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On some consequences of Nadler's fixed point problem

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Abstract. Recently, N. Bunlue al. [N. Bunlue, Y.J. Cho, S. Suantai, Best proximity point theorems for proximal multi-valued contractions, Filomat, 35;6, (2021) 1889-1897] studied the existence of best proximity points for proximal multi-valued contractions as well as proximal multi-valued nonexpansive mappings in the framework of metric and Banach spaces, respectively. In this paper we show that the well-known Nadler's fixed point theorem implies the best proximity point results of such proximal multi-valued contractions and nonexpansive non-self mappings. Moreover, in the case that the considered non-self mapping is proximal multi-valued nonexpansive, we drop the conditions of semi-sharp proximinality as well as q-starshepedness which were assumed in a main result of aforementioned paper.

1. Introduction and Preliminaries

In 1969, Nadler proved the following fixed point theorem for multi-valued contractions as an interesting generalization of the *Banach contraction principle*.

Theorem 1.1. (Nadler's fixed point theorem; Theorem 5 of [7]) *Let (X, d) be a complete metric space and let T be a mapping from X into $\mathcal{CB}(X)$, where $\mathcal{CB}(X)$ is the set of all nonempty, bounded and closed subsets of X. Assume that there exists $\alpha \in (0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X,$$

where $\mathcal{H}$ is a function from $\mathcal{CB}(X)^2$ into $[0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

Then T has a fixed point, that is, there exists an element $p \in X$ for which $p \in Tp$.

Just recently, Bunlue et al., ([1]) presented extensions of Nadler's fixed point theorem. Before stating their main conclusions, we need to recall some related concepts and notations.

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Let (X, d) be a metric space and $A, B \in \mathcal{CB}(X)$. We set

$$\begin{aligned} d(x, B) &= \inf\{d(x, y) : y \in B\}, \\ D(A, B) &= \inf\{d(x, y) : (x, y) \in A \times B\}, \\ A_0 &= \{x \in A : d(x, y) = D(A, B), \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = D(A, B), \text{ for some } x \in A\}. \end{aligned}$$

We recall that the pair (A, B) is called *proximinal* provided that $A_0 = A$ and $B_0 = B$. The set of all proximinal and bounded subsets of B will be denoted by $\mathcal{P}(B)$. Moreover, the pair (A, B) is said to be a *semi-sharp proximinal pair* ([6]) if, for each $x \in A$, there exists at most one $y \in B$ such that $d(x, y) = D(A, B)$. For more details of proximinal pairs we refer to [3–5].

Definition 1.2. *A nonempty subset A of a linear space X is called a p-starshaped set if there exists a point p in A such that*

$$rp + (1 - r)x \in A, \quad \forall (x, r) \in A \times [0, 1].$$

It is worth noticing that if A is a p-starshaped set, B is a q-starshaped set and $\|p - q\| = D(A, B)$, then A_0 is a p-starshaped set and B_0 is a q-starshaped set (see [2]).

Assume that $T : A \to 2^B$ is a multivalued non-self mapping. In case $A \cap B = \emptyset$, the multifunction T has not fixed point. Then $d(x, Tx) > 0$ for all $x \in A$. So, we can explore to find necessary conditions such that the minimization problem

$$\min_{x \in A} D(x, Tx), \tag{1}$$

has at least one solution. Since $d(x, Tx) \geq D(A, B)$ for all $x \in A$, the optimal solution to the problem (1) is obtained in some points of A for which the value $D(A, B)$ is attained. A point $x^* \in A$ is called a *best proximity point* of a multivalued non-self mapping T, if $d(x^*, Tx^*) = D(A, B)$. We note that if $D(A, B) = 0$, then we get a fixed point of T.

