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Description of jet like functors on vector bundles by means of module bundle functors on the bases

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Abstract. Let C be an admissible category over manifolds and $\mathcal{VB}_C$ be the category of vector bundles with bases being C-objects and vector bundle maps with base maps being C-maps. Assume that any C-morphism is a local isomorphism. We describe all jet like functors (i.e. fiber product preserving gauge bundle functors) of order r on $\mathcal{VB}_C$ by means of $J^r(-,\mathbf{R})$-module bundle functors on C. Then we describe all jet like functors of vertical type of order r on $\mathcal{VB}_C$ by means of vector bundle functors on C of order r. As an application we classify jet like functors of some type on $\mathcal{VB}_m$. Finally, we determine all natural $J^r(M,\mathbf{R})$-module bundle structures on the vector bundles $J^rE \to M$ and $J^r_vE \to M$, where $E \to M$ is a vector bundle with m-dimensional basis and $m \geq 2$.

1. Introduction

In [2] we described all fiber product preserving gauge bundle functors on the category $\mathcal{VB}_m$ of vector bundles with m-dimensional bases and their vector bundle homomorphisms with base maps being local diffeomorphisms by means of $J^r(-,\mathbf{R})$-module bundle functors on the category $\mathcal{M}f_m$ of m-manifolds and their local diffeomorphisms.

In this paper we first describe fiber product preserving gauge bundle functors (also called jet like functors) on the category $\mathcal{VB}_C$ of vector bundles with bases being C-objects and vector bundle maps with base maps being C-maps by means of $J^r(-,\mathbf{R})$-module bundle functors on C, where C is an admissible category over manifolds in the sense of [6] such that any C-morphism is a local isomorphism. We have the following examples of such admissible categories:
- The category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms;
- The category $\mathcal{M}f^o$ of all manifolds and their local diffeomorphisms;
- The category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibres and their local fibered diffeomorphisms;
- The subcategory $\mathcal{D}^o$ of an admissible category $\mathcal{D}$ over manifolds with morphisms being local $\mathcal{D}$-isomorphisms.
Clearly, for $C = \mathcal{M}f_m$ we have $\mathcal{VB}_C = \mathcal{VB}_m$. The structure of the paper is as follows.

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In Sections 2-5 we extend the presented in [2] description of jet-like functors defined on the category $\mathcal{VB}_m$ to the description of jet-like functors defined on an arbitrary mentioned above category $\mathcal{VB}_\mathcal{C}$. Then the main results of the paper are presented in Sections 6-8.

In particular, in Section 3 we show that for any jet like functor F on $\mathcal{VB}_\mathcal{C}$ of order r there exists a unique $J^r(-,\mathbf{R})$-module bundle functor G^F on $\mathcal{C}$ such that (shortly written)

$$FE = J^rE \otimes_{J^r(M,\mathbf{R})} G^FM \ \text{ and } \ Ff = J^rf \otimes G^F\underline{f} : FE \to FE' \tag{1}$$

for any $\mathcal{VB}_\mathcal{C}$-object E with basis M and any $\mathcal{VB}_\mathcal{C}$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$. Conversely, any $J^r(-,\mathbf{R})$-module bundle functor G on $\mathcal{C}$ determines a jet like functor F^G on $\mathcal{VB}_\mathcal{C}$ and we have $F = F^{G^F}$. We point out that the expression (1) justifies the notation "jet like functor" for fiber product preserving gauge bundle functors on $\mathcal{VB}_\mathcal{C}$. In Section 4 we solve similar problems for jet like functors on $\mathcal{VB}_\mathcal{C}$ of vertical type. We also show that any jet like functor F on $\mathcal{VB}_\mathcal{C}$ has its vertical version F_v. In Section 5 we study the so called "iteration" problem: we compute $G^{F^1\circ G^{F^2}}$ by means of G^{F^1} and G^{F^2}.

