{[I]}K_{|\Omega} \tilde{=} A^m \times V_1^{n_1} \times \ldots \times V_\kappa^{n_\kappa} \tag{2}$$

defined by

$$\tilde{x}^j(\varphi, \psi_1, \ldots, \psi_\kappa) = \varphi(\mathrm{germ}_x(x^j)) \in A,\ \tilde{y}^{j_k}_k(\varphi, \psi_1, \ldots, \psi_\kappa) = \psi_k(\mathrm{germ}_x(y^{j_k}_k)) \in V_k,$$

$j = 1, \ldots, m$, $k = 1 \ldots, \kappa$, $j_k = 1, \ldots, n_k$. The trivialization (2) is really a bijection. Indeed, any $(\varphi, \psi_1, \ldots, \psi_\kappa) \in F^{[I]}_x K_{|\Omega}$ is uniquely determined by the values

$$\varphi(germ_x(x^j)) \in A, \quad j = 1 \ldots, m$$

together with the values

$$\psi_k(germ_x(y_q^{j_q})) \in V_k, \quad k = 1, \dots, \kappa, \quad q = k, \dots, \kappa, \quad j_q = 1, \dots, n_q$$

because the module $\mathcal{G}_x(K, \mathbf{R}^{0;0_1,\dots,0_{k-1},1_k,0_{k+1},\dots,0_\kappa})$ *is free with the basis*

$$germ_x(y_q^{j_q}), \quad q = k, \dots, \kappa, \quad j_q = 1, \dots, n_q.$$

So, using the condition (1) one can easily show that any $(\varphi, \psi_1, \dots, \psi_\kappa)$ *as above is uniquely determined by the values*

$$\varphi(germ_x(x^j)) \in A,\ j = 1 \dots, m \quad \text{and} \quad \psi_k(germ_x(y_k^{j_k})) \in V_k,\ k = 1, \dots, \kappa,\ j_k = 1, \dots, n_k$$

as well.

Any $\mathcal{F}_\kappa\mathcal{VB}$*-map* $f : K \to K^1$ *induces a* $\mathcal{FM}$*-map* $F^{[I]}f : F^{[I]}K \to F^{[I]}K^1$ *such that*

$$F^{[I]}(f)(\varphi, \psi_1, \dots, \psi_\kappa) := (\varphi \circ f_x^*, \psi_1 \circ f_x^*, \dots, \psi_\kappa \circ f_x^*)\,,$$

$(\varphi, \psi_1, \dots, \psi_\kappa) \in F_x^{[I]}K$, $x \in M$, *where* f_x^* *is the pull-back with respect to* f. *Clearly, the resulting correspondence* $F^{[I]} : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{FM}$ *is a ppgb-functor.*

If $I' = (I'_1, \dots, I'_{\kappa-1})$ *is another system in question and* $\mu = (\alpha, \beta_1, \dots, \beta_\kappa) : I \to I'$ *is a morphism (i.e.* $\alpha : A \to A'$ *is a Weil algebra homomorphism and* $\beta_i : V_i \to V'_i$ *are module maps over* α *for* $i = 1, \dots, \kappa$ *such that* $I'_i \circ \beta_{i+1} = \beta_i \circ I_i$ *for* $i = 1, \dots, \kappa - 1$*) then we have the natural transformation* $\eta^{[\mu]} : F^{[I]} \to F^{[I']}$ *given by* $(\varphi, \psi_1, \dots, \psi_\kappa) \mapsto (\alpha \circ \varphi, \beta_1 \circ \psi_1, \dots, \beta_\kappa \circ \psi_\kappa)$.

Lemma 2.2. *(i) The functor* $F^{[I]}$ *has values in* $\mathcal{F}_\kappa\mathcal{VB}$*, i.e.* $F^{[I]} : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{F}_\kappa\mathcal{VB}$.
(ii) The natural transformation $\eta^{[\mu]} : F^{[I]}K \to F^{[I']}K$ *is a* $\mathcal{F}_\kappa\mathcal{VB}$*-morphism for any* $\mathcal{F}_\kappa\mathcal{VB}$*-object* K.

Proof. Let $K = (K; K_1, \dots, K_\kappa)$ be a $\mathcal{F}_\kappa\mathcal{VB}$-object with the basis M. It is clear that $F^{[I]}K$ is the vector bundle with basis T^AM with the projection $(\varphi, \psi_1, \dots, \psi_\kappa) \mapsto \varphi$. For $i = 1, \dots, \kappa$ we have vector sub-bundle $(F^{[I]}K)_i := \{(\varphi, \psi_1, \dots, \psi_\kappa) \in F^{[I]}K \mid \psi_{i+1} = \dots = \psi_\kappa = 0\}$. □

A rather simple but non-trivial system I in question is given by the projections $I_i : A^{i+1} \to A^i$ for $i = 1, \dots, \kappa - 1$, where A is a Weil algebra, $A^i = A \times \dots \times A$ (i times) is A-module with the multiplication $a(a_1, \dots, a_i) = (aa_1, \dots, aa_i)$ for $a \in A$ and $(a_1, \dots, a_i) \in A^i$, and $I_i(a_1, \dots, a_{i+1}) = (a_1, \dots, a_i)$ for $(a_1, \dots, a_{i+1}) \in A^{i+1}$. Another system in question can be obtained from this one by replacing A on an ideal in A.

3. The system $I^{[F]}$

Example 3.1. *Let* $F : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{FM}$ *be a ppgb-functor. Let*

$$A^{[F]} := F\mathbf{R}^{1;0_1,\dots,0_\kappa} \text{ and } V_i^{[F]} := F\mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1}\dots,0_\kappa}$$

for $i = 1, \dots, \kappa$. *Then* $A^{[F]}$ *is a Weil algebra and* $V_i^{[F]}$ *are* $A^{[F]}$*-modules. Indeed, the algebra operations of* $A^{[F]}$ *are* $F(+) : F(\mathbf{R}^{1;0_1,\dots,0_\kappa} \times \mathbf{R}^{1;0_1,\dots,0_\kappa}) = A^{[F]} \times A^{[F]} \to F\mathbf{R}^{1;0_1,\dots,0_\kappa} = A^{[F]}$ *and* $F(\cdot) : A^{[F]} \times A^{[F]} \to A^{[F]}$, *where the sum map* $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *and the multiplication map* $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *are treated as* $\mathcal{F}_\kappa\mathcal{VB}$*-maps* $+, \cdot : \mathbf{R}^{1;0_1,\dots,0_\kappa} \times \mathbf{R}^{1;0_1,\dots,0_\kappa} \to \mathbf{R}^{1;0_1,\dots,0_\kappa}$, *the unity of* $A^{[F]}$ *is* $F(1)$ *and the null is* $F(0)$. *Similarly, the* $A^{[F]}$*-module operations of* $V_i^{[F]}$ *are* $F(+) : V_i^{[F]} \times V_i^{[F]} \to V_i^{[F]}$ *and* $F(\cdot) : A^{[F]} \times V_i^{[F]} \to V_i^{[F]}$, *where the sum and multiplication maps* $+$ *and* $\cdot$ *are treated as* $\mathcal{F}_\kappa\mathcal{VB}$*-maps*

$$+ : \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa} \times \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa} \to \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}\,,$$

$$\cdot : \mathbf{R}^{1;0_1,\dots,0_\kappa} \times \mathbf{R}^{0;0_1,\dots,0_{i_1},1_i,0_{i+1},\dots,0_\kappa} \to \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}.$$

For $i = 1, \dots, \kappa - 1$ we have a $A^{[F]}$-linear map

$$I_i^{[F]} := F(\iota_{(i)}) : V_{i+1}^{[F]} \to V_i^{[F]} ,$$

where $\iota_{(i)} : \mathbf{R}^{0;0_1,\dots,0_i,1_{i+1},0_{i+2},\dots,0_\kappa} \to \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$ is the $\mathcal{F}_\kappa\mathcal{VB}$-map given by the identity map $\mathrm{id}_\mathbf{R} : \mathbf{R} \to \mathbf{R}$. We put $I^{[F]} := (I_1^{[F]}, \dots, I_{\kappa-1}^{[F]})$. Any natural transformation $\eta : F \to F'$ of ppgb-functors induces a morphism $\mu^{[\eta]} := (\eta_{\mathbf{R}^{1;0_1,\dots,0_\kappa}}, \eta_{\mathbf{R}^{0;1_1,0_2,\dots,0_\kappa}}, \dots, \eta_{\mathbf{R}^{0;0_1,\dots,0_{\kappa-1},1_\kappa}}) : I^{[F]} \to I^{[F']}$.

For example, if T is the tangent functor (on $\mathcal{F}_\kappa\mathcal{VB}$) then $A^{[T]} = \mathbf{D}$ is the algebra of dual numbers, $V_i^{[T]} = \mathbf{D}$ with the $\mathbf{D}$-module multiplication being the one of dual numbers for $i = 1, \dots, \kappa$, and $I_i^{[T]} = \mathrm{id}_\mathbf{D} : V_{i+1}^{[T]} \to V_i^{[T]}$ for $i = 1, \dots, \kappa - 1$.

4. The isomorphism $F \tilde{=} F^{[I^{[F]}]}$

Theorem 4.1. *Let $\kappa \geq 2$. We have $F = F^{[I^{[F]}]}$ modulo the natural isomorphism.*

Proof. Let K be a $\mathcal{F}_\kappa\mathcal{VB}$-object with basis M and let $y \in F_xK$ be a point, $x \in M$. We define a map $\varphi_y : \mathcal{G}_x(K, \mathbf{R}^{1;0_1,\dots,0_\kappa}) \to A^{[F]} = F\mathbf{R}^{1;0_1,\dots,0_\kappa}$ by

$$\varphi_y(\mathrm{germ}_x(g)) = F(g)(y) ,$$

where $g : K \to \mathbf{R}^{1;0_1,\dots,0_\kappa}$ is a $\mathcal{F}_\kappa\mathcal{VB}$-map. Similarly, given $i = 1, \dots, \kappa$, we define a map

$$(\psi_y)_i : \mathcal{G}_x(K, \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}) \to V_i^{[F]} = F\mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa} \quad \text{by}$$

$$(\psi_y)_i(\mathrm{germ}_x(g)) = F(g)(y) ,$$

where $g : K \to \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$ is a $\mathcal{F}_\kappa\mathcal{VB}$-map.

