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Some estimates in $L_p(\Gamma)$ for maximal commutator and commutator of maximal function

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Abstract. In this article, maximal commutators and commutators of maximal function through bounded mean oscillation functions in $L_p(\Gamma)$ space examined. New point estimates for these operators have been proven.

1. Introduction

It should be mentioned that, in recent years, there has been an increasing interest in various spaces on Carleson curves, such as Lebesgue spaces, Morrey spaces. We only mention [5],[6],[10],[13](see also the references therein).

With $v(m)$ = arc length measurement, let $\Gamma = \{t \in \mathbb{C} : t = t(m),\ 0 \leq m \leq l \leq \infty\}$ be a rectifiable Jordan curve in the complex plane.

The length Γ is defined as $l = v\Gamma$.

We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma, r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$v(\Gamma(t,r)) \leq c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

Let Γ be a composed locally rectifiable curve and let $\Gamma_1, \ldots, \Gamma_N$ be a finite number of arcs such that $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_N$. Using the embedding

$$\Gamma_j(t,\varepsilon) \subset \Gamma(t,\varepsilon) \subset \Gamma_1(t,\varepsilon) \cup \ldots \cup \Gamma_N(t,\varepsilon),$$

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the following inequality is obtained;

$$\left|\Gamma_j(t,\varepsilon)\right| \leq |\Gamma(t,\varepsilon)| \subset |\Gamma_1(t,\varepsilon)| + \ldots + |\Gamma_N(t,\varepsilon)|. \tag{1}$$

Using the condition (1), Γ is a Carleson curve if and only if every arc Γ_j is a Carleson curve.

Now, we need to give below the necessary definitions for the case of spaces on Carleson curves.

Definition 1.1. *[13]Let $1 \leq p < \infty$, $L_p(\Gamma)$ space of measurable functions on Γ with the finite norm*

$$\left\|f\right\|_{L_p(\Gamma)} = \left(\int_\Gamma \left|f(t)\right|^p dv(t)\right)^{1/p}.$$

Definition 1.2. *[13]The space of functions with bounded mean oscillation $BMO(\Gamma)$ is defined as the set of locally integrable functions f with the finite norm*

$$\left\|f\right\|_{BMO(\Gamma)} = \sup_{r>0, t\in\Gamma} (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau) < \infty,$$

where

$$f_{\Gamma(t,r)} = (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} f(\tau)\, dv(\tau).$$

Lemma 1.3. *[13]Let Γ be a Carleson curve and $1 \leq p < \infty$. Then*

$$L_\infty(\Gamma) = \sup_{t\in\Gamma, r>0} r^{-\frac{1}{p}} \left\|f\right\|_{L_p(\Gamma(t,r))}$$

and

$$\left\|f\right\|_{L_\infty(\Gamma)} \leq \sup_{t\in\Gamma, r>0} r^{-\frac{1}{p}} \left\|f\right\|_{L_p(\Gamma)} \leq c_0^{1/p} \left\|f\right\|_{L_\infty(\Gamma)}.$$

Definition 1.4. *Let $f \in L_1^{loc}(\Gamma)$. For every $\lambda > 0$ number that does not increase in $\Gamma \cap (0,\infty)$ and the f function that provides the following;*

$$\left|\{\xi \in \Gamma \cap (0,\infty) : f^*(\xi) > \lambda\}\right| = \left|\left\{x \in \Gamma : \left|f(x)\right| > \lambda\right\}\right|,$$

it is called a non-increasing rearrangement of f.

If this function is continuous from the right, then the following statement is valid;

$$f^*(\xi) = \inf\left\{\lambda > 0 : \left|\left\{x \in \Gamma : \left|f(x)\right| > \lambda\right\}\right| \leq \xi\right\}.$$

Definition 1.5. *Let $f \in L_1^{loc}(\Gamma)$. In this case, the maximal function of f is*

$$f^{**}(\xi) = \frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} f^*(\tau)\, dv(\tau), \ (t > 0)$$

defined as.

Definition 1.6. *Let $f \in L_1^{loc}(\Gamma)$. In this case, the sharp maximal function of f is*

$$M^{\neq}(\xi) = \sup_{r>0} \frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau)$$

defined as, where $f_{\Gamma(t,r)} = \frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} f(\tau)\, dv(\tau)$.

The maximal operators play a significant role in reel analysis. Maximal operators are not linear. These operators are of great importance in various problems of harmonic analysis. They also contribute to the development of the general class of singular and potential operators (see these references for detailed information, [6],[9],[11],[16]).

Definition 1.7. *Let Γ be a simple Carleson curve and $f \in L_1^{loc}(\Gamma)$. The maximal operator M on Γ is defined by*

$$Mf(t) = \sup_{t>0} (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} \left|f(\tau)\right| dv(\tau).$$

The boundedness of the maximal function in $L_p(\Gamma)$ is given by Guliyev in ([10], Lemma 4.4).

Theorem 1.8. *[10]Let Γ be a Carleson curve, $1 \leq p < \infty$ and $t_0 \in \Gamma$. Then for $p > 1$ and any $r > 0$ in Γ, the following inequality*

$$\left\|Mf\right\|_{L_p(\Gamma(t_{0,r}))} \leq r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-\frac{1}{p}} \left\|f\right\|_{L_p(\Gamma(t_{0,r}))}$$

holds for all $f \in L_p^{loc}(\Gamma)$.

Moreover, for $p = 1$ the following inequality

$$\left\|Mf\right\|_{WL_1(\Gamma(t_{0,r}))} \leq r^1 \sup_{\tau>2r} \tau^{-1} \left\|f\right\|_{L_1(\Gamma(t_{0,r}))}$$

holds for all $f \in L_1^{loc}(\Gamma)$.

The commutator operation and the properties of maximal integrals in various spaces have been studied intensively and so there exist plenty of results about them (see, for example [1]-[4],[7],[8],[12],[14],[15],[17]).

The maximal commutator M_b plays a significant role in the study of commutators of singular integral operators with the symbol BMO, and this topic has attracted the attention of many mathematicians (check it out, for example, [1],[2],[8],[12]).

The commutator of maximal operator $[M, b]$ was studied by many authors (see, for instance [1]-[3],[8]). This operator is the product of two functions from BMO and H^1 Hardy space. It emerged when it was wanted to give meaning (Let us note that the product of these two functions may not be locally integrable). Boundedness of operator $[M, b]$ in L_p using real interpolation techniques made by Milman and Schonbek, at [14].

Although the M_b and $[M, b]$ operators are very similar, they are fundamentally different.

For example, unlike the operator M_b, $[M, b]$ is neither positive nor sublinear.

The definitions of maximal commutator and commutator of maximal function on Carleson curves are as follows, respectively.