Let A_0 be nonempty. For the multivalued non-self mapping $T : A \to 2^B$ we set

$$\mathcal{U}_x := \Big\{y \in A_0 : d(y, Tx) = D(A, B)\Big\}, \quad \forall x \in A_0.$$

Definition 1.3. ([1]) *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that A_0 is nonempty and let $T : A \to 2^B$ be a multivalued non-self mapping.*

(i) *T is called a proximal multivalued contraction with respect to A_0 if there exists $\alpha \in (0, 1)$ such that for each $x_1, x_2 \in A_0$ with $\mathcal{U}_{x_1}, \mathcal{U}_{x_2} \in \mathcal{CB}(X)$ we have*

$$\mathcal{H}(\mathcal{U}_{x_1}, \mathcal{U}_{x_2}) \leq \alpha d(x_1, x_2);$$

(ii) *T is called proximal multivalued nonexpansive with respect to A_0 if for each $x_1, x_2 \in A_0$ with $\mathcal{U}_{x_1}, \mathcal{U}_{x_2} \in \mathcal{CB}(X)$ we have*

$$\mathcal{H}(\mathcal{U}_{x_1}, \mathcal{U}_{x_2}) \leq d(x_1, x_2).$$

The next lemma will be used in our coming discussions.

Lemma 1.4. (see Lemma 3.3 of [1]) *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty. Suppose that $T : A \to 2^B$ is a multivalued mapping such that for $x \in A_0$, the set $Tx \cap B_0$ is nonempty. Then we have the following:*

(1) *for all $x \in A_0$, $\mathcal{U}_x$ is a nonempty set;*

(2) *if A_0 is closed and $x \in A_0$, then $\mathcal{U}_x$ is closed;*

(3) *for each $x \in A_0$, the set $Tx \cap B_0$ is bounded if and only if $\mathcal{U}_x$ is bounded.*

Here, we state the following best proximity point theorems which are the main results of [1].

Theorem 1.5. (Theorem 3.4 of [1]) *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to 2^B$ satisfies the following conditions:*

(i) *T is an α-proximal multivalued contraction with respect to A_0;*

(ii) *for each $x \in A_0$, $Tx \cap B_0$ is nonempty and bounded.*

Then T has a best proximity point.

Theorem 1.6. (Theorem 4.2 of [1]) *Let (A, B) be a nonempty pair of subsets of a Banach space X such that A_0 is a p-starshaped set, and B_0 is a q-starshaped set with $\|p - q\| = D(A, B)$. Assume that A_0 is a compact set and (B_0, A_0) is a semi-sharp proximinal pair. Suppose that a multi-valued mapping $T : A \to \mathcal{P}(B)$ satisfies the following conditions:*

(i) *T is proximal multivalued nonexpansive with respect to A_0;*

(ii) *for each $x \in A_0$, $Tx \cap B_0$ is nonempty and bounded.*

Then T has a best proximity point.

The main purpose of this paper is to show that both Theorems 1.5, 1.6 are particular cases of Theorem 1.1.

2. Main results

We now state our main results of this article.

Theorem 2.1. *Nadler's Theorem implies Theorem 1.5.*

Proof. Define $\Gamma : A_0 \to \mathcal{CB}(A_0)$ by

$$\Gamma(x) = \Big\{y \in A_0 : d(y, Tx) = D(A, B)\Big\},$$

for $x \in A_0$. It follows from Lemma 1.4 that Γx is a nonempty, closed and bounded subset of A_0 for each $x \in A_0$ and so Γ is well-defined. Since T is an α-proximal multivalued contraction with respect to A_0,

$$\mathcal{H}(\Gamma x, \Gamma y) = \mathcal{H}(\mathcal{U}_x, \mathcal{U}_y) \leq \alpha d(x, y), \quad \forall x, y \in A_0,$$

It now follows from Nadler's fixed point theorem that there exists an element $z \in A_0$ for which $z \in \Gamma z$. By the definition of the mapping Γ, the point z satisfies $d(z, Tz) = D(A, B)$ and this completes the proof. □

Theorem 2.2. *Nadler's Theorem implies Theorem 1.6.*

Proof. Define $\Gamma : A_0 \to \mathcal{CB}(A_0)$ by

$$\Gamma(x) = \Big\{y \in A_0 : d(y, Tx) = D(A, B)\Big\},$$

for $x \in A_0$. By Lemma 1.4, Γ is well-defined. Since T is proximal nonexpansive, the mapping Γ is a multivalued nonexpansive self mapping. Let $\{r_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} r_n = 0$. For each $n \in \mathbb{N}$ define the multivalued mapping Γ_n with

$$\Gamma_n(x) = \Big\{u \in A_0 : u = r_n p + (1 - r_n)w, w \in \Gamma x\Big\}, \quad \forall x \in A_0.$$

