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Some general properties of analytic and p-valent functions

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Abstract. Let $\mathcal{A}_p$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots, (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the open unit disc $\mathbb{U}$. In this article, we consider some generalization properties of the functions in $\mathcal{A}_p$ and generalize results by applying fractional derivatives.

1. Introduction

A function $f(z)$ which is analytic or meromorphic in a region D is said to be $p-$valent in D ($p \in \mathbb{N} = \{1, 2, 3, \ldots\}$), if the equation $f(z) = \omega$ has at most p roots in D for each complex ω. Let $\mathcal{A}_p$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \left\{z \in \mathbb{C} : |z| < 1\right\}$.

Let $S_p^*(\alpha)$ denote the subclass of $\mathcal{A}_p$ which satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}) \tag{2}$$

for some real α $(0 \leq \alpha < p)$. Also $C_p(\alpha)$ be the subclass of $\mathcal{A}_p$ consisting of $f(z)$ which satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}) \tag{3}$$

for some real α $(0 \leq \alpha < p)$. The subclasses $S_p^*(\alpha)$ and $C_p(\alpha)$ will be said to the class of $p-$valently starlike of order α in $\mathbb{U}$ and $p-$valently convex of order α in $\mathbb{U}$. Especially, for $p = 1$, we have the well-known classes of normalized starlike and convex functions of order α, respectively.

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The function

$$f(z)=\frac{z^p}{(1-z)^{2(p-\alpha)}}$$

is in the class $S_p^*(\alpha)$ and the function $f(z)$ given by

$$\frac{1}{p}zf'(z)=\frac{z^p}{(1-z)^{2(p-\alpha)}}$$

belongs to the class $C_p(\alpha)$.

Although p-valent functions are examined in terms of studies such as determining the bounds of coefficient estimates, the Fekete-Szegö problem, inclusion relationships, integral means and neighborhoods, it continues to attract attention as a current research topic with recent studies using operators such as the q-derivative operator, q-Bernardi integral operator; see, for example, [1–6]

For $f(z)\in\mathcal{A}_p$, Nunokawa [7] gave the following lemma.

Lemma 1.1 ([7]). *If $f(z)\in\mathcal{A}_p$ satisfies*

$$j+Re\left\{\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\}>0\,,\ (z\in\mathbb{U})$$

then

$$j-1+Re\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\}>0\,,\ (z\in\mathbb{U})$$

where $1\leq j\leq p$.

We require the following lemma given by Miller and Mocanu [8] to think about our problems.

Lemma 1.2 ([8]). *Let α be real and $M(z)$, $N(z)$ be analytic in $\mathbb{U}$ with the condition $M(0)=N(0)=0$. If $N(z)$ maps $\mathbb{U}$ onto a (possibly many-sheeted) domain which is starlike with respect to the origin, then*

$$Re\left\{\frac{M'(z)}{N'(z)}\right\}>\alpha\,,\ (z\in\mathbb{U})\Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\}>\alpha\,,\ (z\in\mathbb{U}).$$

The case of $\alpha=0$ for Lemma 1.2 is given by Sakaguchi [9] and Libera [10].

In this paper, we consider a few generalizations of Lemma 1.1 in the light of Lemma 1.2.

2. Main results

In the following theorem, we will give a proof of well-known inclusion relation for $f(z)\in\mathcal{A}_p$ using Lemma 1.1 and Lemma 1.2.

Theorem 2.1. *If $f(z)\in C_p(\alpha)$, then $f(z)\in\mathcal{S}_p^*(\alpha)$.*

Proof. The theorem means that if $f(z)\in\mathcal{A}_p$ satisfies

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\,,\ (z\in\mathbb{U}),$$

then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\,,\ (z\in\mathbb{U}).$$

Considering $j = 1$ in Lemma 1.1, we see that if $f(z) \in \mathcal{A}_p$ satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0\,,\ (z \in \mathbb{U}),$$

then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0\,,\ (z \in \mathbb{U}).$$