Definition 1.9. *Given a locally integrable function b, the maximal commutator is defined by*

$$M_b(f)(t) := \sup_{t>0} \frac{1}{v\Gamma(t,r)} \int_{\Gamma(t,r)} |b(t) - b(\tau)||f(\tau)| dv(\tau), \ for\ all\ t \in \Gamma.$$

Definition 1.10. *Given a locally integrable function b, the commutator of the Hardy-Littlewood maximal operator M and b is defined by*

$$[M, b] f(t) := M(bf)(t) - b(t)Mf(t), \ for\ all\ t \in \Gamma.$$

The main purpose of this article is to examine the boundedness of the maximal commutator and the commutator of maximal function in L_p spaces define on Carleson curves.

2. Auxiliary Results

To get the main results claimed in the article, we first need some auxiliary results.

Theorem 2.1. *There exist constants* C_1, C_2, *such that for every* $f \in BMO(\Gamma)$ *every* $\Gamma(t,r)$ *and every* $s > 0$*:*

$$\left|\left\{\tau \in \Gamma(t,r) : \left|f(\tau) - f_{\Gamma(t,r)}\right| > s\right\}\right| \leq C_1 e^{-\left(C_2/\|f\|_{BMO(\Gamma)}\right)s} \left|\Gamma(t,r)\right|. \qquad (2)$$

Proof. Observe, first of all, that we can assume $\|f\|_{BMO(\Gamma)} = 1$. We fix $\Gamma(t,r)$ and take $\alpha > 1$. We know that

$$\frac{1}{|\Gamma(t,r)|}\int_{\Gamma(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau) \leq 1 < \alpha.$$

We make the Calderón-Zygmund decomposition of $\Gamma(t,r)$ for the function $f - f_{\Gamma(t,r)}$ relative to α, obtaining $\Gamma_{1,j}(t,r)$ (dyadic curves of Γ) for each of which:

$$\alpha < \frac{1}{\left|\Gamma_{1,j}(t,r)\right|}\int_{\Gamma_{1,j}(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau) \leq 2\alpha.$$

Besides, for a.e. $\tau \notin \underset{j}{\cup}\Gamma_{1,j}(t,r)$ is $\left|f(\tau) - f_{\Gamma(t,r)}\right| \leq \alpha$. It follows that for each $\Gamma_{1,j}(t,r)$ is

$$\left|f_{\Gamma_{1,j}(t,r)} - f_{\Gamma(t,r)}\right| \leq 2\alpha.$$

Also:

$$\begin{aligned}\textstyle\sum_j \left|\Gamma_{1,j}(t,r)\right| &\leq \tfrac{1}{\alpha}\textstyle\int_{\Gamma_{1,j}(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau)\\ &\leq \tfrac{1}{\alpha}\textstyle\int_{\Gamma(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right| dv(\tau)\\ &\leq \tfrac{1}{\alpha}\left|\Gamma(t,r)\right|.\end{aligned}$$

On each $\Gamma_{1,j}(t,r)$ we make the Calderón-Zygmund decomposition for the function of $f - f_{\Gamma_{1,j}(t,r)}$ relative to α. Thus we obtain a family $\Gamma_{2,k}(t,r)$ of dyadic subcurves of $\Gamma_{1,j}(t,r)$, for each of which is function $\left|f_{\Gamma_{2,k}(t,r)} - f_{\Gamma_{1,j}(t,r)}\right| \leq 2\alpha$, and also for a.e. $\tau \in \Gamma_{1,j}(t,r) \setminus \left(\underset{k}{\cup}\Gamma_{2,k}(t,r)\right)$ is $\left|f(x) - f_{\Gamma_{1,j}(t,r)}\right| \leq \alpha$. Besides, $\sum_k \left|\Gamma_{2,k}(t,r)\right| \leqq \frac{1}{\alpha}\left|\Gamma_{1,j}(t,r)\right|$. Now we put together all the families $\{\Gamma_{2,k}(t,r)\}$ corresponding to different $\Gamma_{1,j}(t,r)$'s and call the resulting family also $\{\Gamma_{2,k}(t,r)\}$. Then, outside the union of the $\Gamma_{2,k}(t,r)$'s we have:

$$\begin{aligned}\left|f(\tau) - f_{\Gamma(t,r)}\right| &\leq \left|f(\tau) - f_{\Gamma_{1,j}(t,r)}\right| + \left|f_{\Gamma_{1,j}(t,r)} - f_{\Gamma(t,r)}\right|\\ &\leq 3\alpha\end{aligned}$$

and also

$$\sum_k \left|\Gamma_{2,k}(t,r)\right| \leq \left(\frac{1}{\alpha}\right)^2 \left|\Gamma(t,r)\right|.$$

Subsequently, we obtain for each natural number N, a family of nonoverlapping curves $\left\{\Gamma_{N,j}(t,r)\right\}$ in such a way that outside of their union is $\left|f(\tau) - f_{\Gamma(t,r)}\right| \leq N.2\alpha$ and such that $\sum_j \left|\Gamma_{N,j}(t,r)\right| \leq \alpha^{-N}\left|\Gamma(t,r)\right|$.

If $N.2\alpha \leq t < (N+1).2\alpha$ with $N = 1, 2, \ldots$, then

$$\begin{aligned}&\left|\left\{\tau \in \Gamma(t,r) : \left|f(\tau) - f_{\Gamma(t,r)}\right| > s\right\}\right|\\ &\leq \textstyle\sum_j \left|\Gamma_{N,j}(t,r)\right| \leq \alpha^{-N}\left|\Gamma(t,r)\right|\\ &= e^{-N\log\alpha}\left|\Gamma(t,r)\right|\\ &\leq e^{-C_2 s}\left|\Gamma(t,r)\right|\end{aligned}$$

(with $C_2 = 2^{-n-1} (\log \alpha) / \alpha$) since $t < (N+1) 2\alpha \leq N4\alpha$. On the other hand, if $t < 2\alpha$, then $C_2 s < (\log \alpha) / 2$, and we use the trivial majorization

$$\begin{aligned} &\left|\left\{\tau \in \Gamma(t,r) : \left|f(\tau) - f_{\Gamma(t,r)}\right| > s\right\}\right| \\ &\leq |\Gamma(t,r)| < e^{(\log \alpha)/2 - C_2 s} |\Gamma(t,r)| . \end{aligned}$$

Thus, we get (2) for every s by choosing C_2 as above and $C_1 = \sqrt{\alpha}$. Finally, α can be chosen to get an optimal value of the constant C_2 ($\alpha = e$). □

Lemma 2.2. *For every* p *with* $0 < p < \infty$, $BMO(\Gamma), p = BMO(\Gamma)$ *where*

$$\|f\|_{BMO(\Gamma),p} := \sup_{\Gamma(t,r)} \left(\frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left|f(\tau) - f_{\Gamma(t,r)}\right|^p dv(\tau) \right)^{1/p} .$$