Recalling the definitions of operations in $A^{[F]}$ and $V_i^{[F]}$ (from Example 3.1), since F is a functor, we get that φ_y is an algebra homomorphism and $(\psi_y)_i$ is a module map over φ_y.

Using similar arguments, given $i = 1, \dots, \kappa - 1$ we get

$$(\psi_y)_i(\iota_{(i)} \circ g) = I_i^{[F]} \circ (\psi_y)_{i+1}(g)$$

for all $g \in \mathcal{G}_x(K, \mathbf{R}^{0;0_1,\dots,0_i,1_{i+1},0_{i+2},\dots,0_\kappa})$. Consequently,

$$\Theta_K^F(y) := (\varphi_y, (\psi_y)_1, \dots, (\psi_y)_\kappa) \in F_x^{[I^{[F]}]}K .$$

So, we have the resulting $\mathcal{F}_\kappa\mathcal{VB}$-natural transformation

$$\Theta^F : F \to F^{[I^{[F]}]} .$$

We prove that Θ_K^F is a diffeomorphism for any $\mathcal{F}_\kappa\mathcal{VB}$-object K.

Applying $\mathcal{F}_\kappa\mathcal{VB}$-trivialization, we can assume that $K = \mathbf{R}^{m;n_1,\dots,n_\kappa}$. Since F and $F^{[I^{[F]}]}$ are product preserving and K is a (multi) product of $\mathbf{R}^{1;0_1,\dots,0_\kappa}$ and $\mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$ for $i = 1, \dots, \kappa$, we can assume that K is $\mathbf{R}^{1;0_1,\dots,0_\kappa}$ or $\mathbf{R}^{0;0_1,\dots,0_{i_1},1_i,0_{i+1},\dots,0_\kappa}$ with $i = 1, \dots, \kappa$.

If $K = \mathbf{R}^{1;0_1,\dots,0_\kappa}$, we consider $\tilde{x}^1 \circ \Theta_K^F : F\mathbf{R}^{1;0_1,\dots,0_\kappa} \to A^F = F\mathbf{R}^{1;0_1,\dots,0_\kappa}$, where $\tilde{x}^1$ is induced by $x^1 = \mathrm{id}_\mathbf{R} : \mathbf{R}^{1;0_1,\dots,0_\kappa} \to \mathbf{R}^{1;0_1,\dots,0_\kappa}$, see Example 2.1. This composition is the identity map of $F\mathbf{R}^{1;0_1,\dots,0_\kappa} = A^{[F]}$. That is why, Θ_κ^F is a diffeomorphism in this case.

If $K = \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$, Θ_K^F is a diffeomorphism by the reason as above with $\mathbf{R}^{1;0_1,\dots,0_\kappa}$ replaced by $\mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$ and with $\tilde{x}^1$ replaced by $\tilde{y}_i^1$, where $\tilde{y}_i^1$ is induced by $y_i^1 = \mathrm{id}_\mathbf{R} : \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa} \to \mathbf{R}^{0;0_1,\dots,0_{i-1},1_i,0_{i+1},\dots,0_\kappa}$. □

From Theorem 4.1 and Lemma 2.2, it follows immediately

Proposition 4.2. *(i) Any ppgb-functor $F : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{FM}$ on $\mathcal{F}_\kappa\mathcal{VB}$ has values in $\mathcal{F}_\kappa\mathcal{VB}$, i.e. $F : \mathcal{F}_\kappa\mathcal{VB} \to \mathcal{F}_\kappa\mathcal{VB}$.*

(ii) For any natural transformation $\eta : F \to F'$ of ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$, the fibred map $\eta : FK \to F'K$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism for any $\mathcal{F}_\kappa\mathcal{VB}$-object K.

5. Local expression

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. By Theorem 4.1, we may assume $F = F^{[I]}$, where $I = (I_1, \ldots, I_{\kappa-1})$ is a system consisting of A-module homomorphisms $I_i : V_{i+1} \to V_i$, where A is a Weil algebra and $V_1, \ldots, V_\kappa$ are finite dimensional (over $\mathbf{R}$) A-modules. Then we can write

$$F\mathbf{R}^{m;n_1,\ldots,n_\kappa} = A^m \times V_1^{n_1} \times \cdots \times V_\kappa^{n_\kappa} \text{ (modulo the trivialization)} .$$

Consider a $\mathcal{F}_\kappa\mathcal{VB}$-map $f : \mathbf{R}^{m;n_1,\ldots,n_\kappa} \to \mathbf{R}^{m';n'_1,\ldots,n'_\kappa}$. It is of the form

$$f(x, y_1, \ldots, y_\kappa) = (a^{j'}(x), \sum_{q=k}^{\kappa} \sum_{j_q=1}^{n_q} a^{q,j'_k}_{k,j_q}(x) y_q^{j_q})_{j'=1,\ldots,m';k=1,\ldots,\kappa;j'_k=1,\ldots,n'_k},$$

$x = (x^1, \ldots, x^m) \in \mathbf{R}^m$, $y_1 = (y_1^1, \ldots, y_1^{n_1}) \in \mathbf{R}^{n_1}$,..., $y_\kappa = (y_\kappa^1, \ldots, y_\kappa^{n_\kappa}) \in \mathbf{R}^{n_\kappa}$, where $a^{j'} : \mathbf{R}^m \to \mathbf{R}$ and $a^{q,j'_k}_{k,j_q} : \mathbf{R}^m \to \mathbf{R}$ are some smooth maps. Then we can see that the induced map $F^{[I]}f : A^m \times V_1^{n_1} \times \cdots \times V_\kappa^{n_\kappa} \to A^{m'} \times V_1^{n'_1} \times \cdots \times V_\kappa^{n'_\kappa}$ is of the similar form

$$F^{[I]}f(x, y_1, \ldots, y_\kappa) = ((a^{j'})^A(x), \sum_{q=k}^{\kappa} \sum_{j_q=1}^{n_q} (a^{q,j'_k}_{k,j_q})^A(x) \cdot I_k^{q-1}(y_q^{j_q}))_{j'=1,\ldots,m';k=1,\ldots,\kappa;j'_k=1,\ldots,n'_k} ,$$

$x = (x^1, \ldots, x^m) \in A^m$, $y_1 = (y_1^1, \ldots, y_1^{n_1}) \in V_1^{n_1}, \ldots, y_\kappa = (y_\kappa^1, \ldots, y_\kappa^{n_\kappa}) \in V_\kappa^{n_\kappa}$, where $I_k^{q-1} := I_k \circ \cdots \circ I_{q-1} : V_q \to V_k$, $(a^{j'})^A := T^A a : T^A\mathbf{R}^m = A^m \to T^A\mathbf{R} = A$, $(a^{q,j'_k}_{k,j_q})^A := T^A a^{q,j'_k}_{k,j_q} : A^m \to A$, T^A is the Weil functor of Weil algebra A and $\cdot$ is the multiplication of the A-module V_k. (If $q = k$ then I_k^{k-1} is the identity map of V_k.)

If $\mu = (\alpha, \beta_1, \ldots, \beta_\kappa) : I \to I'$ is a morphism then $\eta^{[\mu]}_{\mathbf{R}^{m;n_1,\ldots,n_\kappa}} : A^m \times V_1^{n_1} \times \cdots \times V_\kappa^{n_\kappa} \to A'^m \times V_1'^{n_1} \times \cdots \times V_\kappa'^{n_\kappa}$ is of the form

$$\eta^{[\mu]}_{\mathbf{R}^{m;,n_1,\ldots,n_\kappa}}(x, y_1, \ldots, y_\kappa) = ((\alpha(x^1), \ldots, \alpha(x^m)), (\beta_1(y_1^1), \ldots, \beta_1(y_1^{n_1})), \ldots, (\beta_\kappa(y_\kappa^1), \ldots, \beta_\kappa(y_\kappa^{n_\kappa}))) ,$$

where $x = (x^1, \ldots, x^m) \in A^m$ and $y_1 = (y_1^1, \ldots, y_1^{n_1}) \in V_1^{n_1}$, ... , $y_\kappa = (y_\kappa^1, \ldots, y_\kappa^{n_\kappa}) \in V_\kappa^{n_\kappa}$.

Proposition 5.1. *We have*

$$F(K_1 \times_M K_2) = FK_1 \times_{FM} FK_2 \text{ modulo } (Fpr_1, Fpr_2) \tag{3}$$

for any $\mathcal{F}_\kappa\mathcal{VB}$-objects K_1 and K_2 with the same basis M, i.e. if $pr_i : K_1 \times_M K_2 \to K_i$ are the fiber product projections, then so are $Fpr_i : F(K_1 \times_M K_2) \to FK_i$.

Proof. It follows easily from the above "local expression". □

6. Iteration

Let F and F' be ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$. Since F and F' have values in $\mathcal{F}_\kappa\mathcal{VB}$, we can compose F and F'. It is clear that the composition $F'' = F' \circ F$ is again a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. We are going to compute $I^{[F'']}$ by means of $I^{[F]}$ and $I^{[F']}$.