We consider $M(z) = zf'(z)$ and $N(z) = f(z)$. Then

$$Re\left\{\frac{zN'(z)}{N(z)}\right\} = Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0\,,\ (z \in \mathbb{U}),$$

by $f(z) \in C_p(\alpha)$. This means that $N(z)$ is starlike in $\mathbb{U}$. Further, we have

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} = Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

by $f(z) \in C_p(\alpha)$. Therefore, applying Lemma 1.2, we prove that

$$Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

that is, that $f(z) \in \mathcal{S}_p^*(\alpha)$. □

Remark 2.2. *Theorem 2.1 is generalization of Lemma 1.1 for $j = 1$.*

Next, we have the following theorem.

Theorem 2.3. *Let $F(z) = \frac{1}{p}zf'(z)$ for $f(z) \in \mathcal{A}_p$. If $F(z) \in C_p(\alpha)$ then $f(z) \in C_p(\alpha)$.*

Proof. We consider functions $M(z) = zF'(z)$ and $N(z) = F(z)$. Then

$$Re\left\{\frac{zN'(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

by $F(z) \in C_p(\alpha)$ and Theorem 2.1. Thus $N(z)$ is starlike in $\mathbb{U}$. It follows that

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} = 1 + Re\left\{\frac{zF''(z)}{F'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

with $F(z) \in C_p(\alpha)$. Thus using Lemma 1.2, we have

$$Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} = 1 + Re\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

and that $f(z) \in C_p(\alpha)$. □

Using the same method, we have the following theorem.

Theorem 2.4. *Let $F(z) = \frac{(p-j)!}{p!}z^j f^{(j)}(z)$ for $f(z) \in \mathcal{A}_p$ and $j = 0, 1, 2, \cdots, p$. If $F(z) \in C_p(\alpha)$, then*

$$j + Re\left\{\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}). \tag{4}$$

Proof. Considering $M(z) = zF'(z)$ and $N(z) = F(z)$, we see

$$Re\left\{\frac{zN'(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

with $F(z) \in \mathcal{C}_p(\alpha) \subset \mathcal{S}_p^*(\alpha)$. Also, we see that

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} = 1 + Re\left\{\frac{zF''(z)}{F'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

by $F(z) \in \mathcal{C}_p(\alpha)$. Thus, Lemma 1.2 implies that

$$Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} = j + Re\left\{\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}).$$

□

3. Applications for fractional derivatives

For $f(z) \in \mathcal{A}_p$, we define

$$\begin{aligned} D_z^{\lambda} f(z) &= \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz}\left\{\int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt\right\}, \quad (0 \leq \lambda < 1) \\ &= \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} z^{p-\lambda} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda}, \end{aligned}$$

where $\Gamma(z)$ is the Gamma function. Further, we see that

$$\begin{aligned} D_z^{1+\lambda} f(z) &= \frac{d}{dz}(D_z^{\lambda} f(z)) \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\lambda)} z^{p-\lambda-1} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} a_k z^{k-\lambda-1}, \end{aligned}$$

and

$$\begin{aligned} D_z^{j+\lambda} f(z) &= \frac{d^j}{dz^j}(D_z^{\lambda} f(z)) \\ &= \frac{\Gamma(p+1)}{\Gamma(p+1-j-\lambda)} z^{p-\lambda-j} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-j-\lambda)} a_k z^{k-j-\lambda}, \end{aligned}$$

where $j = 0, 1, 2, \cdots, p$. The function $D_z^{j+\lambda} f(z)$ is the fractional derivative of order $j + \lambda$ of $f(z) \in \mathcal{A}_p$, and is defined by Owa [11] and by Srivastava and Owa [12] (see also [13–16]).

Applying the fractional derivatives, we obtain the following theorem.