Proof. Let $0 < p < 1$. Then from Hölder inequality, the following is obtained easily;

$$\|f\|_{BMO(\Gamma),p} \leq \|f\|_{BMO(\Gamma)} .$$

Now let's denote;

$$\|f\|_{BMO(\Gamma)} \leq \|f\|_{BMO(\Gamma),p} .$$

For $0 < t < \frac{|\Gamma(t,r)|}{6}$, the following is valid;

$$\begin{aligned} \left(f\chi_{\Gamma(t,r)}\right)^{**}(t) &= \left[\left(f - f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)} + f_{\Gamma(t,r)}\chi_{\Gamma(t,r)}\right]^{**}(t) \\ &\leq \left[\left(f - f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right]^{**}(t) + \left(f_{\Gamma(t,r)}\chi_{\Gamma(t,r)}\right)^{**}(t) \\ &\leq c \int_t^{|\Gamma(t,r)|} \left(f^{\neq}_{\Gamma(t,r)}\right)(s) \frac{dv(s)}{s} + |f|_{\Gamma(t,r)} . \end{aligned}$$

Let $f \in BMO(\Gamma)$. Then the following inequalies are valid;

$$\begin{aligned} &\left\| |f|^p \right\|_{BMO(\Gamma)} \\ &= \sup_{\Gamma(t,r)} \tfrac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left| |f(y)|^p - \left(|f|^p\right)_{\Gamma(t,r)} \right| dv(y) \\ &= \sup_{\Gamma(t,r)} \inf_{c\in\mathbb{R}} \tfrac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left| |f(y)|^p - c \right| dv(y) \\ &\leq \sup_{\Gamma(t,r)} \tfrac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left| |f(y)|^p - \left|f_{\Gamma(t,r)}\right|^p \right| dv(y) \\ &\leq \sup_{\Gamma(t,r)} \tfrac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left|f(y) - f_{\Gamma(t,r)}\right|^p dv(y) . \end{aligned}$$

From here, it can be seen that it is $| \, |f|^p \in BMO((\Gamma))$. Thus,

$$\begin{aligned} &\left(|f|^p \chi_{\Gamma(t,r)}\right)^{**}(t) \\ \leq \quad & c \int_t^{|\Gamma(t,r)|} \left(\left(|f|^p\right)^{\neq}_{\Gamma(t,r)}\right)^{*}(s) \frac{dv(s)}{s} + \left(|f|^p\right)_{\Gamma(t,r)} \end{aligned}$$

is obtained.

On the other hand

$$\begin{aligned}\left(|f|^{p}\right)^{\neq}_{\Gamma(t,r)}(x) &= \sup_{\Gamma(t_0,r_0)\subseteq\Gamma(t,r),\, x\in\Gamma(t_0,r_0)} \left\{\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)} \left||f(y)|^{p}-\left(|f|^{p}\right)_{\Gamma(t_0,r_0)}\right| dv(y)\right\}\chi_{\Gamma(t,r)}(x)\\ &= \sup_{\Gamma(t_0,r_0)\subseteq\Gamma(t,r),\, x\in\Gamma(t_0,r_0)} \inf_{c\in\mathbb{R}}\left\{\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)} \left||f(y)|^{p}-c\right| dv(y)\right\}\chi_{\Gamma(t,r)}(x)\\ &\leq \sup_{\Gamma(t_0,r_0)\subseteq\Gamma(t,r),\, x\in\Gamma(t_0,r_0)} \left\{\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)} \left||f(y)|^{p}-\left|f_{\Gamma(t_0,r_0)}\right|^{p}\right| dv(y)\right\}\chi_{\Gamma(t,r)}(x)\\ &\leq \sup_{\Gamma(t_0,r_0)\subseteq\Gamma(t,r),\, x\in\Gamma(t_0,r_0)} \left\{\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)} \left|f(y)-f_{\Gamma(t_0,r_0)}\right|^{p} dv(y)\right\}\chi_{\Gamma(t,r)}(x)\\ &= \left(f^{p}_{\Gamma(t,r)}(x)\right)^{p}\end{aligned}$$

is true for $0<t<\frac{|\Gamma(t,r)|}{6}$ when

$$\left(|f|^{p}\chi_{\Gamma(t,r)}\right)^{**}(t)\leq c\int_{t}^{|\Gamma(t,r)|}\left[\left(f^{p}_{\Gamma(t,r)}\right)^{*}\right]^{p}(s)\,\frac{dv(s)}{s}+\left(|f|^{p}\right)_{\Gamma(t,r)}.$$

From here

$$\begin{aligned}&\left(\left(\left(f-f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\right)^{p}\\ \leq\;& c\int_{t}^{|\Gamma(t,r)|}\left[\left(f^{p}_{\Gamma(t,r)}\right)^{*}\right]^{p}(s)\,\frac{dv(s)}{s}+\frac{1}{|\Gamma(t,r)|}\int_{\Gamma(t,r)}\left(f(y)-f_{\Gamma(t,r)}\right)^{p}dv(y),\end{aligned}$$

and so the inequality

$$\begin{aligned}&\left(\left(\left(f-f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\right)^{p}\\ \leq\;& c\,\|f\|^{p}_{BMO,p}(\Gamma(t,r))\log\frac{|\Gamma(t,r)|}{t}\\ &+\frac{1}{|\Gamma(t,r)|}\int_{\Gamma(t,r)}\left(f(y)-f_{\Gamma(t,r)}\right)^{p}dv(y),\end{aligned}$$

is true. In that case,

$$\begin{aligned}&\left(\left(\left(f-f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\right)^{p}\\ \leq\;& c\,\|f\|^{p}_{BMO,p}(\Gamma(t,r))\left(1+\log\frac{|\Gamma(t,r)|}{t}\right)\end{aligned}$$

is true. Then we get the following inequality

$$\begin{aligned}&\left(\left(f-f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\\ \leq\;& c\,\|f\|^{p}_{BMO,p}(\Gamma(t,r))\left(1+\log\frac{|\Gamma(t,r)|}{t}\right)^{\frac{1}{p}}.\end{aligned}$$