Then we have the following observations:
♣ Γ_n *maps A_0 to $\mathcal{CB}(A_0)$.*

Proof. Since the set A_0 is p-starshaped, $\emptyset \neq \Gamma_n(x) \subseteq A_0$ for all $x \in A_0$. Besides, for each $x \in A_0$, $\Gamma_n x$ is a continuous image of the compact set Γx and therefore, is a closed and bounded subset of A_0. □

♣ *Γ_n is a multivalued contraction.*

Proof. For each $u \in \Gamma_n x$, there is a point $w \in \Gamma x$ such that $u = r_n p + (1 - r_n)w$. For any $y \in A_0$, choose an element $v \in \Gamma y$ such that $\|w - v\| \leq \mathcal{H}(\Gamma x, \Gamma y)$ and let $z = r_n p + (1 - r_n)v$. Then $z \in \Gamma_n y$ and

$$\|u - z\| = \|\Big(r_n p + (1 - r_n)w\Big) - \Big(r_n p + (1 - r_n)v\Big)\| = (1 - r_n)\|w - v\|,$$

which deduces that

$$\mathcal{H}(\Gamma_n x, \Gamma_n y) \leq (1 - r_n)\mathcal{H}(\Gamma x, \Gamma y) \leq (1 - r_n)\|x - y\|,$$

that is, Γ_n is a $(1 - r_n)$-contraction. □

Hence from Nadler's Theorem, Γ_n has a fixed point, say z_n. Using the compactness of A_0, we may assume that $\{z_n\}$ converges to an element $z \in A_0$.
♣ *If z_n is a fixed point of Γ_n, then $d(z_n, \Gamma z_n) \to 0$.*

Proof. By the definition of Γ_n and by the fact that $z_n \in \Gamma_n(z_n)$, we have $z_n = r_n p + (1 - r_n)w_n$ for some $w_n \in \Gamma z_n$, and $\|z_n - w_n\| = r_n\|p - w_n\|$. Since A_0 is bounded, there is a constant $M > 0$ such that $\mathrm{diam}(A_0) \leq M$. Thus $\|z_n - w_n\| \leq r_n M$ for each $n \in \mathbb{N}$ which concludes that

$$d(z_n, \Gamma z_n) \leq \|z_n - w_n\| \leq r_n M \to 0.$$

□

♣ *The point $z \in A_0$ is a fixed point of Γ.*

Proof. Considering the inequality

$$d(z, \Gamma z) \leq \|z - z_n\| + d(z_n, \Gamma z_n) + \mathcal{H}(\Gamma z_n, \Gamma z),$$

we see that all terms on the right side converge to 0. □

Finally from the definition of Γ, we have $d(z, Tz) = D(A, B)$. □

Remark 2.3. *In the proof of Theorem 2.2 we note that the result follows without the assumption made in Theorem 1.5, that B_0 is a q-starshaped set with $\|p - q\| = D(A, B)$.*

Remark 2.4. *It is worth mentioning that we do not use the condition of semi-sharp proximinality of the pair (B_0, A_0) in the proof of Theorem 2.2 and so, this condition should be removed of Theorem 1.5.*

Let us illustrate Remark 2.3 and Remark 2.4 with the following example.

Example 2.5. *Consider the Banach space ℓ_∞ with the supremum norm and let*

$$A = \Big\{te_1 : t \in [-1, 1]\Big\}, \quad B = \Big\{se_2 : s \in [-3, -2] \cup [2, 3]\Big\},$$

where $\{e_n\}$ stands for the canonical basis of ℓ_∞. Then $D(A, B) = 2$ and $A_0 = A, B_0 = \Big\{-2e_2, 2e_2\Big\}$. Clearly A_0 is a p-strashaped set whereas B_0 is not q-strashaped for any $q \in B_0$. Moreover, A_0 is a compact set, but (B_0, A_0) is not a semi-sharp proximinal pair. Now define $T : A \to \mathcal{P}(B)$ with

$$T(te_1) = \begin{cases} \{2e_2\}, & \forall t \in [-1, 0] \\ \{-2e_2\}, & \forall t \in (0, 1]. \end{cases}$$

Then for any $x \in A$ we have $\mathcal{U}_x = A$ and so, T is a proximal multivalued nonexpansive mapping. Note that every point of the set A is a best proximity point of T.

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