Lemma 6.1. *We have $A^{[F'']} = A^{[F]} \otimes A^{[F']}$ (the tensor product over $\mathbf{R}$). Moreover, the algebra multiplication of $A^{[F'']}$ satisfies $(a \otimes a')(b \otimes b') = (ab) \otimes (a'b')$ for any $a, b \in A^{[F]}$ and $a', b' \in A^{[F']}$.*

Proof. Of course, $A^{[F]}$, $A^{[F']}$ and $A^{[F'']}$ are the Weil algebras of the Weil functors $\tilde{F}, \tilde{F}', \tilde{F}'' : \mathcal{M}f \to \mathcal{FM}$ given by $\tilde{F}M = FM$, $\tilde{F}'M = F'M$, $\tilde{F}''M = F''M$, where manifolds M are treated as the $\mathcal{F}_\kappa\mathcal{VB}$-objects with bases M. We also see that $\tilde{F}'' = \tilde{F}' \circ \tilde{F}$. So, our result in question is the well-known one for Weil functors on manifolds, see [8, 9]. □

Lemma 6.2. *Let $i = 1, \dots, \kappa$. Then $V_i^{[F'']} = V_i^{[F]} \otimes V_i^{[F']}$ (the tensor product over $\mathbf{R}$). Moreover, the multiplication of $A^{[F'']} = A^{[F]} \otimes A^{[F']}$ on $V_i^{[F'']}$ satisfies $(a \otimes a')(u \otimes u') = (au) \otimes (a'u')$ for any $a \in A^{[F]}$, $a' \in A^{[F']}$, $u \in V_i^{[F]}$ and $u' \in V_i^{[F']}$.*

Proof. Put $p = \dim_{\mathbf{R}}(A^{[F]})$, $p' = \dim_{\mathbf{R}}(A^{[F']})$, $q = \dim_{\mathbf{R}}(V_i^{[F]})$ and $q' = \dim_{\mathbf{R}}(V_i^{[F']})$. Choose the basis $\{e_{i_1}\}_{i_1=1,\dots,p}$ of $A^{[F]}$ over $\mathbf{R}$, the basis $\{e'_j\}_{j=1,\dots,p'}$ of $A^{[F']}$ over $\mathbf{R}$, the basis $\{f_k\}_{k=1,\dots,q}$ of $V_i^{[F]}$ over $\mathbf{R}$ and the basis $\{f'_l\}_{l=1,\dots,q'}$ of $V_i^{[F']}$ over $\mathbf{R}$. Then we can write $A^{[F]} = \mathbf{R}^p$, $A^{[F']} = \mathbf{R}^{p'}$, $V_i^{[F]} = \mathbf{R}^q$ and $V_i^{[F']} = \mathbf{R}^{q'}$. We have $e_{i_1} f_k = \sum_a c^a_{i_1 k} f_a$ and $e'_j f'_l = \sum_b d^b_{jl} f'_b$, where $c^a_{i_1 k}$ and d^b_{jl} are the real numbers. Then $F(\cdot) : A^{[F]} \times V_i^{[F]} = \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^q = V_i^{[F]}$ satisfies $F(\cdot)(x, y) = (\sum_{i_1,k} c^a_{i_1 k} x^{i_1} y^k)_{a=1,\dots,q}$ for any $x = (x^{i_1}) \in \mathbf{R}^p$ and any $(y^k) \in \mathbf{R}^q$. Of course, $A^{[F]} = \mathbf{R}^p = \mathbf{R}^{p;0_1,\dots,0_\kappa}$ and $V_i^{[F]} = \mathbf{R}^q = \mathbf{R}^{0;0_1,\dots 0_{i-1},q,0_{i+1},\dots,0_\kappa}$ are the trivial $\mathcal{VB}$-objects (and similarly for $A^{[F']}$ and $V_i^{[F']}$). Then $F''(\cdot) = F'(F(\cdot)) : (A^{[F']})^p \times (V_i^{[F']})^q \to (V_i^{[F']})^q$ satisfies the similar formula

$$F''(\cdot)(x, y) = (\sum_{i_1,k} c^a_{i_1 k} x^{i_1} y^k)_{a=1,\dots,q}$$

for any $x = (x^{i_1}) \in (A^{[F']})^p$ and $y = (y^k) \in (V_i^{[F']})^q$, see Section 5. So, $F''(\cdot) : \mathbf{R}^{p'p} \times \mathbf{R}^{q'q} \to \mathbf{R}^{q'q}$ satisfies $F''(\cdot)((x^{i_1\alpha}), (y^{k\beta})) = (\sum_{i_1,k,\alpha,\beta} c^a_{i_1 k} d^b_{\alpha\beta} x^{i_1\alpha} y^{k\beta})$. So, $F''(\cdot) : (A^{[F]} \otimes A^{[F']}) \times (V_i^{[F]} \otimes V_i^{[F']}) \to V_i^{[F]} \otimes V_i^{[F']}$ satisfies $F''(\cdot)(x \otimes x', y \otimes y') = (xy) \otimes (x'y')$ for any $x \in A^{[F]}$, $x' \in A^{[F']}$, $y \in V_i^{[F]}$ and $y' \in V_i^{[F']}$, where $A^{[F]} \otimes A^{[F']} = \mathbf{R}^{pp'}$ modulo the basis $(e_{i_1} \otimes e'_j)$ and $V_i^{[F]} \otimes V_i^{[F']} = \mathbf{R}^{qq'}$ modulo the basis $(f_k \otimes f'_l)$. □

Lemma 6.3. *Let $i = 1, \dots, \kappa - 1$. Then $I_i^{[F'']}(u \otimes u') = I_i^{[F]}(u) \otimes I_i^{[F']}(u')$ for any $u \in V_{i+1}^{[F]}$ and $u' \in V_{i+1}^{[F']}$.*

Proof. The proof is similar to the one of the previous lemma. More precisely, we analyze local expression of $F''(\iota_{(i)})$. □

Summing up we have

Theorem 6.4. *Let F and F' be ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$. Then $I^{[F' \circ F]} = I^{[F]} \otimes I^{[F']}$, where the tensor product is explained in Lemmas 6.1-6.3. Consequently, the exchange isomorphism $ex : I^{[F']} \otimes I^{[F]} \to I^{[F]} \otimes I^{[F']}$ induces the isomorphism $\eta^{[ex]} : FF' \to F'F$ of ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$. Roughly speaking, any two ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$ commute.*

7. The natural affinors on ppgb-functors

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. Composing the tangent functor T with F we get TF. It is a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. After Example 3.1 we remarked that $A^{[T]} = \mathbf{D}$, $V_i^{[T]} = \mathbf{D}$ for $i = 1, \dots, \kappa$, and $I^{[T]} = (\mathrm{id}_{\mathbf{D}}, \dots, \mathrm{id}_{\mathbf{D}})$. Then $A^{[TF]} = A^{[F]} \otimes \mathbf{D} = A^{[F]} \times A^{[F]}$ with the algebra multiplication

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_1 + a_1 b_2)$$

for any $a_1, a_2, b_1, b_2 \in A^{[F]}$, see Theorem 6.4. Moreover, given $i \in \{1, \dots, \kappa\}$, $V_i^{[TF]} = V_i^{[F]} \otimes \mathbf{D} = V_i^{[F]} \times V_i^{[F]}$ with the $A^{[F]} \times A^{[F]}$-module multiplications

$$(a_1, a_2)(v_1, v_2) = (a_1 v_1, a_2 v_1 + a_1 v_2)$$

for any $a_1, a_2 \in A^{[F]}$, $v_1, v_2 \in V_i^{[F]}$. Moreover, given $i \in \{1, ..., \kappa - 1\}$,

$$I_i^{[TF]}(v_1, v_2) = (I_i^{[F]}(v_1), I_i^{[F]}(v_2))$$

for any $v_1, v_2 \in V_{i+1}^{[F]}$.

For any $c \in A^{[F]}$ we define $\alpha_c : A^{[F]} \times A^{[F]} \to A^{[F]} \times A^{[F]}$ by $\alpha_c(a_1, a_2) = (a_1, ca_2)$ for any $a_1, a_2 \in A^{[F]}$ and given $i \in \{1, \dots, \kappa\}$ we define $\beta^i_c : V^{[F]}_i \times V^{[F]}_i \to V^{[F]}_i \times V^{[F]}_i$ by $\beta_c(v_1, v_2) = (v_1, cv_2)$ for any $v_1, v_2 \in V^{[F]}_i$. Then $(\alpha_c, \beta^1_c, \dots, \beta^\kappa_c)$ is a morphism $I^{[TF]} \to I^{[TF]}$. Hence we have the corresponding natural transformation

$$\mathrm{af}(c) : TFK \to TFK\,.$$

Locally,

$$\mathrm{af}(c) : T((A^{[F]})^m \times (V^{[F]}_1)^{n_1} \times ... \times (V^{[F]}_\kappa)^{n_\kappa}) \to T((A^{[F]})^m \times (V^{[F]}_1)^{n_1} \times ... \times (V^{[F]}_\kappa)^{n_\kappa}) \quad \text{satisfies}$$

$$\mathrm{af}(c)((a, v_1, \dots, v_\kappa), (b, u_1, \dots, u_\kappa)) = ((a, v_1, \dots, v_\kappa), c(b, u_1, \dots, u_\kappa)) \tag{4}$$

(modulo the obvious identification) for $a, b \in (A^{[F]})^m$ and $v_i, u_i \in (V^{[F]}_i)^{n_i}$, $i = 1, \dots, \kappa$. So, $\mathrm{af}(c)$ is an affinor on FK. Since $\mathrm{af}(c)$ is a natural transformation of ppgb-functors on $\mathcal{F}_\kappa\mathcal{VB}$, then $\mathrm{af}(c) : TFK \to TFK$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism.

Let $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$ be the category of all $\mathcal{F}_\kappa\mathcal{VB}$-objects K being locally isomorphic with $\mathbf{R}^{m;n_1,\dots,n_\kappa}$ with local $\mathcal{F}_\kappa\mathcal{VB}$-isomorphisms between them as morphisms.