On the other hand, for $\frac{|\Gamma(t,r)|}{6}<t<|\Gamma(t,r)|$, if

$$tg^{*}(t)\leq tg^{**}(t)\leq\|g\|_{1}$$

is applied to the function $g = \left(f - f_{\Gamma(t,r)}\right)^{p}_{\chi_{\Gamma(t,r)}}$,

$$\begin{aligned}
&\left(\left(\left(f - f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\right)^{p} \\
&= \left(\left(\left(f - f_{\Gamma(t,r)}\right)^{p}\chi_{\Gamma(t,r)}\right)^{*}(t)\right) \\
&\leq \left(\left(\left(f - f_{\Gamma(t,r)}\right)^{p}\chi\Gamma(t,r)\right)^{*}\left(\frac{|\Gamma(t,r)|}{6}\right)\right) \\
&\leq \frac{6}{|\Gamma(t,r)|}\int_{\Gamma(t,r)}\left|f - f_{\Gamma(t,r)}\right|^{p} dv(x) \\
&\leq 6\left\|f\right\|^{p}_{BMO(\Gamma(t,r))}
\end{aligned}$$

the above expression is obtained. From here,

$$\left(\left(f - f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t) \leq \left\|f\right\|_{BMO(\Gamma(t,r))}.$$

It is easily seen from the obtained ones;

$$\int_{\Gamma(t,r)}\left|f - f_{\Gamma(t,r)}\right| dv(x) = \int_{0}^{|\Gamma(t,r)|}\left(\left(f - f_{\Gamma(t,r)}\right)_{\chi\Gamma(t,r)}\right)^{*}(t)\, dv(t)$$

$$\begin{aligned}
&= \left(\int_{0}^{\frac{|\Gamma(t,r)|}{6}} + \int_{\frac{|\Gamma(t,r)|}{6}}^{|\Gamma(t,r)|}\right)\left(\left(f - f_{\Gamma(t,r)}\right)\chi_{\Gamma(t,r)}\right)^{*}(t)\, dv(t) \\
&\leq c\left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}\left(\int_{0}^{\frac{|\Gamma(t,r)|}{6}}\left(1 + \log\frac{|\Gamma(t,r)|}{t}\right)^{\frac{1}{p}} dv(t) + \int_{\frac{|\Gamma(t,r)|}{6}}^{|\Gamma(t,r)|} dv(t)\right) \\
&\leq c\left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}\left(\int_{0}^{\frac{|\Gamma(t,r)|}{6}}\left(1 + \log\frac{|\Gamma(t,r)|}{t}\right)^{\frac{1}{p}} dv(t) + |\Gamma(t,r)|\right) \\
&= c\,|\Gamma(t,r)|\left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}\left(\int_{6}^{\infty}\frac{\log^{\frac{1}{p}} y}{y^{2}} dv(y) + 1\right) \\
&= c\,|\Gamma(t,r)|\left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}.
\end{aligned}$$

Hence, we get the following;

$$\left\|f\right\|_{BMO(\Gamma(t,r))} \leq \left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}.$$

Now let's assume that $1 < p < \infty$. In this situation

$$\left\|f\right\|_{BMO(\Gamma(t,r))} \leq \left\|f\right\|_{BMO_{rp}(\Gamma(t,r))}$$

inequality is found directly from the Hölder inequality. In that case,

$$\begin{aligned}
\frac{1}{|\Gamma(t,r)|}\int_{\Gamma(t,r)}\left|f(y)\, f_{\Gamma(t,r)}\right|^{p} dv(y) &= \frac{1}{|\Gamma(t,r)|}\int_{0}^{|\Gamma(t,r)|}\left\{\left(\left(f - f_{\Gamma(t,r)}\right)_{\chi_{\Gamma(t,r)}}\right)^{*}\right\}^{p}(t)\, dv(t) \\
&\leq c\frac{\left\|f\right\|^{p}_{BMO(\Gamma(t,r))}}{|\Gamma(t,r)|}\int_{0}^{|\Gamma(t,r)|}\log\left(\frac{6|\Gamma(t,r)|}{t}\right)^{p} dv(t) \\
&\leq 6c\left\|f\right\|^{p}_{BMO(\Gamma(t,r))}\left(\int_{6}^{\infty}\log^{p} u\,\frac{dv(u)}{u^{2}}\right) \\
&= c\left\|f\right\|^{p}_{BMO(\Gamma(t,r))}.
\end{aligned}$$

.

Both sides $\frac{1}{p}$. first. If the forces are taken and then all $\Gamma(t,r)$ curves are passed from both sides to the supremum;

$$\|f\|_{BMO,_p(\Gamma(t,r))} \le \|f\|_{BMO(\Gamma(t,r))}$$

is obtained, which completes the proof. □

Lemma 2.3. *If $f \in BMO(\Gamma)$ then:*

i) For every p with $0 < p < \infty$:

$$\|f\|_{BMO(\Gamma),p} \equiv \sup_{\Gamma(t,r)} \left(\frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left| f(\tau) - f_{\Gamma(t,r)} \right|^p dv(\tau) \right)^{1/p} \le C_p \|f\|_{BMO(\Gamma)}$$

with C_p dependent of f, in such a way that, for $1 < p < \infty$, $f \mapsto \|f\|_{BMO(\Gamma),p}$ is *a norm equivalent to* $f \mapsto \|f\|_{BMO(\Gamma)}$ on B.M.O.

ii) *For every λ such that $0 < \lambda < / \|f\|_{BMO(\Gamma),p}$, where C_2 is the same constant appearing* in (2), *we* have:

$$\sup_{\Gamma(t,r)} \frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} e^{\lambda \left| f(\tau) - f_{\Gamma(t,r)} \right|} dv(\tau) < \infty.$$

Proof. i)

$$\begin{aligned} \int_{\Gamma(t,r)} \left| f(x) - f_{\Gamma(t,r)} \right|^p dx &= \int_0^\infty p\tau^{p-1} \left| \left\{ \tau \in \Gamma(t,r) : \left| f(x) - f_{\Gamma(t,r)} \right| > \tau \right\} \right| dv(\tau) \\ &\le C_1 \int_0^\infty p\tau^{p-1} e^{-\left(C_2/\|f\|_{BMO(\Gamma)}\right)\tau} dv(\tau) . |\Gamma(t,r)| . \end{aligned}$$

After a change of variables

$$\begin{aligned} \tfrac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} \left| f(x) - f_{\Gamma(t,r)} \right|^p dx &\le C_1 . p \left(\|f\|_{BMO(\Gamma)} / C_2 \right)^p \int_0^\infty \tau^{p-1} e^{-\tau} dv(\tau) \\ &= C_p^p \|f\|_{BMO(\Gamma)}^p \end{aligned}$$

which gives i) with $C_p := \left(C_1 p \int_0^\infty \tau^{p-1} e^{-\tau} dv(\tau)\, C_2^{-p} \right)^{1/p}$.