In Section 6, if $m \geq 2$ and $r \geq 1$, we describe (in particular) all $J^r(-,\mathbf{R})$-module bundle functors G on $\mathcal{C} = \mathcal{M}f_m$ such that $G = J^r(-,\mathbf{R})$ (as vector bundle functors). Consequently, in Section 7, if $m \geq 2$ and $r \geq 1$, we get the classification of jet like functors F on $\mathcal{VB}_m$ such that the corresponding $J^r(-,\mathbf{R})$-module bundle functors $G = G^F$ on $\mathcal{C} = \mathcal{M}f_m$ satisfy $G = J^r(-,\mathbf{R})$ (as vector bundle functors). In addition to the classical jet functors J^r and J^r_v, we discover some new jet like functors F on $\mathcal{VB}_m$ satisfying the above property.

Finally, in Theorems 8.8 and 8.20 we describe all natural $J^r(M,\mathbf{R})$-module bundle structures on the vector bundles $J^rE \to M$ and $J^r_vE \to M$, where J^rE is the classical jet bundle in the sense of Ehresmann [4] and J^r_vE is the vertical jet bundle.

We remark that fiber product preserving (gauge) bundle functors on other important categories are completely described e.g. in [2, 3, 5–7, 9, 10, 13, 14]. The third author [12] also characterized all fiber product preserving gauge bundle functors on the category $\mathcal{VB}_m$ by means of some admissible triples. In contrast to [12], the description of jet like functors from the present paper is simple and fully geometrical.

We point out that some of jet like functors on $\mathcal{VB}_m$ are very important. In particular, the rth holonomic jet functor J^r plays an important role in the theory of global differential systems and the rth semiholonomic jet functor $\overline{J}^r$ in the sense of Libermann [11] is useful e.g. in prolongation of connections.

2. Preliminaries

Any manifold considered in the paper is assumed to be Hausdorff, second countable, finite dimensional, without boundary and smooth, i.e. of class C^∞. All maps between manifolds are assumed to be smooth.

Let $\mathcal{M}f$ be the category of manifolds and their maps and $\mathcal{FM}$ be the category of fibered manifolds and fibered maps.

We fix an arbitrary positive integer r. We also fix an admissible category $\mathcal{C}$ over manifolds and let $m : \mathcal{C} \to \mathcal{M}f$ be its faithful functor. Assume that any $\mathcal{C}$-morphism is a local isomorphism.

Let $J^r(-,\mathbf{R}) : \mathcal{C} \to \mathcal{FM}$ be the covariant functor sending any $\mathcal{C}$-object M into the bundle $J^r(M,\mathbf{R}) := J^r(m(M),\mathbf{R})$ of (usual) r-jets $m(M) \to \mathbf{R}$ and any $\mathcal{C}$-map $\varphi : M \to M'$ into $J^r(\varphi,\mathrm{id}_\mathbf{R}) : J^r(M,\mathbf{R}) \to J^r(M',\mathbf{R})$ given by $j^r_x\gamma \mapsto j^r_{m(\varphi)(x)}(\gamma \circ m(\varphi)^{-1})$. Clearly, $J^r_x(M,\mathbf{R})$ is a ring (even Weil algebra) and $J^r_x(\varphi,\mathrm{id}_\mathbf{R})$ is a ring (algebra) morphism for any $\mathcal{C}$-map $\varphi : M \to M'$ and any $x \in m(M)$.

Definition 2.1. *Let r be the fixed positive integer. A $J^r(-,\mathbf{R})$-module bundle functor ($J^r(-,\mathbf{R})$-mb-functor) on $\mathcal{C}$ is a regular bundle functor $G : \mathcal{C} \to \mathcal{FM}$ in the sense of [6] of order r (i.e. of the minimal order not more than r) such that:*

(i) GM is a $J^r(M,\mathbf{R})$-module bundle for any $\mathcal{C}$-object M, i.e. G_xM is a $J^r_x(M,\mathbf{R})$-module for any $x \in m(M)$ and the resulting maps $+ : GM \times_{m(M)} GM \to GM$ and $\cdot : J^r(M,\mathbf{R}) \times_{m(M)} GM \to GM$ and $0 : M \to GM$ are smooth;

(ii) $G\varphi : GM \to GM'$ is a $J^r(M,\mathbf{R})$-module bundle map for any $\mathcal{C}$-map $\varphi : M \to M'$, i.e. $G_x\varphi : G_xM \to G_{m(\varphi)(x)}M'$ is a module map over ring map $J^r_x(\varphi,\mathrm{id}_\mathbf{R}) : J^r_x(M,\mathbf{R}) \to J^r_{m(\varphi)(x)}(M',\mathbf{R})$ for any $x \in m(M)$.