Definition 7.1. *A $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-natural affinor on F is a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-invariant family B of affinors $B : TFK \to TFK$ on FK for any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-object K. It means that $TFf \circ B = B \circ TFf$ for any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-map $f : K \to K'$.*

Theorem 7.2. *Let $m, n_1, \dots, n_\kappa$ be non-negative integers with $m \geq 2$. Any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-natural affinor B on F is $\mathrm{af}(c)$ for some $c \in A^{[F]}$.*

Proof. Of course, B is determined by affinor $B : TF\mathbf{R}^{m;n_1,\dots,n_\kappa} \to TF\mathbf{R}^{m;n_1,\dots,n_\kappa}$ on $F\mathbf{R}^{m;n_1,\dots,n_\kappa} = (A^{[F]})^m \times (V^{[F]}_1)^{n_1} ... \times (V^{[F]}_\kappa)^{n_\kappa}$. Then (modulo the standard identification) we have $B : F\mathbf{R}^{m;n_1,\dots,n_\kappa} \times F\mathbf{R}^{m;n_1,\dots,n_\kappa} \to F\mathbf{R}^{m;n_1,\dots,n_\kappa} \times F\mathbf{R}^{m;n_1,\dots,n_\kappa}$ and we can write

$$B(x, y) = (x, \tilde{B}(x, y))$$

for all $x, y \in F\mathbf{R}^{m;n_1,\dots,n_\kappa}$, where $\tilde{B}(x, y) \in F\mathbf{R}^{m;n_1,\dots,n_\kappa}$ is linear in y. Using the invariance of B with respect to the homotheties $t \cdot \mathrm{id}_{\mathbf{R}^{m;n_1,\dots,n_\kappa}}$, $t > 0$, we get the homogeneity condition $\tilde{B}(tx, ty) = t\tilde{B}(x, y)$, i.e. $\tilde{B}(tx, y) = \tilde{B}(x, y)$. Consequently, $\tilde{B}(x, y)$ is independent of x. So, we can write

$$B((a, u_1, \dots, u_\kappa), (b, v_1, \dots, v_\kappa))$$
$$= ((a, u_1, \dots, u_\kappa), (\alpha(b, v_1, \dots, v_\kappa), \beta_1(b, v_1, \dots, v_\kappa), \dots, \beta_\kappa(b, v_1, \dots, v_\kappa)))$$

for all $a, b \in (A^{[F]})^m$, $u_1, v_1 \in (V^{[F]}_1)^{n_1}, \dots, u_\kappa, v_\kappa \in (V^{[F]}_\kappa)^{n_\kappa}$, where $\alpha(b, v_1, \dots, v_\kappa) \in (A^{[F]})^m$ is linear in $(b, v_1, \dots, v_\kappa)$ and $\beta_1(b, v_1, \dots, v_\kappa) \in (V^{[F]}_1)^{n_1}$ is linear in $(b, v_1, \dots, v_\kappa)$ and ... and $\beta_\kappa(b, v_1, \dots, v_\kappa) \in (V^{[F]}_\kappa)^{n_\kappa}$ is linear in $(b, v_1, \dots, v_\kappa)$.

Let $\varphi_{t,t_1,\dots,t_\kappa} : \mathbf{R}^{m;n_1,\dots,n_\kappa} \to \mathbf{R}^{m;n_1,\dots,n_\kappa}$ be given by

$$\varphi_{t,t_1,\dots,t_\kappa}(x, y_1, \dots, y_\kappa) = (tx, t_1y_1, \dots, t_\kappa y_\kappa)$$

for all $x \in \mathbf{R}^m$ and $y_1 \in \mathbf{R}^{n_1}$ and ... and $y_\kappa \in \mathbf{R}^{n_\kappa}$, where $t, t_1, \dots, t_\kappa$ are positive real numbers. It is a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-map. Then, using the invariance of B with respect to $\varphi_{t,t_1,\dots,t_\kappa}$, we get the homogeneity condition

$$\alpha(tb, t_1v_1, \dots, t_\kappa v_\kappa) = t\alpha(b, v_1, \dots, v_\kappa)\,.$$

Consequently, $\alpha(b, v_1, \dots, v_\kappa)$ is linear in b and independent of $v_1, \dots, v_\kappa$. Similarly, $\beta_1(b, v_1, \dots, v_\kappa)$ is linear in v_1 and independent of $b, v_2, \dots, v_\kappa$, and ... and $\beta_\kappa(b, v_1, \dots, v_\kappa)$ is linear in v_κ and independent of $b, v_1, \dots, v_{\kappa-1}$.

So, we can write

$$B((a, u_1, \dots, u_\kappa), (b, v_1, \dots, v_\kappa)) = ((a, u_1, \dots, u_\kappa), (\alpha(b), \beta_1(v_1), \dots, \beta_\kappa(v_\kappa)))$$

for all $a, b \in (A^{[F]})^m$, $u_1, v_1 \in (V_1^{[F]})^{n_1}$, ..., $u_\kappa, v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where $\alpha(b) \in (A^{[F]})^m$ is linear in b and $\beta_1(v_1) \in (V_1^{[F]})^{n_1}$ is linear in v_1 and ... and $\beta_\kappa(v_\kappa) \in (V_\kappa^{[F]})^{n_\kappa}$ is linear in v_κ.

Let $\varphi : \mathbf{R}^{m;n_1,\dots,n_\kappa} \to \mathbf{R}^{m;n_1,\dots,n_\kappa}$ be given by

$$\varphi(x, y_1, \dots, y_\kappa) = (x + x^1 x, y_1 + x^1 y_1, \dots, y_\kappa + x^1 y_\kappa)$$

for all $x = (x^1, \dots, x^m) \in \mathbf{R}^m$ and $y_1 \in \mathbf{R}^{n_1}$ and ... and $y_\kappa \in \mathbf{R}^{n_\kappa}$. It is a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-map on the open and dense subset in $\mathbf{R}^{m;n_1,\dots,n_\kappa}$ satisfying $x^1 \neq -1$. Then, using the invariance of B with respect to φ and the local expression for $TF\varphi$, we get the conditions

$$((a + a^1 a, u_1 + a^1 u_1, \dots), (\alpha(b + a^1 b + b^1 a), \beta_1(v_1 + a^1 v_1 + b^1 u_1), \dots))$$

$$= ((a + a^1 a, u_1 + a^1 u_1, \dots), (\alpha(b) + a^1\alpha(b) + \alpha^1(b)a, \beta_1(v_1) + a^1\beta_1(v_1) + \alpha^1(b)u_1, \dots))$$

for all $a, b \in (A^{[F]})^m$ and $u_1, v_1 \in (V_1^{[F]})^{n_1}$ and ... and $u_\kappa, v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where we write $(\alpha^1(a), \dots, \alpha^m(a)) = \alpha(a) \in (A^{[F]})^m$ and $(\alpha^1(b), \dots, \alpha^m(b)) = \alpha(b) \in (A^{[F]})^m$ and $(a^1, \dots, a^m) = a \in (A^{[F]})^m$ and $(b^1, \dots, b^m) = b \in (A^{[F]})^m$. Then

$$\alpha(a^1 b) + \alpha(b^1 a) = a^1\alpha(b) + \alpha^1(b)a\ ,$$

$$\beta_1(a^1 v_1) + \beta_1(b^1 u_1) = a^1\beta_1(v_1) + \alpha^1(b)u_1\ ,$$

....

$$\beta_\kappa(a^1 v_\kappa) + \beta_\kappa(b^1 u_\kappa) = a^1\beta_\kappa(v_\kappa) + \alpha^1(b)u_\kappa\ .$$

Putting $a^1 = 1$, we get

$$\alpha(b^1 a) = \alpha^1(b)a\ ,\ \beta_1(b^1 u_1) = \alpha^1(b)u_1\ , \dots, \beta_\kappa(b^1 u_\kappa) = \alpha^1(b)u_\kappa\ .$$

Then putting $b = (1, 0, \dots, 0) \in (A^{[F]})^m$, we get

$$\alpha(a) = c_1 a\ ,\ \beta_1(u_1) = c_1 u_1\ , \dots, \beta_\kappa(u_\kappa) = c_1 u_\kappa$$

for any $a = (a^1, \dots, a^m) \in (A^{[F]})^m$ with $a^1 = 1$ and any $u_1 \in (V_1^{[F]})^{n_1}$ and ... and $u_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where $c_1 := \alpha^1(1, 0, \dots, 0) \in A^{[F]}$. Quite similarly (replacing 1 by $i \in \{1, \dots, m\}$) we get

$$\alpha(a) = c_i a\ ,\ \beta_1(u_1) = c_i u_1\ , \dots, \beta_\kappa(u_\kappa) = c_i u_\kappa$$

for any $a = (a^1, \dots, a^m) \in (A^{[F]})^m$ with $a^i = 1$ and any $u_1 \in (V_1^{[F]})^{n_1}$ and ... and $u_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where $c_i := \alpha^i(0, \dots, 1, \dots, 0) \in A^{[F]}$ (1 in i-th position). Then

$$\alpha(a) = ca\ ,\ \beta_1(u_1) = cu_1\ , \dots, \beta_\kappa(u_\kappa) = cu_\kappa$$

for any $a = (a^1, \dots, a^m) \in (A^{[F]})^m$ and any $u_1 \in (V_1^{[F]})^{n_1}$ and ... and $u_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where $c = c_1 = \dots = c_m \in A^{[F]}$. That $c_1 = \dots = c_m$ follows from the invariance of B with respect to the permutations of the base coordinates. Then

$$B((a, u_1, \dots, u_\kappa), (b, v_1, \dots, v_\kappa)) = ((a, u_1, \dots, u_\kappa), (cb, cv_1, \dots, cv_\kappa))$$

for all $a, b \in (A^{[F]})^m$, $u_1, v_1 \in (V_1^{[F]})^{n_1}, \dots, u_\kappa, v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$, where $c \in A^{[F]}$ is as above. Then $B = \mathrm{af}(c)$, as well. □

8. The natural vector fields on ppgb-functors

Let $I = (I_1, \dots, I_{\kappa-1})$ be a system (as in Section 1) consisting of A-module homomorphisms $I_i : V_{i+1} \to V_i$ for $i = 1, ..., \kappa - 1$, where A is a Weil algebra and $V_1, ..., V_\kappa$ are finite dimensional (over $\mathbf{R}$) A-modules.

Definition 8.1. *A derivation of I is a system $D = (\tilde{\alpha}, \tilde{\beta}_1, ..., \tilde{\beta}_\kappa)$ of $\mathbf{R}$-linear maps $\tilde{\alpha} : A \to A$ and $\tilde{\beta}_i : V_i \to V_i$ for $i = 1, ..., \kappa$ such that*

$$\tilde{\alpha}(ab) = a\tilde{\alpha}(b) + \tilde{\alpha}(a)b\ ,\ \tilde{\beta}_i(av_i) = a\tilde{\beta}_i(v_i) + \tilde{\alpha}(a)v_i$$

for all $a, b \in A$, $v_i \in V_i$ and $i = 1, ..., \kappa$ and

$$\tilde{\beta}_i \circ I_i = I_i \circ \tilde{\beta}_{i+1}$$

for $i = 1, ..., \kappa - 1$.