If $p > 1$, we have $\|f\|_{BMO(\Gamma)} \le \|f\|_{BMO(\Gamma),p} \le C_p \|f\|_{BMO(\Gamma)}$, so that the norms $\|.\|_{BMO(\Gamma)}$ and $\|.\|_{BMO(\Gamma),p}$ are equivalent over *BMO* from Lemma 2.2. Also, if $p > 1$, Stirling's formula can be used to conclude that $C_p \le C.p$ with an absolute constant C.

ii) From Theorem 2.1

$$\begin{aligned} \int_{\Gamma(t,r)} e^{\lambda \left| f(x) - f_{\Gamma(t,r)} \right|} dv(\tau) &= \int_0^\infty \lambda e^{\lambda\tau} \left| \left\{ x \in \Gamma(t,r) : \left| f(x) - f_{\Gamma(t,r)} \right| > \tau \right\} \right| dv(\tau) \\ &\le \int_0^\infty \lambda e^{\lambda\tau} C_1 e^{-\left(C_2/\|f\|_{BMO(\Gamma)}\right)\tau} dv(\tau) . |\Gamma(t,r)| \\ &= C_1 \lambda \int_0^\infty e^{\left(\lambda - C_2/\|f\|_{BMO(\Gamma)}\right)\tau} dv(\tau) . |\Gamma(t,r)| \\ &= C_1 \lambda \left(C_2 / \|f\|_{BMO(\Gamma)} - \lambda \right)^{-1} |\Gamma(t,r)| \text{ if } 0 < \lambda < C/\|f\|_{BMO(\Gamma)} . \end{aligned}$$

□

Theorem 2.4. *Let $b \in BMO(\Gamma)$ and let $0 < \delta < 1$. Then, there exists a positive constant $C = C_\delta$, the following inequality*

$$M_\delta(M_b(f))(\varsigma) \le C \|b\|_{BMO(\Gamma)} M^2 f(\varsigma), \ \varsigma \in \Gamma$$

holds for all $f \in L_1^{loc}(\Gamma)$. Where $M_\delta f(\varsigma) := \left[M\left(|f|^\delta \right)(\varsigma) \right]^{1/\delta}$

Proof. Let $\varsigma \in \Gamma(t,r)$ and $\Gamma(t_0,r_0)$ be a fixed Carleson curve $\varsigma \in \Gamma(t_0,r_0)$. Let $f = f_1 + f_2$, where $f_1 = f_{\chi_{3\Gamma(t_0,r_0)}}$. Since for any $\tau \in \Gamma(t,r)$

$$\begin{aligned} M_b(f)(\tau) &= M((b-b(\tau))f)(\tau) \\ &= M\left(\left(b-b_{3\Gamma(t_0,r_0)}+b_{3\Gamma(t_0,r_0)}-b(\tau)\right)f\right)(\tau) \\ &\leq M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)(\tau) \\ &+M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_2\right)(\tau) \\ &+\left|b(\tau)-b_{3\Gamma(t_0,r_0)}\right|Mf(\tau), \end{aligned}$$

we have

$$\begin{aligned} &\left(\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)}(M_b(f)(\tau))^\delta dv(\tau)\right)^{1/\delta} \\ &\leq \left(\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)}\left|M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)(\tau)\right|^\delta dv(\tau)\right)^{1/\delta} \\ &+\left(\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)}\left|M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_2\right)(\tau)\right|^\delta dv(\tau)\right)^{1/\delta} \\ &+\left(\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)}\left|b(\tau)-b_{3\Gamma(t_0,r_0)}\right|^\delta (Mf(\tau))^\delta dv(\tau)\right)^{1/\delta} \\ &= A+B+C \end{aligned} \tag{3}$$

Since

$$\begin{aligned} &\int_{\Gamma(t_0,r_0)}\left|M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)(\tau)\right|^\delta dv(\tau) \\ &\leq \int_0^{|\Gamma(t_0,r_0)|}\left[M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)^*(s)\right]^\delta dv(s) \\ &\leq \left[\sup_{0<s<|\Gamma(t_0,r_0)|} sM\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)^*(s)\right]^\delta \int_0^{|\Gamma(t_0,r_0)|} s^{-\delta}dv(s), \end{aligned}$$

from Theorem 1.8

$$\begin{aligned} &\int_{\Gamma(t_0,r_0)}\left|M\left(\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right)(\tau)\right|^\delta dv(\tau) \\ &\leq \left\|\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right\|^\delta_{L_1(3\Gamma(t,r))}|\Gamma(t_0,r_0)|^{-\delta+1} \\ &= \left\|\left(b-b_{3\Gamma(t_0,r_0)}\right)f_1\right\|^\delta_{L_1(3\Gamma(t_0,r_0))}|\Gamma(t_0,r_0)|^{-\delta+1}. \end{aligned}$$

Thus

$$A \leq \frac{1}{|\Gamma(t_0,r_0)|}\int_{3\Gamma(t_0,r_0)}\left|b(\tau)-b_{3\Gamma(t_0,r_0)}\right|\left|f(\tau)\right|dv(\tau).$$

From Hölder inequality, we get

$$\begin{aligned} A &\leq \left\|b(\tau)-b_{3\Gamma(t_0,r_0)}\right\|_{\exp L,3\Gamma(t_0,r_0)}\left\|f\right\|_{L(\log L),3\Gamma(t_0,r_0)} \\ &= \left\|b(\tau)-b_{3\Gamma(t_0,r_0)}\right\|_{\exp L,3\Gamma(t_0,r_0)} \\ &\times\inf\left\{\lambda>0:\frac{1}{|3\Gamma(t_0,r_0)|}\int_{3\Gamma(t_0,r_0)}\frac{|f(\tau)|}{\lambda}\right. \\ &\left.\times\log\left(e+\frac{|f(\tau)|}{\lambda}\right)dv(\tau)\leq 1\right\}. \end{aligned}$$

For any constant $\Gamma(t_0, r_0)$ curve, there is such a fixed $C > 0$ that it is easily seen from Lemma 2.3 that the following inequality will be obtained;

$$\left\| b(\tau) - b_{3\Gamma(t_0,r_0)} \right\|_{\exp L, 3\Gamma(t_0,r_0)} \leq C \|b\|_{BMO(\Gamma)} .$$

Thus, we get the following inequality

$$\begin{aligned} A &\leq \|b\|_{BMO(\Gamma)} \\ &\times \sup_{\tau\in\Gamma(t_0,r_0)} \inf \left\{ \lambda > 0 : \tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \tfrac{|f(\tau)|}{\lambda} \right. \\ &\left. \times \log\left(e + \tfrac{|f(\tau)|}{\lambda}\right) dv(\tau) \leq 1 \right\}. \end{aligned} \tag{4}$$

for A is obtained.