Example 2.2. *(a) A simple example of $J^r(-,\mathbf{R})$-mb-functor on C is the described above $J^r(-,\mathbf{R})$-mb-functor $J^r(-,\mathbf{R})$. Many specific examples of $J^r(-,\mathbf{R})$-mb-functors G on C with $G = J^r(-,\mathbf{R})$ or $G = J^r(-,\mathbf{R})_0$ (as vector bundle functors) will be presented in Section 6, where $J^r(M,\mathbf{R})_0$ is the vector sub-bundle in $J^r(M,\mathbf{R})$ of all r-jets $j_x\gamma$ with $\gamma(x) = 0$ and $x \in m(M)$.*

(b) If G^1 and G^2 are $J^r(-,\mathbf{R})$-mb-functors on C, then we have $J^r(-,\mathbf{R})$-mb-functor $G = G^1 \oplus G^2$ on C such that $G_xM = G^1_xM \times G^2_xM$ for any C-object M and $x \in m(M)$, and with $G_x\varphi = G^1_x\varphi \times G^2_x\varphi : G_xM \to G_{m(\varphi)(x)}M^1$ for any C-map $\varphi : M \to M^1$ and $x \in m(M)$.

(c) If G^1 and G^2 are $J^r(-,\mathbf{R})$-mb-functors on C, then we have $J^r(-,\mathbf{R})$-mb-functor $G = G^1 \otimes_{J^r(-,\mathbf{R})} G^2$ on C such that $G_xM = G^1_xM \otimes_{J^r_x(M,\mathbf{R})} G^2_xM$ for any C-object M and $x \in m(M)$, the tensor product over $J^r_x(M,\mathbf{R})$ of $J^r_x(M,\mathbf{R})$-modules G^1_xM and G^2_xM, and with $G_x\varphi = G^1_x\varphi \otimes G^2_x\varphi : G_xM \to G_{m(\varphi)(x)}M^1$ (naturally induced from the module maps $G^1_x\varphi$ and $G^2_x\varphi$ over $J^r_x(\varphi, \mathrm{id}_{\mathbf{R}})$) for any C-map $\varphi : M \to M^1$ and $x \in m(M)$.

(d) Given a $J^r(-,\mathbf{R})$-mb-functor G on C, we have the dual (to G) $J^r(-,\mathbf{R})$-mb-functor G^* on C, see Example 6.11 below.

Definition 2.3. *If G^1 is an another $J^r(-,\mathbf{R})$-mb-functor, then a natural transformation $G \to G^1$ of $J^r(-,\mathbf{R})$-mb-functors is a natural transformation $\nu : G \to G^1$ of bundle functors such that $\nu_M : GM \to G^1M$ is $J^r(M,\mathbf{R})$-module bundle map for any C-object M, i.e. $(\nu_M)_x : G_xM \to G^1_xM$ is $J^r_x(M,\mathbf{R})$-linear for any C-object M and $x \in m(M)$.*

Clearly, any $\mathcal{VB}_C$-object E with basis M is a pair $(E \to m(M), M)$, where M is a C-object (called the basis) and $E \to m(M)$ is a vector bundle, and any $\mathcal{VB}_C$-morphism $f : E \to E_1$ with base map $\underline{f} : M \to M_1$ is a pair $((f, m(\underline{f})), \underline{f})$, where $(f, m(\underline{f}))$ is a vector bundle homomorphism and $\underline{f}$ is a C-morphism. Obviously, if $C = \mathcal{M}f_m$, then $\mathcal{VB}_C$ is the well-known category $\mathcal{VB}_m$, and for $C = \mathcal{FM}_{m,n}$ the $\mathcal{VB}_C$-objects are couples $(E \to Y, Y)$, where $E \to Y$ is a vector bundle and $Y \to X$ is a fibered manifold from $\mathcal{FM}_{m,n}$.