Let $F = F^{[I]}$ be the ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$ from I. Using a derivation $D = (\tilde{\alpha}, \tilde{\beta}_1, ..., \tilde{\beta}_\kappa)$ of I we can define canonical vector field $\mathrm{Op}(D)$ on FK for any $\mathcal{F}_\kappa\mathcal{VB}$-object K as follows. We define $\alpha : A \to A \times A$ and $\beta_i : V_i \to V_i \times V_i$ for $i = 1, ..., \kappa$ by

$$\alpha(a) = (a, \tilde{\alpha}(a))\ ,\ \beta_i(v_i) = (v_i, \tilde{\beta}_i(v_i))\ ,$$

$a \in A$, $v_i \in V_i$, $i = 1, ..., \kappa$.

It is easy to see that $(\alpha, \beta_1, ..., \beta_\kappa)$ is a morphism $I \to I \otimes I^{[T]}$. So, we have the corresponding natural transformation $\mathrm{Op}(D) : FK \to TFK$ for any $\mathcal{F}_\kappa\mathcal{VB}$-object K. Locally $\mathrm{Op}(D) : A^m \times V_1^{n_1} \times ... \times V_\kappa^{n_\kappa} \to T(A^m \times V_1^{n_1} \times ... \times V_\kappa^{n_\kappa})$ satisfies the formula

$$\mathrm{Op}(D)((a^j), (v_1^{j_1}), ..., (v_\kappa^{j_\kappa})) = (((a^j), (v_1^{j_1}), ..., (v_\kappa^{j_\kappa})), ((\tilde{\alpha}(a^j)), (\tilde{\beta}_1(v_1^{j_1})), ..., (\tilde{\beta}_\kappa(v_\kappa^{j_\kappa}))))$$

(modulo the standard identification) for any $(a^j) \in A^m$, $(v_1^{j_1}) \in V_1^{n_1}$,..., $(v_\kappa^{j_\kappa}) \in (V_\kappa^{[F]})^{n_\kappa}$. Hence $\mathrm{Op}(D)$ is a vector field on FK for any $\mathcal{F}_\kappa\mathcal{VB}$-object K.

Definition 8.2. *A $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-natural vector field on F is a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-invariant family L of vector fields*

$$L \in \mathcal{X}(FK)$$

for any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-object K. It means that $TFf \circ L = L \circ Ff$ for any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-map $f : K \to K'$.

Proposition 8.3. *Let $m, n_1, ..., n_\kappa$ be positive integers. Any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-natural vector field L on F is of the form*

$$L = \mathrm{Op}(D)$$

for some derivation D of $I^{[F]}$.

Proof. Of course, L is determined by the vector field L on $F\mathbf{R}^{m;n_1,...,n_\kappa} = (A^{[F]})^m \times (V_1^{[F]})^{n_1} ... \times (V_\kappa^{[F]})^{n_\kappa}$, i.e. $L : F\mathbf{R}^{m;n_1,...,n_\kappa} \to (A^{[F]} \times A^{[F]})^m \times (V_1^{[F]} \times V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]} \times V_\kappa^{[F]})^{n_\kappa}$. We can write

$$L = ((\alpha^j), (\beta_1^{j_1}), ..., (\beta_\kappa^{j_\kappa}))\ ,$$

where

$$\alpha^j : (A^{[F]})^m \times (V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]})^{n_\kappa} \to A^{[F]} \times A^{[F]}$$

and

$$\beta_k^{j_k} : (A^{[F]})^m \times (V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]})^{n_\kappa} \to V_k^{[F]} \times V_k^{[F]}\ ,$$

$j = 1, ..., m$, $j_k = 1, ..., n_k$, $k = 1, ..., \kappa$.

Let $((x^j), (y_1^{j_1}), ..., (y_\kappa^{j_\kappa}))$ be the usual coordinates on $\mathbf{R}^{m;n_1,...,n_\kappa}$. By the invariance of L with respect to the $\mathcal{F}_\kappa\mathcal{VB}_{m;n_1,...,n_\kappa}$-maps

$$((t^j x^j), (t_1^{j_1} y_1^{j_1}), ..., (t_\kappa^{j_\kappa} y_\kappa^{j_\kappa})) : \mathbf{R}^{m;n_1,...,n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$$

for all real numbers $t^j \neq 0$ and $t_k^{j_k} \neq 0$ and by the homogeneous function theorem, given $j \in \{1, ..., m\}$ we have

$$\alpha^j(a, v_1, ..., v_\kappa) = (a^j, \tilde{\alpha}^j(a^j)) ,$$

where $a = (a^1, ..., a^m) \in A^m$, $v_1 \in (V_1^{[F]})^{n_1}, ..., v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$ and where $\tilde{\alpha}^j : A \to A$ is the $\mathbf{R}$-linear map. Moreover, given $k \in \{1, ..., \kappa\}$ and $j_k \in \{1, ..., n_k\}$,

$$\beta_k^{j_k}(a, v_1, ..., v_\kappa) = (v_k^{j_k}, \tilde{\beta}_k^{j_k}(v_k^{j_k})) ,$$

where $a \in (A^{[F]})^m$, $v_1 \in (V_1^{[F]})_{n_1}, ..., v_k = (v_k^1, ..., v_k^{n_k}) \in (V_k^{[F]})^{n_k}, ..., v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$ and where $\tilde{\beta}_k^{j_k} : V_k^{[F]} \to V_k^{[F]}$ is the $\mathbf{R}$-linear map.

Applying the invariance of L with respect to the permutations of coordinates, we deduce that all $\tilde{\alpha}^j$ are equal and all $\tilde{\beta}_1^{j_1}$ are equal and ... and all $\tilde{\beta}_\kappa^{j_\kappa}$ are equal. Then we can write

$$L(a, v_1, ..., v_\kappa) = ((a^j, \tilde{\alpha}(a^j)), (v_1^{j_1}, \tilde{\beta}_1(v_1^{j_1})), ..., (v_\kappa^{j_\kappa}, \tilde{\beta}_\kappa(v_\kappa^{j_\kappa})))$$

for $a = (a^j) \in (A^{[F]})^m$, $v_1 = (v_1^{j_1}) \in (V_1^{[F]})^{n_1}$ $, ..., v_\kappa = (v_\kappa^{j_\kappa}) \in (V_\kappa^{[F]})^{n_\kappa}$, where $\tilde{\alpha} : A^{[F]} \to A^{[F]}$, $\tilde{\beta}_1 : V_1^{[F]} \to V_1^{[F]}$ $, ..., \tilde{\beta}_\kappa : V_\kappa^{[F]} \to V_\kappa^{[F]}$ are the $\mathbf{R}$-linear maps.

Next, applying the invariance of L with respect to the (locally defined) $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-map $((x^j + (x^j)^2), (y_1^{j_1}), (y_\kappa^{j_\kappa})) : \mathbf{R}^{m;,n_1,...,n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$, we derive that

$$\alpha(a + a^2) = \alpha(a) + (\alpha(a))^2$$

for any $a \in A^{[F]}$, where $\alpha : A^{[F]} \to A^{[F]} \times A^{[F]}$ is given by $\alpha(a) = (a, \tilde{\alpha}(a))$ for $a \in A^{[F]}$, and where $A^{[F]} \times A^{[F]}$ is the Weil algebra $A^{[TF]} = A^{[F]} \otimes \mathbf{D}$. Then $\alpha(a^2) = (\alpha(a))^2$ for any $a \in A^{[F]}$. By the polarization, $\alpha(ab) = \alpha(a)\alpha(b)$ for any $a, b \in A^{[F]}$. Then

$$(ab, \tilde{\alpha}(ab)) = (a, \tilde{\alpha}(a))(b, \tilde{\alpha}(b)) = (ab, a\tilde{\alpha}(b) + \tilde{\alpha}(a)b) .$$

Hence $\tilde{\alpha}(ab) = a\tilde{\alpha}(b) + \tilde{\alpha}(a)b$ for any $a, b \in A^{[F]}$.

Similarly, given $k \in \{1, ..., \kappa\}$, applying the invariance of L with respect to the (locally defined) $\mathcal{F}_\kappa\mathcal{VB}_{m;n_1,...,n_\kappa}$-map $((x^j), (y_1^{j_1}), ..., (y_k^{j_k} + x^1 y_k^{j_k}), ..., (y_\kappa^{j_\kappa})) : \mathbf{R}^{m;n_1,...,n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$, we derive that $\tilde{\beta}_k(av) = a\tilde{\beta}_k(v) + \tilde{\alpha}(a)v$ for any $a \in A^{[F]}$ and $v \in V_k^{[F]}$.

Similarly, given $k \in \{1, .., \kappa - 1\}$, applying the invariance of L with respect to the $\mathcal{F}_\kappa\mathcal{VB}_{m;n_1,...,n_\kappa}$-map $((x^j), (y_1^{j_1}), ..., (y_k^{j_k} + y_{k+1}^1), ..., (y_\kappa^{j_\kappa})) : \mathbf{R}^{m;n_1,...,n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$, we obtain that $I_k^{[F]} \circ \tilde{\beta}_{k+1}(v) = \tilde{\beta}_k \circ I_k^{[F]}(v)$ for any $v \in V_{k+1}^{[F]}$.

Hence $D := (\tilde{\alpha}, \tilde{\beta}_1, ..., \tilde{\beta}_\kappa)$ is a derivation of $I^{[F]}$, and $L = \mathrm{Op}(D)$. □

9. Lifting κ-flag-linear vector fields to ppgb-functors

Let K be a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-object.