Now let's make predictions for B. Since B is comparable to $\inf\limits_{\tau\in\Gamma(t_0,r_0)} M\left(\left(b - b_{3\Gamma(t_0,r_0)}\right) f\right)(\tau)$, we have

$$B \leq M\left(\left(b - b_{3\Gamma(t_0,r_0)}\right) f\right)(\varsigma) .$$

Again by Lemma 2.3, we get

$$\begin{aligned} B &\leq \left\| b(\tau) - b_{3\Gamma(t_0,r_0)} \right\|_{\exp L, 3\Gamma(t_0,r_0)} \\ &\times \inf \left\{ \lambda > 0 : \tfrac{1}{|3\Gamma(t_0,r_0)|} \int\limits_{3\Gamma(t_0,r_0)} \tfrac{|f(\tau)|}{\lambda} \right. \\ &\left. \times \log\left(e + \tfrac{|f(\tau)|}{\lambda}\right) dv(\tau) \leq 1 \right\} \\ &\leq \|b\|_{BMO(\Gamma)} \\ &\times \sup_{\tau\in\Gamma(t_0,r_0)} \inf \left\{ \lambda > 0 : \tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \tfrac{|f(\tau)|}{\lambda} \right. \\ &\left. \times \log\left(e + \tfrac{|f(\tau)|}{\lambda}\right) dv(\tau) \leq 1 \right\}. \end{aligned} \tag{5}$$

Let $\delta < \varepsilon < 1$. To obtain an estimate for C, let's use the Hölder inequality with exponents p and p', $p = \frac{\varepsilon}{\delta} > 1$:

$$\begin{aligned} C &\leq \left(\tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \left| b(\tau) - b_{3\Gamma(t_0,r_0)} \right|^{\delta p'} dv(\tau) \right)^{1/\delta p'} \\ &\times \left(\tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} (Mf(\tau))^{\delta p} dv(\tau) \right)^{1/\delta p} . \end{aligned}$$

Thus we get

$$\begin{aligned} C &\leq \|b\|_{BMO(\Gamma)} \left(\tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} (Mf(\tau))^{\varepsilon} dv(\tau) \right)^{1/\varepsilon} \\ &\leq \|b\|_{BMO(\Gamma)} M_\varepsilon (Mf)(\varsigma) . \end{aligned} \tag{6}$$

Finally, since

$$\begin{aligned} M^2 &= M_{L(\log L), \Gamma(t_0,r_0)} \\ &= \sup_{\tau\in\Gamma(t_0,r_0)} \inf \left\{ \lambda > 0 : \tfrac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \tfrac{|f(\tau)|}{\lambda} \right. \\ &\left. \times \log\left(e + \tfrac{|f(\tau)|}{\lambda}\right) dv(\tau) \leq 1 \right\}. \end{aligned}$$

and by (3) and (6), we get

$$M_\delta\left(M_b\left(f\right)\right)(\varsigma) \leq C\left\|b\right\|_{BMO(\Gamma)}\left(M_\varepsilon\left(Mf\right)(\varsigma) + M^2 f\left(\varsigma\right)\right).$$

Since

$$M_\varepsilon\left(M\left(f\right)\right)(\varsigma) \leq M^2 f\left(\varsigma\right),\ 0<\varepsilon<1,$$

the proof of the theorem is completed. □

Theorem 2.5. *Let $b \in BMO(\Gamma)$ and let $0<\delta<1$. Then, there exists a positive constant $C = C_\delta$, for all $f \in L_1^{loc}(\Gamma)$ the following inequality holds;*

$$M_b\left(f\right)(\varsigma) \leq C\left\|b\right\|_{BMO(\Gamma)} M^2 f\left(\varsigma\right),\ \varsigma\in\Gamma.$$

Proof. Since by the Lebesgue differentiation theorem

$$M_b\left(f\right)(\varsigma) \leq M_\delta\left(Mf\right)(\varsigma), \tag{7}$$

the statement follows from (7) and Theorem 2.4. □

Lemma 2.6. *Let b be any non-negative locally integrable function. Then*

$$\left|\left[M,b\right]f(t)\right| \leq M_b(f)(t),\ t\in\Gamma \tag{8}$$

holds for all $f \in L_1^{loc}(\Gamma)$.

Let b is any locally integrable function on Γ. Then

$$\left|\left[M,b\right]f(t)\right| \leq M_b(f)(t) + 2b^-(t)Mf(t),\ t\in\Gamma$$

holds for all $f \in L_1^{loc}(\Gamma)$.

Proof. For $f, g \in L_1^{loc}(\Gamma)$, the following inequality

$$\left|Mf\left(t\right) - Mg\left(t\right)\right| \leq M\left(f-g\right)(t) \tag{9}$$

holds for all $t\in\Gamma$. From $b\geq 0$ and the inequality (9)

$$\begin{aligned}\left|\left[M,b\right]f(t)\right| &= \left|M\left(bf\right)(t) - b\left(t\right)Mf\left(t\right)\right| \\ &= \left|M\left(bf\right)(t) - M\left(b\left(t\right)f\right)(t)\right| \\ &\leq M\left(bf - b\left(t\right)f\right)(t) \\ &= M\left(\left(b-b\left(t\right)\right)f\right)(t) \\ &= M_b\left(f\right)(t)\end{aligned} \tag{10}$$

is obtained.

On the other hand, from $b \in L_1^{loc}(\Gamma) \Rightarrow |b| \in L_1^{loc}(\Gamma)$ and $b^- \in L_1^{loc}(\Gamma)$. Denote by $b^+ = \max\{b(t), 0\}$ and $b^- = -\min\{b(t), 0\}$, consequently $b = b^+ - b^-$ and $|b| = b^+ + b^-$. So, the following inequality

$$\left|\left[M,b\right](f)(t) - \left[M,|b|\right](f)(t)\right| \leq 2b^-(t)Mf(t)$$

holds for all $t\in\Gamma$. For all $t\in\Gamma$ $M_{|b|}(f)(t) \leq M_b(f)(t)$ and the inequality (10)

$$\begin{aligned}\left|\left[M,b\right]f(t)\right| &\leq M_{|b|}\left(f\right)(t) + 2b^-(t)Mf(t) \\ &\leq M_b\left(f\right)(t) + 2b^-(t)Mf(t).\end{aligned}$$

□

Theorem 2.7. *Let $b \in BMO(\Gamma)$ such that $b^- \in L_\infty(\Gamma)$. Then, there exists a positive constant C, for all $f \in L_1^{loc}(\Gamma)$ the following inequality holds;*

$$\left|[M,b]\,f(t)\right| \leq C\left(\left\|b^+\right\|_{BMO(\Gamma)} + \left\|b^-\right\|_\infty\right) M^2 f(t).$$

Proof. From Lemma 2.6 and Theorem 2.5

$$\left|[M,b]\,f(t)\right| \leq C\left(\left\|b^+\right\|_{BMO(\Gamma)} M^2 f(t) + \left\|b^-\right\|_\infty Mf(t)\right).$$

To remind again, $b = b^+ - b^-$ and $|b| = b^+ + b^-$ is obtained from $b^+ = \max\{b(t), 0\}$ and $b^- = -\min\{b(t), 0\}$.

$$\begin{aligned} \|b\|_{BMO(\Gamma)} &\leq \|b^+\|_{BMO(\Gamma)} + \|b^-\|_{BMO(\Gamma)} \\ &\leq \|b^+\|_{BMO(\Gamma)} + \|b^-\|_\infty . \end{aligned} \tag{11}$$

Using $f \leq Mf$ and (11), the statement of the theorem is obtained. □

3. Main Results

In this section, the limitedness of the commutator of maximal $[M,b]$ and the maximal commutator M_b on $L_p(\Gamma)$ will be examined. To examine the limitedness of the commutator of maximal, it is more useful to first examine the limitation of the maximal commutator, which is easier to examine. For this reason, the main results section will begin with the limitation of the maximal commutator.