Definition 2.4. *A gauge bundle functor (gb-functor) on $\mathcal{VB}_C$ is a covariant functor $F : \mathcal{VB}_C \to \mathcal{FM}$ such that the conditions (a)-(d) are satisfied.*

(a) every $\mathcal{VB}_C$-object E with basis M is transformed into fibered manifold $\pi_E : FE \to m(M)$,

(b) every $\mathcal{VB}_C$-morphism $f : E \to E_1$ with the base map $\underline{f} : M \to M_1$ is transformed into fibered map $Ff : FE \to FE_1$ over $m(\underline{f}) : m(M) \to m(M_1)$,

(c) for every $\mathcal{VB}_C$-object E with basis M and every open subset $U \subset m(M)$ the inclusion $i : E_{|U} \to E$ induces diffeomorphism $Fi : F(E_{|U}) \to \pi_E^{-1}(U)$,

(d) F transforms smoothly parametrized families of $\mathcal{VB}_C$-maps into smoothly parametrized families.

Definition 2.5. *Given gb-functors F_1 and F_2 on $\mathcal{VB}_C$, a natural transformation $\mu : F_1 \to F_2$ is a system of base preserving fibered maps $\mu_E : F_1E \to F_2E$ for every $\mathcal{VB}_C$-object E satisfying $F_2f \circ \mu_E = \mu_{E'} \circ F_1f$ for every $\mathcal{VB}_C$-map $f : E \to E'$.*

Definition 2.6. *A gb-functor F on $\mathcal{VB}_C$ is of order r if for any $\mathcal{VB}_C$-maps $f, g : E \to E_1$ with the base maps $\underline{f}, \underline{g} : M \to M_1$ and any point $x \in m(M)$ from $j^r_xf = j^r_xg$ (i.e. from $j^r_zf = j^r_zg$ for any $z \in E_x$) it follows $F_xf = F_xg$.*

Definition 2.7. *A jet like functor on $\mathcal{VB}_C$ is a gb-functor F on $\mathcal{VB}_C$ of finite order such that $F(E_1 \times_{m(M)} E_2) = FE_1 \times_{m(M)} FE_2$ modulo (Fpr_1, Fpr_2) for every $\mathcal{VB}_C$ objects E_1 and E_2 with base M, where $pr_i : E_1 \times_{m(M)} E_2 \to E_i$ is the fibered projection.*

Example 2.8. *(a) A simple example of jet like functor on $\mathcal{VB}_C$ of order r is the r-jet prolongation functor $J^r : \mathcal{VB}_C \to \mathcal{FM}$ sending any $\mathcal{VB}_C$-object E with basis M into the bundle J^rE of r-jets $j^r_x\sigma$ of local sections $\sigma : m(M) \to E$ and any $\mathcal{VB}_C$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$ into $J^rf : J^rE \to J^rE'$ given by $j^r_x\sigma \mapsto j^r_{m(\underline{f})(x)}(f \circ \sigma \circ m(\underline{f})^{-1})$.*

(b) Another example of jet like functor on $\mathcal{VB}_C$ of order r is the vertical r-jet prolongation functor $J^r_v : \mathcal{VB}_C \to \mathcal{FM}$ sending any $\mathcal{VB}_C$-object E with basis M into the bundle J^r_vE of r-jets $j^r_x\sigma$ of maps $\sigma : m(M) \to E_x$ and any $\mathcal{VB}_C$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$ into $J^r_vf : J^r_vE \to J^r_vE'$ given by $j^r_x\sigma \mapsto j^r_{m(\underline{f})(x)}(f_x \circ \sigma \circ m(\underline{f})^{-1})$.

(c) There are many other examples of jet like functors (e.g. non-holonomic and semi-holonomic jet prolongation functors, iteration of jet like functors and others).

From now on, for the sake of simplicity we will write M instead of $m(M)$ and $\varphi : M \to M'$ instead of $m(\varphi) : m(M) \to m(M')$ for any C-map $\varphi : M \to M'$.

We have the ring inclusion $\mathbf{R} \subset J^r_x(M, \mathbf{R})$ given by $\lambda \to j^r_x\lambda$. So, any $J^r(M, \mathbf{R})$-module bundle $E \to M$ is (in obvious way) a vector bundle with the fiber multiplication $\lambda v := j^r_x\lambda \cdot v$, $\lambda \in \mathbf{R}$, $v \in E_x$, $x \in M$. Consequently, any $J^r(-, \mathbf{R})$-mb-functor is also a vector bundle functor. From now on, if we consider a $J^r(-, \mathbf{R})$-mb-functor as a vector bundle functor, then we consider it in this way, only.