Theorem 3.1. *Let*

$$F(z) = \frac{\Gamma(p+1-j-\lambda)}{\Gamma(p+1)} z^{j+\lambda} D_z^{j+\lambda} f(z)$$

for $f(z) \in \mathcal{A}_p$, $j = 0, 1, 2, \cdots, p$ *and* $0 \leq \lambda < 1$. *If* $F(z) \in \mathcal{C}_p(\alpha)$ *then*

$$j + \lambda + Re\left\{\frac{zD_z^{j+\lambda+1} f(z)}{D_z^{j+\lambda} f(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}).$$

Proof. We define $M(z) = zF'(z)$ and $N(z) = F(z)$. It follows that

$$Re\left\{\frac{zN'(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

by $F(z) \in \mathcal{C}_p(\alpha) \subset \mathcal{S}^*_p(\alpha)$. This implies that $N(z)$ is starlike in $\mathbb{U}$. Note that

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} = 1 + Re\left\{\frac{zF''(z)}{F'(z)}\right\} > \alpha\,,\ (z \in \mathbb{U})$$

by $F(z) \in \mathcal{C}_p(\alpha)$. Thus, applying Lemma 1.2, we get

$$Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{\frac{zF'(z)}{F(z)}\right\} = j + \lambda + Re\left\{\frac{zD_z^{j+\lambda+1}f(z)}{D_z^{j+\lambda}f(z)}\right\} > \alpha\,,\ (z \in \mathbb{U}).$$

□

Remark 3.2. *If we take $\lambda = 0$ in Theorem 3.1, then we have Theorem 2.4.*

In order to consider our next problem, we must first remember the following lemma by Nunokawa, Sokol and Tuneski [18].

Lemma 3.3 ([18]). *Let $f(z) \in \mathcal{A}_p$ for $p \geq 2$. If $f(z)$ satisfies*

$$Re\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad (z \in \mathbb{U})$$

then, we have

$$Re\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad (|z| < \sqrt{2} - 1).$$

Further, we need the following lemma by MacGregor [17].

Lemma 3.4 ([17]). *If $f(z) \in \mathcal{A}_1$ satisfies $Re f'(z) > 0$ $(z \in \mathbb{U})$, $f(z)$ maps $|z| < \sqrt{2} - 1$ onto a convex domain.*

Theorem 3.5. *Let $f(z) \in \mathcal{A}_p$ for $p \geq 2$. If $f(z)$ satisfies*

$$Re\left\{\frac{D_z^{p-1+\lambda}f(z)}{z^{1-\lambda}}\right\} > 0, \quad (z \in \mathbb{U}), \tag{5}$$

then

$$Re\left\{\frac{zD_z^{p+\lambda}f(z)}{D_z^{p-1+\lambda}f(z)}\right\} > 0, \quad \left(|z| < \frac{\sqrt{1 + (1-\lambda)^2} - 1}{1 - \lambda}\right).$$

Proof. Let us consider a function $g(z)$ by

$$g(z) = \frac{\Gamma(2-\lambda)}{\Gamma(p+1)z^{1-\lambda}} D_z^{p-1+\lambda} f(z),$$

then $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = 1$. This implies that $g(z)$ satisfies the condition (5). Thus, by using the manner in the proof of Lemma 3.4, we see that

$$\left|\frac{zg'(z)}{g(z)}\right| = \left|\frac{zD_z^{p+\lambda}f(z)}{D_z^{p-1+\lambda}f(z)} - (1-\lambda)\right| \leq \frac{2|z|}{1-|z|^2}. \tag{6}$$

Since

$$\frac{2|z|}{1-|z|^2} < 1-\lambda$$

for

$$|z| < \frac{\sqrt{1+(1-\lambda)^2}-1}{1-\lambda},$$

we see that the inequality (6) shows

$$Re\left\{\frac{zD_z^{p+\lambda}f(z)}{D_z^{p-1+\lambda}f(z)}\right\} > 0, \quad \left(|z| < \frac{\sqrt{1+(1-\lambda)^2}-1}{1-\lambda}\right).$$

□

Remark 3.6. *If we consider $\lambda = 0$ in Theorem 3.5, then Theorem 3.5 becomes Lemma 3.3 by Nunokawa, Sokol and Tuneski [18].*

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