Theorem 3.1. *Let $1 < p < \infty$. The operator M_b is bounded on $L_p(\Gamma)$ if and only if $b \in BMO(\Gamma)$.*

Proof. (:⇒) $1 < p < \infty$. Suppose that $b \in BMO(\Gamma)$. By Theorem 2.5 and Theorem 1.8 the following inequality holds:

$$\|M_b(f)\|_{L_p(\Gamma)} \leq \|b\|_{BMO(\Gamma)} \|f\|_{L_p(\Gamma)}.$$

(⇐:) Let $f \in L_p(\Gamma)$. In this case

$$\|M_b(f)\|_{L_p(\Gamma)} \leq c\|f\|_{L_p(\Gamma)} \tag{12}$$

there is a constant $c > 0$ that satisfies the inequality (12). The following expression obviously can be written

$$\|f\|_{L_p(\Gamma)} = \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{-1} \int_{\Gamma(t,r)} |f(\tau)|^p dv(\tau) \right)^{\frac{1}{p}}. \tag{13}$$

Let $\Gamma(t_0, r_0)$ be a fixed Carleson curve. If $x_{\Gamma(t_0,r_0)}$ is written instead of the f function in (13), the following expression is easily written;

$$\begin{aligned} \|\chi_{\Gamma(t_0,r_0)}\|_{L_p(\Gamma)} &= \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{-1} \int_{\Gamma(t,r)} x_{\Gamma(t_0,r_0)}(\tau)\, dv(\tau) \right)^{\frac{1}{p}} \\ &= \sup_{\Gamma(t,r)} \left((|\Gamma(t,r) \cap \Gamma(t_0,r_0)|) \, |\Gamma(t,r)|^{-1} \right)^{\frac{1}{p}} \\ &= \sup_{\Gamma(t,r) \subset \Gamma(t_0,r_0)} \left(|\Gamma(t,r)| \, |\Gamma(t,r)|^{-1} \right)^{\frac{1}{p}} \\ &= |\Gamma(t_0,r_0)|^{\frac{\lambda}{p}} . \end{aligned} \tag{14}$$

On the other hand, since

$$M_b\left(\chi_{\Gamma(t_0,r_0)}\right)(t) \geq \frac{1}{|\Gamma(t_0,r_0)|}\int\limits_{\Gamma(t_0,r_0)}\left|b(\tau)-b_{\Gamma(t_0,r_0)}\right|dv(\tau), \text{ for all } t\in\Gamma(t_0,r_0)$$

then

$$\begin{aligned}\left\|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)\right\|_{L_p(\Gamma)} &= \sup_{\Gamma(t,r)}\left(|\Gamma(t,r)|^{-1}\int\limits_{\Gamma(t,r)}\left|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)(\tau)\right|^p dv(\tau)\right)^{\frac{1}{p}} \\ &\geq \frac{1}{|\Gamma(t_0,r_0)|}\int\limits_{\Gamma(t_0,r_0)}\left|b(\tau)-b_{\Gamma(t_0,r_0)}\right|dv(\tau).\end{aligned} \tag{15}$$

Since by assumption

$$\left\|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)\right\|_{L_p(\Gamma)} \leq \left\|\chi_{\Gamma(t_0,r_0)}\right\|_{L_p(\Gamma)},$$

by (14) and (15), we get that

$$\frac{1}{|\Gamma(t_0,r_0)|}\int_{\Gamma(t_0,r_0)}|b(\tau)-b_{\Gamma(t_0,r_0)}|dv(\tau)\leq c.$$

Thus, the desired result is obtained. □

Theorem 3.2. *Suppose that $1<p<\infty$, $[M,b]$ is bounded on $L_p(\Gamma)$ if and only if $b\in BMO(\Gamma)$ and $b^-\in L_\infty(\Gamma)$. The operators M_b and $[M,b]$ enjoy weak-type $L(1+\log^+ L)$ estimate.*

Proof. (:⇒) Let's accept that $[M,b]$ is bounded on $L_p(\Gamma)$. $\Gamma(t_0,r_0)$ being the constant Carleson curve, we will denote the local maximal function M_Γ of f as follows;

$$M_{\Gamma(t_0,r_0)}f(x) := \sup_{t\in\Gamma(t,r):\ \Gamma(t,r)\subset\Gamma(t_0,r_0)}\frac{1}{|\Gamma(t,r)|}\int\limits_{\Gamma(t,r)}\left|f(\tau)\right|dv(\tau),$$

Since

$$M(b\chi_{\Gamma(t_0,r_0)})\chi_{\Gamma(t_0,r_0)} = M_{\Gamma(t_0,r_0)}(b)$$

and

$$M\left(\chi_{\Gamma(t_0,r_0)}\right)\chi_{\Gamma(t_0,r_0)} = \chi_{\Gamma(t_0,r_0)},$$

then the below inequality is valid;

$$\begin{aligned}\left|M_{\Gamma(t_0,r_0)}(b)-b\chi_{\Gamma(t_0,r_0)}\right| &= \left|M(b\chi_{\Gamma(t_0,r_0)})\chi_{\Gamma(t_0,r_0)}-bM\left(\chi_{\Gamma(t_0,r_0)}\right)\chi_{\Gamma(t_0,r_0)}\right| \\ &\leq \left|M(b\chi_{\Gamma(t_0,r_0)})-bM\left(\chi_{\Gamma(t_0,r_0)}\right)\right| \\ &= \left|[M,b]\,\chi_{\Gamma(t_0,r_0)}\right|.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{|\Gamma(t_0,r_0)|}\int\limits_{\Gamma(t_0,r_0)}\left|\left(b-M_{\Gamma(t_0,r_0)}(b)\right)(\tau)\right|dv(\tau) &\leq \left(\frac{1}{\Gamma(t_0,r_0)}\int\limits_{\Gamma(t_0,r_0)}\left|\left(b-M_{\Gamma(t_0,r_0)}(b)\right)(\tau)\right|^p dv(\tau)\right)^{\frac{1}{p}} \\ &\leq |\Gamma(t_0,r_0)|^{-\frac{1}{p}}\left\|b\chi_{\Gamma(t_0,r_0)}-M_{\Gamma(t_0,r_0)}(b)\right\|_{L_p(\Gamma)} \\ &\leq |\Gamma(t_0,r_0)|^{-\frac{1}{p}}\left\|[M,b]\,\chi_{\Gamma(t_0,r_0)}\right\|_{L_p(\Gamma)} \\ &\leq c\,|\Gamma(t_0,r_0)|^{-\frac{1}{p}}\left\|\chi_{\Gamma(t_0,r_0)}\right\|_{L_p(\Gamma)} \\ &= c\,|\Gamma(t_0,r_0)|^{-\frac{1}{p}}\,|\Gamma(t_0,r_0)|^{\frac{1}{p}} = c,\end{aligned}$$