3. The description of jet like functors by $J^r(-, \mathbf{R})$-mb-functors

Let C be the fixed category and r be the fixed positive integer.

Proposition 3.1. *(i) Let F be a jet like functor on $\mathcal{VB}_C$ of order r. Then FE is a $J^r(M, \mathbf{R})$-module bundle with basis M for any $\mathcal{VB}_C$-object E with basis M, and $Ff : FE \to FE'$ is a module bundle map for any $\mathcal{VB}_C$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$.*

(ii) If F^1 is another jet like functor on $\mathcal{VB}_C$ of order r and $\mu : F \to F^1$ is a natural transformation, then $\mu_E : FE \to F^1E$ is a $J^r(M, \mathbf{R})$-module bundle morphism for any $\mathcal{VB}_C$-object E with basis M.

Proof. The proof of this proposition is substantially the same as the one of Proposition 1 in [2]. We inform only that the fiber sum map of FE is

$$F(+) : FE \times_M FE = F(E \times_M E) \to FE\,,$$

where $+ : E \times_M E \to E$ is the fiber sum map of E treated as a $\mathcal{VB}_C$-map, and the fiber multiplication map $\cdot : J^r(M, \mathbf{R}) \times_M FE \to FE$ is given by

$$\sigma \cdot v = F_x\tilde{\gamma}(v)\,,\ \sigma \in J^r_x(M, \mathbf{R})\,, v \in F_xE\,,\ x \in M\,,$$

where $\gamma : M \to \mathbf{R}$ is a map such that $j^r_x\gamma = \sigma$ and $\tilde{\gamma} : E \to E$ is a base preserving $\mathcal{VB}_C$-map defined by $\tilde{\gamma}(w) = \gamma(x)w$, $w \in E_x$, $x \in M$. □

Remark 3.2. *We can easily see that for $F = J^r$, the usual $J^r_x(M, \mathbf{R})$-module multiplication in J^r_xE is the same as the one of the proof of Proposition 3.1 for $F = J^r$.*

Because of Proposition 3.1, any jet like functor on $\mathcal{VB}_C$ of order r induces a $J^r(-, \mathbf{R})$-mb-functor on C. More precisely, we have the following example.

Example 3.3. *Let F be a jet like functor on $\mathcal{VB}_C$ of order r. We put*

$$G^FM = F(M \times \mathbf{R})\,,\ G^F\varphi = F(\varphi \times \mathrm{id}_{\mathbf{R}})$$

for any C-object M and any C-map $\varphi : M \to M'$. Applying Proposition 3.1, we can see that G^F is a $J^r(-, \mathbf{R})$-mb-functor on C. If F' is an another jet like functor on $\mathcal{VB}_C$ of order r and $\mu : F \to F'$ is a $\mathcal{VB}_C$-natural transformation, then we have $J^r(M, \mathbf{R})$-module bundle map

$$\nu^\mu_M := \mu_{M\times\mathbf{R}} : G^FM \to G^{F'}M$$

for any C-object M. So, $\nu^\mu : G^F \to G^{F'}$ is a natural transformation of $J^r(-, \mathbf{R})$-mb-functors.

Conversely, we have

Example 3.4. *Let G be a $J^r(-,\mathbf{R})$-mb-functor on C. For any $\mathcal{VB}_C$-object E with basis M we define a fibered manifold $F^G E$ with basis M by*

$$F^G E := \bigcup_{x\in M} J^r_x E \otimes_{J^r_x(M,\mathbf{R})} G_x M$$

(or simply by $F^G E := J^r E \otimes_{J^r(M,\mathbf{R})} GM$). For any $\mathcal{VB}_C$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$ we define a fibered map $F^G f : F^G E \to F^G E'$ by

$$F^G f = \bigcup_{x\in M} J^r_x f \otimes G_x \underline{f} : F^G E \to F^G E'$$