Definition 9.1. *A vector field Z on K is called κ-flag-linear if the map $Z : K \to TK$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism.*

Lemma 9.2. *A κ-flag-linear vector field $Z : K \to TK$ on K is projectable.*

Proof. We have the underlying map $\underline{Z} : M \to TM$. It is a vector field on M. □

Using local expression of $\mathcal{F}_\kappa\mathcal{VB}$-morphisms one can easily get

Lemma 9.3. *A vector field Z on K is κ-flag-linear if and only if in any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$ coordinate system $(x^j, y_k^{j_k})_{j=1,...,m;k=1,...,\kappa;j_k=1,...,n_k}$ it is of the form*

$$Z = \sum_{j=1}^m b^j(x^1, ...x^m)\frac{\partial}{\partial x^j} + \sum_{k=1}^\kappa \sum_{q=k}^\kappa \sum_{j'_k=1}^{n_k} \sum_{j_q=1}^{n_q} b_{k,j_q}^{q,j'_k}(x^1, ..., x^m) y_q^{j_q} \frac{\partial}{\partial y_k^{j'_k}} , \tag{5}$$

where $b^j, b_{k,j_q}^{q,j'_k} : \mathbf{R}^m \to \mathbf{R}$.

Then we immediately obtain

Lemma 9.4. *Let $\lambda \in \mathbf{R}$. If Z_1, Z_2 are κ-flag-linear vector fields, then so are $Z_1 + Z_2$ and λZ_1 and $[Z_1, Z_2]$. In other words, the space $\mathcal{X}_{\kappa-FLAG-LIN}(K)$ of κ-flag-linear vector fields Z on K is the Lie subalgebra in $\mathcal{X}(K)$.*

Lemma 9.5. *If Z is a κ-flag-linear vector field on K and $f : M \to \mathbf{R}$ is a map, then $f \circ \pi \cdot Z$ is κ-flag-linear.*

From Lemma 9.3 we else obtain

Lemma 9.6. *A vector field Z on K is κ-flag-linear if and only if the flow of Z is formed by (local) $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-maps.*

Similarly as in the manifold case we have

Lemma 9.7. *Let Z be a κ-flag-linear vector field on a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-object K such that the underlying vector field $\underline{Z}$ on basis M is non-zero at a point $x_o \in M$. Then there exists a local $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-coordinate system $(x^1, ...)$ on K with centrum x_o such that $Z = \frac{\partial}{\partial x^1}$.*

Proof. We can assume $K = \mathbf{R}^{m;n_1,\dots,n_\kappa}$ and $x_o = 0$ and $\underline{Z}_{|0} = \frac{\partial}{\partial x^1}_{|0}$. Let $\{\varphi_t\}$ be the flow of Z. Then $\Phi : K \to K$ defined by $\Phi(x^1, ...) = \varphi_{x_1}(0, x^2, ...)$ is a local $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-isomorphism transforming $\frac{\partial}{\partial x^1}$ to Z. □

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$.

Proposition 9.8. *Let $Z : K \to TK$ be a κ-flag-linear vector field on a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-object K. Then*

$$\mathcal{F}Z := \eta^{[ex]} \circ FZ : FK \to TFK$$

is a κ-flag-linear vector field on FK. Moreover, $\mathcal{F}Z$ depends linearly on Z.

Proof. That $\mathcal{F}Z$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism follows from Proposition 4.2. The rest follows easily from the local expression of $FZ : FK \to FTK$ and $\eta^{[ex]} : FTK \to TFK$. □

Definition 9.9. *An $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-gauge-natural operator lifting κ-flag-linear vector fields Z on $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-objects K into vector fields $C(Z)$ on FK is a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-invariant family C of regular operators (functions)*

$$C : \mathcal{X}_{\kappa-FLAG-LIN}(K) \to \mathcal{X}(FK)$$

for any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-object K. The $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-invariance of C means that if $Z \in \mathcal{X}_{\kappa-FLAG-LIN}(K)$ and $Z' \in \mathcal{X}_{\kappa-FLAG-LIN}(K')$ are f-related (i.e. $Tf \circ Z = Z' \circ f$) for $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-map $f : K \to K'$, then $C(Z)$ and $C(Z')$ are Ff-related. The regularity of C means that C transforms smoothly parametrized families of κ-flag-linear vector fields into smoothly parametrized families of vector fields.

Theorem 9.10. *Let $m, n_1, ..., n_\kappa$ be positive integers. Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. Any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-gauge-natural operator C in the sense of Definition 9.9 is of the form*

$$C(Z) = \mathrm{af}(c) \circ \mathcal{F}Z + \mathrm{Op}(D)$$

for a (uniquely determined by C) element $c \in A^{[F]}$ and a (uniquely determined by C) derivation D of $I^{[F]}$.

Proof. Consider an operator C in question. Because of Proposition 8.3, $C(0) = \mathrm{Op}(D)$. So, replacing C by $C - C(0)$, we may assume $C(0) = 0$.

Define $\overline{C} : \mathbf{R} \times (A^{[F]})^m \times (V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]})^{n_\kappa} \to (A^{[F]})^m \times (V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]})^{n_\kappa}$ by

$$((a, v_1..., v_\kappa), \overline{C}(t, a, v_1, ..., v_\kappa)) = C\left(t\frac{\partial}{\partial x^1}\right)(a, v_1, ..., v_\kappa) ,$$

$t \in \mathbf{R}$, $a = (a^j) \in (A^{[F]})^m$, $v_1 \in (V_1^{[F]})^{n_1}$, ..., $v_\kappa \in (V_\kappa^{[F]})^{n_\kappa}$. Because of Lemma 9.7, C is uniquely determined by $\overline{C}(1, -, -, -, -)$. Because of the invariance of C with respect to the homotheties $\tau\mathrm{id} : \mathbf{R}^{m;n_1,...n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$ for $\tau \neq 0$ and the homogeneous function theorem, $\overline{C}$ is $\mathbf{R}$-linear. Then, since $C(0) = 0$, we have

$$\overline{C}(1, a, v_1, ..., v_\kappa) = \overline{C}(1) \in (A^{[F]})^m \times (V_1^{[F]})^{n_1} \times ... \times (V_\kappa^{[F]})^{n_\kappa} .$$

Now, because of the invariance of C with respect to the $\mathcal{F}_\kappa\mathcal{VB}_{m;n_1,...,n_\kappa}$-maps

$$(x^1, \tau x^2, ..., \tau x^m, (\tau y_1^{j_1}), ..., (\tau y_\kappa^{j_\kappa})) : \mathbf{R}^{m;,n_1,...,n_\kappa} \to \mathbf{R}^{m;n_1,...,n_\kappa}$$

for $\tau \neq 0$, where $((x^j), (y_1^{j_1}), ..., (y_\kappa^{j_\kappa}))$ are the usual coordinates on $\mathbf{R}^{m;,n_1,...,n_\kappa}$, we derive that

$$\overline{C}(1) \in A^{[F]} \times \{0\} \times ... \times \{0\} .$$

So, the vector space of all such C is of dimension $\leq \dim_{\mathbf{R}}(A^{[F]})$. Then the dimension argument ends the proof.

□

Lemma 9.11. *Let Z be a κ-flag-linear vector field on K and $f : M \to \mathbf{R}$ be a map. Then*

$$\mathcal{F}(f \circ \pi \cdot Z) = Ff \circ F\pi \cdot \mathcal{F}Z \,, \tag{6}$$

where $\pi : K \to M$ is the projection being $\mathcal{F}_\kappa\mathcal{VB}$-map (we treated M as the trivial $\mathcal{F}_\kappa\mathcal{VB}$-object) and $Ff : FM \to F\mathbf{R} = A^{[F]}$ and where $a \cdot y := \mathrm{af}(a)(y)$ for $a \in A^{[F]}$ and $y \in TFK$.

Proof. By Lemma 9.5, both sides of (6) have sense. By the linearity of $\mathcal{F}$, we can assume Z is not π-vertical. Then by Lemma 9.7 we can assume $K = \mathbf{R}^{m;n_1,...,n_\kappa}$ and $Z = \frac{\partial}{\partial x^1}$. Then we can assume $K = M$ is a manifold and Z is a vector field on M. Then our formula is the well-known one $\mathcal{F}(fZ) = Ff \cdot \mathcal{F}Z$ for Weil functors F on manifolds. □

If Z and Z_1 are κ-flag- linear vector fields on K then so is $[Z, Z_1]$, see Lemma 9.4.

Proposition 9.12. *For any κ-flag-linear vector fields Z and Z_1 on K and any $a, a_1 \in A^{[F]}$ it holds*

$$[\mathrm{af}(a) \circ \mathcal{F}Z, \mathrm{af}(a_1) \circ \mathcal{F}Z_1] = \mathrm{af}(aa_1) \circ \mathcal{F}([Z, Z_1]) \tag{7}$$

Proof. We can assume that $K = \mathbf{R}^{m;n_1,...,n_\kappa}$, $Z = \frac{\partial}{\partial x^1}$ and $Z_1 = f(x^1, ..., x^m)Z_2$, where $Z_2 = \frac{\partial}{\partial x^j}$ or $u_q^{j_q} \frac{\partial}{\partial u_k^{j'_k}}$, where $k = 1, ..., \kappa$, $q = k, ..., \kappa$, $j'_k = 1, ..., n_k$, $j_q = 1, ..., n_q$, $j = 1, ..., m$.

If $Z_2 = \frac{\partial}{\partial x^j}$, then the formula is the well-know one for Weil functors on manifolds.

For others Z_2, using formula (6) and the well-known formula $a\mathcal{F}Z(a_1Ff) = aa_1F(Z(f))$ for Weil functor on manifolds, we derive

$$\begin{aligned}&[\mathrm{af}(a) \circ \mathcal{F}Z, \mathrm{af}(a_1) \circ \mathcal{F}(fZ_2)] = [a \cdot \mathcal{F}Z, a_1Ff \cdot \mathcal{F}Z_2] = \\ &= a\mathcal{F}Z(a_1Ff) \cdot \mathcal{F}Z_2 = aa_1F(Z(f)) \cdot \mathcal{F}Z_2 = aa_1 \cdot \mathcal{F}(Z(f)Z_2) = \mathrm{af}(aa_1) \circ \mathcal{F}([Z, Z_1]).\end{aligned}$$

□

Lemma 9.13. *For any κ-flag-linear vector field Z on FK and any $a \in A^{[F]}$, the vector field $\mathrm{af}(a) \circ Z$ is also a κ-flag-linear vector field on FK.*

Proof. Since $\mathrm{af}(a) : TFK \to TFK$ is a $\mathcal{F}_\kappa\mathcal{VB}$-natural transformation, then it is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism. So, since $Z : FK \to TFK$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism, then so is $\mathrm{af}(a) \circ Z : FK \to TFK$. Since $\mathrm{af}(a) : TFK \to TFK$ is an affinor on FK and Z is a vector field on FK, then $\mathrm{af}(a) \circ Z$ is a vector field on FK. □

10. The complete lifting of κ-flag-linear semi-basic tangent valued p-forms

Definition 10.1. *If $\pi : K \to M$ is a fibred manifold, a projectable semi-basic tangent valued p-form on K is a section $\varphi : K \to \wedge^p T^*M \otimes TK$ such that $\varphi(X_1, ..., X_p)$ is a projectable vector field on K for any vector fields $X_1, ..., X_p$ on M.*

Given a projectable semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ we have the underlying tangent valued p-form $\underline{\varphi} : M \to \wedge^p T^*M \otimes TM$ on M such that $\underline{\varphi}(X_1, ..., X_p)$ is the underlying vector field of $\varphi(X_1, ..., X_p)$ for any vector fields $X_1, ..., X_p$ on M. Let K be a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-object with basis M.