the above expression is obtained. Denote by

$$E := \left\{t \in \Gamma(t_0, r_0) : b(t) \leq b_{\Gamma(t_0,r_0)}\right\}, \; F := \left\{t \in \Gamma(t_0, r_0) : b(t) > b_{\Gamma(t_0,r_0)}\right\}.$$

Since

$$\int_E |b(\tau) - b_{\Gamma(t_0,r_0)}| dv(\tau) = \int_F |b(\tau) - b_{\Gamma(t_0,r_0)}| dv(\tau),$$

and considering the following inequality

$$b(t) \leq b_{\Gamma(t_0,r_0)} \leq M_{\Gamma(t_0,r_0)}(b), t \in E,$$

we get that

$$\begin{aligned} \frac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} |\left(b - b_{\Gamma(t_0,r_0)}\right)(\tau)| dv(\tau) &= \frac{2}{|\Gamma(t_0,r_0)|} \int_E |\left(b - b_{\Gamma(t_0,r_0)}\right)(\tau)| dv(\tau) \\ &\leq \frac{2}{|\Gamma(t_0,r_0)|} \int_E |\left(b - M_{\Gamma(t_0,r_0)}(b)\right)(\tau)| dv(\tau) \\ &\leq \frac{2}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} |\left(b - M_{\Gamma(t_0,r_0)}(b)\right) dv(\tau)| dv(\tau) \\ &\leq c. \end{aligned}$$

Consequently, $b \in BMO(\Gamma)$.

For the first step of the proof, it remains to show that $b^- \in L^\infty(\Gamma)$. For this, we will use the inequality $M_{\Gamma(t_0,r_0)}(b) \geq |b|$. Thus, we easily obtain the following inequality

$$0 \leq b^- = |b| - b^+ \leq M_{\Gamma(t_0,r_0)}(b) - b^+ + b^- = M_{\Gamma(t_0,r_0)}(b) - b.$$

That is, the inequality

$$(b^-)_{\Gamma(t_0,r_0)} \leq c$$

is obtained. From this inequality (16) and the Lebesgue differentiation theorem, the following statement

$$b^-(t) \leq c, \; \forall t \in \Gamma \tag{16}$$

is obtained, which shows that we get the desired expression.

($\Leftarrow$:) Suppose that $b \in BMO(\Gamma)$ *and* $b^- \in L_\infty(\Gamma)$. From Theorem 2.7, it is obvious that

$$\left|[M, b] f(t)\right| \leq c\left(\left\|b^+\right\|_{BMO(\Gamma)} + \left\|b^-\right\|_{L_\infty(\Gamma)}\right) M^2 f(t). \tag{17}$$

From (17) and Theorem 1.8, the boundedness of $[M, b]$ on $L_p(\Gamma)$ result is achieved for all $1 < p < \infty$. □

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Availability of Data and Materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
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Authors' contributions
ME prepared the original draft and MM checked, edited and prepared the final draft of the manuscript. All authors read and approved the final manuscript.

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References

[1] M. Agcayazi, A. Gogatishvili, K. Koca, R. Mustafayev, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan. 67 (2), 581-593 (2015).

[2] C. Aykol, H. Armutcu, M.N. Omarova, *Maximal commutator and commutator of maximal function on modified Morrey spaces*. Trans. of NAS of Azerbaijan, Issue Mathematics, 36, 29-35 (2016).

[3] J. Bastero, M. Milman, F.J. Ruiz, *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc. 128 (11), 3329-3334 (electronic) (2000).

[4] A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister, *On the product of functions in BMO and* H_1 , Ann. Inst. Fourier (Grenoble). 57 (5), 1405-1439 (2007).

[5] A. Böttcher, Y.I. Karlovich, J.I. Karlovics, Y.I. Karlovich, (1997). *Carleson curves, Muckenhoupt weights, and Toeplitz operators* (Vol. 154). Springer Science & Business Media.

[6] I.B. Dadashova, C. Aykol, Z. Cakir, A. Serbetci, Potential operators in modified Morrey spaces defined on Carleson curves. Transactions of A. Razmadze Mathematical Institute, 172(1), 15-29 (2018).

[7] J. Garcia-Cuerva E. Harboure, C. Segovia, J.L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. 40 (4), 1397-1420 (1991).

[8] A. Gogatishvili, R. Ch. Mustafayev, M. Agcayazi, *Weak-type estimates in Morrey spaces for maximal commuatator and commutator of maximal function*, ArXiv: 1504.04509v2 [math.FA] 19 June (2015).

[9] L. Grafakos, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, 250, Springer, New York (2009).

[10] V. Guliyev, H. Armutcu, T. Azeroglu, *Characterizations for the potential operators on Carleson curves in local generalized Morrey spaces.* Open Mathematics, 18(1), 1317-1331 (2020).

[11] A. Kufner, O. John, S. Fucik, *Function Spaces*, Noordhoff, Leyden, and Academia, Prague (1977).

[12] D. Li, G. Hu, X. Shi, *Weighted norm inequalities for the maximal commutators of singular integral operators*, J. Math. Anal. Appl. 319 (2), 509-521 (2006).

[13] J.I. Mamedkhanov, I.B. Dadashova, *Some properties of the potential operators in Morrey spaces defined on Carleson curves. Complex Variables and Elliptic Equations*, 55(8-10), 937-945 (2010).

[14] M. Milman, T. Schonbek, *Second order estimates in interpolation theory and applications*, Proc. Amer. Math. Soc. 110 (4), 961-969 (1990).

[15] C. Segovia, J.L. Torrea, *Higher order commutators for vector-valued Calderon-Zygmund operators*, Trans. Amer. Math. Soc. 336 (2), 537-556 (1993).

[16] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ (1993).

[17] C.P. Xie, *Some estimates of commutators*, Real Anal. Exchange 36 (2010/11), no. 2, 405–415 (2010).