(or simply $F^G f = J^r f \otimes G\underline{f}$), where (of course) $J^r_x f \otimes G_x \underline{f}$ is the module map naturally induced from module maps $J^r_x f$ and $G_x \underline{f}$ (over ring map $J^r_x(\underline{f}, \mathrm{id}_{\mathbf{R}})$) for any $x \in M$. Clearly, F^G is a jet like functor on $\mathcal{VB}_C$ of order r. If $\nu : G \to G'$ is a natural transformation of $J^r(-,\mathbf{R})$-mb-functors on C, then we put

$$\mu^\nu_E := \bigcup_{x\in M} \mathrm{id}_{J^r_x E} \otimes (\nu_M)_x : F^G E \to F^{G'} E$$

for any $\mathcal{VB}_C$-object E with basis M. Clearly, $\mu^\nu : F^G \to F^{G'}$ is a natural transformation.

Obviously, for any $\mathcal{VB}_C$-object E with basis M and any $x \in M$, the fibre $J^r_x E$ of $J^r E$ over x is a $J^r_x(M,\mathbf{R})$-module (the multiplication is induced by the multiplication of sections by real-valued maps). We used this fact in the above Example 3.4.

Theorem 3.5. *Let F be a jet like functor on $\mathcal{VB}_C$ of order r. Then*

$$F = F^{G^F}$$

modulo canonical (in F) natural isomorphism. If $F = F^G$ then $G = G^F$.

Proof. The proof is substantially the same as the one of Theorem 1 in [2]. We inform only that for any $\mathcal{VB}_C$-object E with basis M and $x \in M$, we have a $J^r_x(M,\mathbf{R})$-bilinear map $\alpha_x : J^r_x E \times G^F_x M \to F_x E$ given by

$$\alpha_x(\eta, v) = F_x \widehat{\sigma}(v)$$

for any $\eta \in J^r_x E$, $v \in G^F_x M = F_x(M \times \mathbf{R})$, where $\sigma : M \to E$ is a section such that $j^r_x \sigma = \eta$, where $\widehat{\sigma} : M \times \mathbf{R} \to E$ is a $\mathcal{VB}_C$-map defined by $\widehat{\sigma}(u,t) = t\sigma(u)$, $(u,t) \in M \times \mathbf{R}$. Equivalently, we have the $J^r_x(M,\mathbf{R})$-linear map

$$\alpha_x : F^G_x E = J^r_x E \otimes_{J^r_x(M,\mathbf{R})} G^F_x M \to F_x E \, .$$

Consequently we have the resulting map $\alpha_E : F^G E \to FE$, which is an isomorphism.
□

Because of Theorem 3.5, any jet like functor F on $\mathcal{VB}_C$ of order r is of the form (1).

Proposition 3.6. *Let F and F' be jet like functors on $\mathcal{VB}_C$ of order r. The correspondence $\mu \mapsto \nu^\mu$ is a bijection between natural transformations $F \to F'$ and morphisms $G^F \to G^{F'}$ between corresponding $J^r(-,\mathbf{R})$-mb-functors on C. The inverse correspondence is $\nu \mapsto \mu^\nu$ (modulo the identification from Theorem 3.5).*

Proof. It is clear. □

Corollary 3.7. *The category $\mathcal{K}^r$ of jet like functors F of order r on $\mathcal{VB}_C$ and their natural transformations is equivalent to the category $\mathcal{L}^r$ of $J^r(-,\mathbf{R})$-mb-functors G on C and their natural transformations.*

Proof. There exist functor $I : \mathcal{K}^r \to \mathcal{L}^r$ given by $I(F) = G^F$ and $I(\mu) = \nu^\mu$ (as in Example 3.3) and functor $J : \mathcal{L}^r \to \mathcal{K}^r$ given by $J(G) = F^G$ and $J(\nu) = \mu^\nu$ (as in Example 3.4). We have $J \circ I = \mathrm{id}_{\mathcal{K}^r}$ (modulo isomorphism) and $I \circ J = \mathrm{id}_{\mathcal{L}^r}$ (modulo isomorphism). The mentioned isomorphisms are immediate consequences of Theorem 3.5. □

4. The jet like functors of vertical type

Definition 4.1. *A jet like functor F on $\mathcal{VB}_C$ is called to be of vertical type if for any $\mathcal{VB}_C$-objects E and E' with the same basis M, any $x_o \in M$ and any base preserving $\mathcal{VB}_C$-morphism $f : E \to E'$, the restriction $F_{x_o}f : F_{x_o}E \to F_{x_o}E'$ of Ff depends on $f_{x_o} : E_{x_o} \to E'_{x_o}$, only.*