Definition 10.2. *A κ-flag-linear semi-basic tangent valued p-form on K is a projectable semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on (fibered manifold) K such that $\varphi(X_1, ..., X_p)$ is a κ-flag-linear vector field on K for any vector fields $X_1, ..., X_p$ on the basis M of K.*

Because of Lemma 9.3, any κ-flag-linear semi-basic tangent valued p-form φ on K has (in any $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,...,n_\kappa}$-coordinates $(x^j, y_k^{j_k})_{j=1,...,m;k=1,...,\kappa;j_k=1,...,n_k}$ on K) the expression

$$\varphi = \sum_{j=1}^m \varphi^j \otimes \frac{\partial}{\partial x^j} + \sum_{k=1}^\kappa \sum_{q=k}^\kappa \sum_{j'_k=1}^{n_k} \sum_{j_q=1}^{n_q} \varphi_{k,j_q}^{q,j'_k} \otimes_{\mathbf{R}} y_q^{j_q} \frac{\partial}{\partial y_k^{j'_k}} \tag{8}$$

for (uniquely determined) real valued p-forms φ^j and φ_{k,j_q}^{q,j'_k} (and vice-versa), where $(\omega \otimes_{\mathbf{R}} Z)(X_1, ..., X_p) := \omega(X_1, ..., X_p) \circ \pi \cdot Z$.

Lemma 10.3. *A section $\varphi : K \to \wedge^p T^*M \otimes TK$ is a κ-flag-linear semi-basic tangent valued p-form on K if and only if $\varphi : TM \times_M ... \times_M TM \times_M K \to TK$ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism from the $\mathcal{F}_\kappa\mathcal{VB}$-object $TM \times_M ... \times_M TM \times_M K$ (with basis $TM \times_M ... \times_M TM \times_M M$) to TK (with basis TM).*

Proof. We may assume $K = \mathbf{R}^{m;n_1,...,n_\kappa}$. Then $\varphi : \mathbf{R}^{m+pm;n_1,...,n_\kappa} \to \mathbf{R}^{2m;2n_1,...,2n_\kappa}$. Now, the lemma is an immediate consequence of the following clear fact (being the consequence of the local expression of $\mathcal{F}_\kappa\mathcal{VB}$-morphisms): φ is a $\mathcal{F}_\kappa\mathcal{VB}$-morphism if and only if $\varphi(-, x_o, -) : \mathbf{R}^{m;n_1,...,n_\kappa} \to \mathbf{R}^{2m;2n_1,...,2n_\kappa}$ is $\mathcal{F}_\kappa\mathcal{VB}$-morphism for any $x_o \in \mathbf{R}^{pm}$. □

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$. Consider a κ-flag-linear semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on K. Applying F to the $\mathcal{F}_\kappa\mathcal{VB}$-morphism $\varphi : TM \times_M ... \times_M TM \times_M K \to TK$, we produce a $\mathcal{F}_\kappa\mathcal{VB}$-morphism $F\varphi : FTM \times_{FM} ... \times_{FM} FTM \times_{FM} FK \to FTK$. Then applying the exchange isomorphism $\eta^{[ex]}$, we obtain a $\mathcal{F}_\kappa\mathcal{VB}$-morphism

$$\mathcal{F}\varphi := \eta^{[ex]} \circ F\varphi \circ ((\eta^{[ex]})^{-1} \times ... \times (\eta^{[ex]})^{-1} \times \mathrm{id}_{FK}) : TFM \times_{FM} ... \times_{FM} TFM \times_{FM} FK \to TFK\,.$$

Theorem 10.4. *The above morphism $\mathcal{F}\varphi$ is the unique κ-flag-linear semi-basic tangent valued p-form $\mathcal{F}\varphi : FK \to \wedge^p T^*FM \otimes TFK$ on FK such that*

$$\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, ..., \mathrm{af}(a_p) \circ \mathcal{F}X_p) = \mathrm{af}(a_1 \cdot ... \cdot a_p) \circ \mathcal{F}(\varphi(X_1, ..., X_p)) \tag{9}$$

for any vector fields $X_1, ..., X_p$ on M and any $a_1, ..., a_p \in A^{[F]}$.

Proof. We may assume $K = \mathbf{R}^{m;n_1,...,n_\kappa}$ and φ is of the form (8). Then

$$\mathcal{F}\varphi := \sum_{j=1}^m \mathcal{F}\varphi^j \otimes_{A^{[F]}} \mathcal{F}\frac{\partial}{\partial x^j} + \sum_{k=1}^\kappa \sum_{q=k}^\kappa \sum_{j'_k=1}^{n_k} \sum_{j_q=1}^{n_q} \mathcal{F}\varphi_{k,j_q}^{q,j'_k} \otimes_{A^{[F]}} \mathcal{F}\left(y_q^{j_q} \frac{\partial}{\partial y_k^{j'_k}}\right),$$

where $\mathcal{F}\omega := F\omega \circ ((\eta^{[ex]})^{-1} \times ... \times (\eta^{[ex]})^{-1}) : TFM \times_{FM} ... \times_{FM} TFM \to A^{[F]}$ is the so called complete lift of a p-form $\omega : TM \times_M ... \times_M TM \to \mathbf{R}$ on M to F, $(\mathcal{F}\omega \otimes_{A^{[F]}} \mathcal{F}Z)(Y_1, ..., Y_p) := \mathcal{F}\omega(Y_1, ..., Y_p) \circ F\pi \cdot \mathcal{F}Z$ for $Y_1, ..., Y_p \in \mathcal{X}(F\mathbf{R}^m)$, and $c \cdot v := \mathrm{af}(c)(v)$, $c \in A^{[F]}$, $v \in TFK$.

It is a well-known fact (from the theory of usual Weil functors F on manifolds) that $\mathcal{F}\omega$ is a $A^{[F]}$-valued p-form on FM such that

$$\mathcal{F}\omega(\mathrm{af}(a_1)\circ\mathcal{F}X_1,...,\mathrm{af}(a_p)\circ\mathcal{F}X_p) = a_1\cdot...\cdot a_p\cdot F(\omega(X_1,...,X_p))$$

for any vector fields $X_1,...,X_p$ on M and any $a_1,...,a_p\in A^{[F]}$. That is why we have (9) for any vector fields $X_1,...,X_p$ on M and any $a_1,...,a_p\in A^{[F]}$. It is also a well-known fact (from the theory of usual Weil functors F on manifolds) that the vector fields $\mathrm{af}(c)\circ\mathcal{F}(X)$ for all $X\in\mathcal{X}(M)$ and all $c\in A^{[F]}$ generate (over $C^\infty(FM)$) the module $\mathcal{X}(FM)$. So, the unique part of the theorem holds, too. □

Definition 10.5. *The κ-flag-linear semi-basic tangent valued p-form $\mathcal{F}\varphi : FK\to\wedge^pT^*FM\otimes TFK$ on FK satisfying condition (9) from Theorem 10.4 is called the complete lift of φ to F.*

11. The F-N-bracket and κ-flag-linear (semi-basic) tangent valued p-forms

Lemma 11.1. *Let $\pi : K\to M$ be a fibred manifold. Given a projectable semi-basic tangent valued p-form $\varphi : K\to\wedge^pT^*M\otimes TK$ on K and a projectable semi-basic tangent valued q-form $\psi : K\to\wedge^qT^*M\otimes TK$ on K the Frölicher-Nijenhuis bracket (F-N-bracket) $[[\varphi,\psi]]$ is (again) a projectable semi-basic tangent valued $(p+q)$-form $[[\varphi,\psi]] : K\to\wedge^{p+q}T^*M\otimes TK$ on K satisfying*

$$\begin{aligned}&[[\varphi,\psi]](X_1,...,X_{p+q}) =\\ &\tfrac{1}{p!q!}\textstyle\sum_\sigma \mathrm{sign}\sigma[\varphi(X_{\sigma1},...,X_{\sigma p}),\psi(X_{\sigma(p+1)},...,X_{\sigma(p+q)})]\\ &+\tfrac{-1}{p!(q-1)!}\textstyle\sum_\sigma \mathrm{sign}\sigma\psi([\underline{\varphi}(X_{\sigma1},...,X_{\sigma p}),X_{\sigma(p+1)}],X_{\sigma(p+2)},...)\\ &+\tfrac{(-1)^{pq}}{(p-1)!q!}\textstyle\sum_\sigma \mathrm{sign}\sigma\varphi([\underline{\psi}(X_{\sigma1},...,X_{\sigma q}),X_{\sigma(q+1)}],X_{\sigma(q+2)},...)\\ &+\tfrac{(-1)^{p-1}}{(p-1)!(q-1)!2!}\textstyle\sum_\sigma \mathrm{sign}\sigma\psi(\underline{\varphi}([X_{\sigma1},X_{\sigma2}],X_{\sigma3},...),X_{\sigma(p+2)},...)\\ &+\tfrac{(-1)^{(p-1)q}}{(p-1)!(q-1)!2!}\textstyle\sum_\sigma \mathrm{sign}\sigma\varphi(\underline{\psi}([X_{\sigma_1},X_{\sigma2}],X_{\sigma3},...),X_{\sigma(q+2)},...)\end{aligned}\tag{10}$$

for any vector fields $X_1,...,X_{p+q}$ on M, where sums are over all permutations $\sigma : \{1,...,p+q\}\to\{1,...,p+q\}$.