Example 4.2. *(a) Let J^r_v be the vertical jet prolongation functor from Section 2. Then it is of vertical type and we have $J^r_vE = E \otimes_{\mathbf{R}} J^r(M, \mathbf{R})$.*

(b) Let V^A be the vertical Weil functor corresponding to a Weil algebra A defined by $V^AE = \bigcup_{x\in M} T^AE_x$, see [1]. Then it is of vertical type and we have $V^AE = E \otimes_{\mathbf{R}} A$.

(c) Let F^1 be the jet like functor on $\mathcal{VB}_m$ given by $F^1E = E \otimes_{\mathbf{R}} TM$ and $F^1f = f \otimes T\underline{f}$ for any $\mathcal{VB}_m$-object $E \to M$ and any $\mathcal{VB}_m$-morphisms $f : E \to E'$ with base map $\underline{f} : M \to M'$, where T is the tangent functor. It is a jet like functor of vertical type on $\mathcal{VB}_m$. We will use this functor in the next section.

Example 4.3. *Let F be a jet like functor of vertical type on $\mathcal{VB}_C$ of order r. We put*

$$G^{(F)}M = F(M \times \mathbf{R}) \ , \ G^{(F)}\varphi = F(\varphi \times \mathrm{id}_{\mathbf{R}})$$

for any C-object M and any C-map $\varphi : M \to M'$. Then $G^{(F)}$ is a vector bundle functor (because it is even $J^r(-, \mathbf{R})$-mb-functor) of order r on C. If F' is an another jet like functor of vertical type on $\mathcal{VB}_C$ of order r and $\mu : F \to F'$ is a $\mathcal{VB}_C$-natural transformation, then we have vector bundle map

$$\nu^{(\mu)}_M := \mu_{M\times\mathbf{R}} : G^{(F)}M \to G^{(F')}M$$

for any C-object M. So, $\nu^{(\mu)} : G^{(F)} \to G^{(F')}$ is a natural transformation of vector bundle functors.

Conversely, we have

Example 4.4. *Let G be a vector bundle functor on C of order r. For any $\mathcal{VB}_C$-object E with basis M we define a fibered manifold $F^{(G)}E$ with basis M by*

$$F^{(G)}E := \bigcup_{x\in M} E_x \otimes_{\mathbf{R}} G_xM$$

(or simply by $F^{(G)}E := E \otimes_{\mathbf{R}} GM$). For any $\mathcal{VB}_C$-map $f : E \to E'$ with base map $\underline{f} : M \to M'$ we define a fibered map $F^{(G)}f : F^{(G)}E \to F^{(G)}E'$ by

$$F^{(G)}f = \bigcup_{x\in M} f_x \otimes G_x\underline{f} : F^{(G)}E \to F^{(G)}E'$$

(or simply $F^{(G)}f = f \otimes G\underline{f}$). Clearly, $F^{(G)}$ is a jet like functor of vertical type on $\mathcal{VB}_C$ of order r. If $\nu : G \to G'$ is a natural transformation of vector bundle functors (of order r) on C, then we put

$$\mu^{(\nu)}_E := \bigcup_{x\in M} \mathrm{id}_{E_x} \otimes (\nu_M)_x : F^{(G)}E \to F^{(G')}E$$

for any $\mathcal{VB}_C$-object E with basis M. Clearly, $\mu^{(\nu)} : F^{(G)} \to F^{(G')}$ is a natural transformation of jet like functors.

Theorem 4.5. *Let F be a jet like functor of vertical type on $\mathcal{VB}_C$ of order r. Then*

$$F = F^{(G^{(F)})}$$

modulo canonical (in F) natural isomorphism. If $F = F^{(G)}$ then $G = G^{(F)}$.