Proof. It is well-known fact, see e.g. [6]. □

Proposition 11.2. *Let K be a $\mathcal{F}_\kappa\mathcal{VB}$-object with basis M. Let $\varphi : K\to\wedge^pT^*M\otimes TK$ be a κ-flag-linear (then projectable) semi-basic tangent valued p-form on K and $\psi : K\to\wedge^qT^*M\otimes TK$ be a κ-flag-linear semi-basic tangent valued q-form on K. Then the Frölicher-Nijenhuis bracket $[[\varphi,\psi]] : K\to\wedge^{p+q}T^*M\otimes TK$ of φ and ψ is a κ-flag-linear semi-basic tangent valued $(p+q)$-form on K.*

Proof. It is a simple consequence of formula (10) and Lemma 9.4 and Definition 10.2. □

Let $\varphi : K\to\wedge^pT^*M\otimes TK$ be a κ-flag-linear semi-basic tangent valued p-form on K and let $\psi : K\to\wedge^qT^*M\otimes TK$ be a κ-flag-linear semi-basic tangent valued q-form on K. Then we have the κ-flag-linear semi-basic tangent valued $(p+q)$-form $[[\varphi,\psi]]$ on K, and then we have the κ-flag-linear semi-basic tangent valued $(p+q)$-form $\mathcal{F}([[\varphi,\psi]])$ on FK. On the other hand, we have the κ-flag-linear semi-basic tangent valued p-form $\mathcal{F}\varphi$ on FK and we have the κ-flag-linear semi-basic tangent valued q-form $\mathcal{F}\psi$ on FK, and then we have the κ-flag-linear semi-basic tangent valued $(p+q)$-form $[[\mathcal{F}\varphi,\mathcal{F}\psi]]$ on FK.

Theorem 11.3. *We have*

$$\mathcal{F}([[\varphi,\psi]]) = [[\mathcal{F}\varphi,\mathcal{F}\psi]]\,.\tag{11}$$

Proof. The proof is almost (algebraically) the same as the one of Theorem 2 in [18]. More detailed, using Theorem 10.4 and Proposition 9.12 and Lemma 11.1 one can easily show that the left hand side of (11) at $(\mathrm{af}(a_1)\circ\mathcal{F}X_1,...,\mathrm{af}(a_{p+q})\mathcal{F}X_{p+q})$ is equal to the right hand side of (11) at $(\mathrm{af}(a_1)\circ\mathcal{F}X_1,...,\mathrm{af}(a_{p+q})\circ\mathcal{F}X_{p+q})$ for any $a_1,...,a_{p+q}\in A^{[F]}$ and any vector fields $X_1,...,X_{p+q}$ on M. □

12. An application to prolongation of κ-flag-linear connections

Let K be a $\mathcal{F}_\kappa\mathcal{VB}$-object with basis M.

Definition 12.1. *A κ-flag-linear connection in $K \to M$ is a κ-flag-linear semi-basic tangent valued 1-form $\Gamma : K \to T^*M \otimes TK$ on K such that the underlying vector field of $\Gamma(X)$ is equal to X for any vector field X on basis M.*

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$.

Lemma 12.2. *Given a κ-flag-linear connection Γ in $K \to M$, its complete lift $\mathcal{F}\Gamma$ is a κ-flag-linear connection in $FK \to FM$.*

Proof. Since $\Gamma(X)$ is a κ-flag-linear vector field on K with the underlying vector field equal to X, then $\mathcal{F}\Gamma(\mathrm{af}(a) \circ \mathcal{F}X) = \mathrm{af}(a) \cdot \mathcal{F}(\Gamma(X))$ is a κ-flag-linear vector field with the underlying vector field equal to $\mathrm{af}(a) \circ \mathcal{F}X$. Then $\mathcal{F}\Gamma(Y)$ is a κ-flag-linear vector field with the underlying vector field equal to Y for any vector field $Y \in \mathcal{X}(FM)$. □

Definition 12.3. *A curvature of a κ-flag-linear connection Γ in $K \to M$ is*

$$\mathcal{R}_\Gamma := \frac{1}{2}[[\Gamma, \Gamma]] : K \to \wedge^2 T^*M \otimes VK\,.$$

Equivalently, $\mathcal{R}_\Gamma(X_1, X_2) = [\Gamma(X_1), \Gamma(X_2)] - \Gamma([X_1, X_2])$ for any $X_1, X_2 \in \mathcal{X}(M)$.

Theorem 12.4. *It holds*

$$\mathcal{R}_{\mathcal{F}\Gamma} = \mathcal{F}(\mathcal{R}_\Gamma)\,. \tag{12}$$

Proof. By (11), $[[\mathcal{F}\Gamma, \mathcal{F}\Gamma]] = \mathcal{F}([[\Gamma, \Gamma]])$.
□

13. An application to torsion of κ-flag-linear connections in $FK \to M$

Let F be a ppgb-functor on $\mathcal{F}_\kappa\mathcal{VB}$ and K be a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-object with basis M. Then we have the fibred manifold $p_K : FK \to M$ (or simply $FK \to M$). We have also the $\mathcal{F}_\kappa\mathcal{VB}$-object FK with basis FM.

Definition 13.1. *A κ-flag-linear semi-basic tangent valued p-form on $FK \to M$ is a projectable semi-basic tangent valued p-form $\varphi : FK \to \wedge^p T^*M \otimes TFK$ on (fibered manifold) FK (with basis M) such that (additionally) $\varphi(X_1, \dots, X_p)$ is a κ-flag-linear vector field on $\mathcal{F}_\kappa\mathcal{VB}$-object FK (with basis FM) for any vector fields $X_1, \dots, X_p$ on M.*

Proposition 13.2. *Let $\varphi : FK \to \wedge^p T^*M \otimes TFK$ be a κ-flag-linear (then projectable) semi-basic tangent valued p-form on $FK \to M$ and $\psi : FK \to \wedge^q T^*M \otimes TFK$ be a κ-flag-linear semi-basic tangent valued q-form on $FK \to M$. Then the F-N bracket $[[\varphi, \psi]] : FK \to \wedge^{p+q} T^*M \otimes TFK$ of φ and ψ is a κ-flag-linear semi-basic tangent valued $(p+q)$-form on $FK \to M$.*

Proof. It is a simple consequence of formula (10) and Lemma 9.4 and Definition 13.1. □

Definition 13.3. *A κ-flag-linear connection in $FK \to M$ is a κ-flag-linear semi-basic tangent valued 1-form $\Gamma : FK \to T^*M \otimes TFK$ on $FK \to M$ such that the underlying vector field of $\Gamma(X)$ is equal to X for any vector field X on basis M.*

Let $\Gamma : FK \to T^*M \otimes TFK$ be a κ-flag-linear connection in $FK \to M$ and let $B : TFK \to TFK$ be a $\mathcal{F}_\kappa\mathcal{VB}_{m,n_1,\dots,n_\kappa}$-natural affinor on FK. If $m \geq 2$, then $B = \mathrm{af}(c)$ for some $c \in A^{[F]}$. Then, because of Lemma 9.13, $B \circ \Gamma(X)$ is a κ-flag-linear vector field on FK for any vector field X on M. Moreover, if $c = \lambda + n$, where $\lambda \in \mathbf{R}$ and n is nilpotent, then $B \circ \Gamma(X)$ is projectable with the underlying vector field λX. So, $B \circ \Gamma$ and Γ are κ-flag-linear semi-basic tangent valued 1-forms on $FK \to M$, where $(B \circ \Gamma)(X) := B \circ \Gamma(X)$ for any vector field X on M.

Definition 13.4. *The F-N bracket*

$$\tau^B(\Gamma) := [[\Gamma, B \circ \Gamma]]$$

is called the torsion of type B of Γ.

Theorem 13.5. *Let F and* Γ *and B be as above. Assume* $m, n_1, ..., n_\kappa$ *are non-negative integers with* $m \geq 2$. *The torsion of type B of* Γ *is a* κ*-flag-linear semi-basic tangent valued* 2*-form* $\tau^B(\Gamma) : FK \to \wedge^2 T^*M \otimes VFK$ *on FK. If* $B = \mathrm{af}(c)$, *where* $c = \lambda + n$, $\lambda \in \mathbf{R}$, $n \in A^{[F]}$ *is a nilpotent, then*

$$\tau^B(\Gamma)(X, Y) = 2\lambda\mathcal{R}_\Gamma(X, Y) + [\Gamma(X), \mathrm{af}(n) \circ \Gamma(Y)] - [\Gamma(Y), \mathrm{af}(n) \circ \Gamma(X)] - \mathrm{af}(n) \circ \Gamma([X, Y])$$

for any vector fields X and Y on M.

Proof. We apply the formulas of the F-N-bracket and of the curvature. □

Remark 13.6. *In particular, if* $K = (M; M, M, ..., M)$ *and* $F = T$ *and* $B = J$ *is the almost tangent structure (i.e.* $A^{[F]} = \mathbf{D}$, $c = n = (0, 1) \in \mathbf{D}$, $\lambda = 0$*), then* $\tau^J(\Gamma)$ *is (almost) the usual torsion of a usual linear connection* Γ *on M. Indeed, if* $x^1, ..., x^m$ *are local coordinates on M and* $x^1, ..., x^m, y^1, ..., y^m$ *the induced coordinates on TM, then* $J = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial y^i}$. *If* $\Gamma(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k}$ *, then* $J \circ \Gamma(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$. *Then* $\tau^J(\Gamma)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = (\Gamma^k_{ij} - \Gamma^k_{ji})\frac{\partial}{\partial y^k}$ *(the Einstein summation convention is used). Therefore our definition of torsion generalizes the classical concept of torsion of (usual) linear connection and joint the classical curvature and the classical torsion of linear connection. In [10], the authors define the torsion of* Γ *of type B as the F-N-bracket* $[[\Gamma, B]]$.

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