Proof. The proof is quite similar to the one of Theorem 3.5. □

Proposition 4.6. *Let F and F' be jet like functors of vertical type on $\mathcal{VB}_C$ of order r. The correspondence $\mu \mapsto \nu^{(\mu)}$ is a bijection between natural transformations $F \to F'$ and morphisms $G^{(F)} \to G^{(F')}$ between corresponding vector bundle functors of order r on $\mathcal{C}$. The inverse correspondence is $\nu \mapsto \mu^{(\nu)}$ (modulo the identification from Theorem 4.5).*

Proof. It is clear. □

Corollary 4.7. *The category of jet like functors F of vertical type of order r on $\mathcal{VB}_C$ and their natural transformations is equivalent to the category of vector bundle functors G of order r on $\mathcal{C}$ and their natural transformations.*

Proof. The proof is quite similar to the one of Corollary 3.7. □

Example 4.8. *Let F be a jet like functor (not necessarily of vertical type) on $\mathcal{VB}_C$ of order r. Let G^F be the $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{C}$ corresponding to F as in Example 3.3, so that*

$$FE = J^rE \otimes_{J^r(M,\mathbf{R})} G^FM.$$

Obviously, we can treat G^F as a vector bundle functor on $\mathcal{C}$ of order r. Thus, by Example 4.4, we have the jet like functor $F_v := F^{(G^F)}$ of vertical type on $\mathcal{VB}_C$ of order r. Clearly,

$$F_vE = E \otimes_{\mathbf{R}} G^FM \quad \text{and} \quad F_vf = f \otimes G^F\underline{f} : F_vE \to F_vE'$$

for any $\mathcal{VB}_C$-object E with basis M and any $\mathcal{VB}_C$-morphism $f : E \to E'$ with base map $\underline{f} : M \to M'$. So any F as above has the vertical version F_v. In particular, if $F = J^r$ is the r-jet prolongation functor on $\mathcal{VB}_C$, then (up to canonical isomorphism) $F_v = J^r_v$=the usual vertical r-jet prolongation functor J^r_v on $\mathcal{VB}_C$.

Proposition 4.9. *The correspondence $F \to F_v$ is a functor from the category $\mathcal{K}^r$ of jet like functors of order r on $\mathcal{VB}_C$ and their natural transformations into the subcategory $\mathcal{K}^r_v$ (in $\mathcal{K}^r$) of jet like functors of vertical type of order r on $\mathcal{VB}_C$ and their natural transformations. Any $\mathcal{K}^r$-object F is a $\mathcal{K}^r_v$-object if and only if $F = F_v$ (modulo the standard isomorphism).*

Proof. It is a simple observation. □

5. Iteration

By Proposition 3.1, any jet like functor (not necessarily of vertical type) on $\mathcal{VB}_C$ has values in $\mathcal{VB}_C$. So, we can compose jet like functors.

Proposition 5.1. *Let $F^1 = F^{G^1}$ and $F^2 = F^{G^2}$ be two jet like functors on $\mathcal{VB}_C$ of (finite) order r^1 and r^2, respectively. Then the composition $F = F^1 \circ F^2$ is a jet like functor on $\mathcal{VB}_C$ of order $r = r^1 + r^2$. Let $F = F^G$, where $G := G^{F^1 \circ F^2}$. Then*

$$GM = J^{r^1}G^2M \otimes_{J^{r^1}(M,\mathbb{R})} G^1M$$

for any $\mathcal{C}$-manifold M, and $G\varphi : GM \to GM_1$ is naturally induced by $J^{r^1}G^2\varphi : J^{r^1}G^2M \to J^{r^1}G^2M_1$ and $G^1\varphi : G^1M \to G^1M_1$ for any $\mathcal{C}$-map $\varphi : M \to M_1$. Moreover, given $x_o \in M$, the multiplication $\cdot : J^r_{x_o}(M,\mathbf{R}) \times G_{x_o}M \to G_{x_o}M$ satisfies

$$j^r_{x_o}\gamma \cdot (j^{r^1}_{x_o}\sigma \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v) = j^{r^1}_{x_o}(x \rightarrowtail j^{r^2}_x\gamma \cdot \sigma(x)) \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v$$

for any section $\sigma : M \to G^2M$ of $G^2M \to M$, any map $\gamma : M \to \mathbb{R}$ and any $v \in G^1_{x_o}M$, where $\cdot$ (on the right of the equality) is the multiplication of G^2M.

Proof. The proof is substantially the same as the one of Proposition 2 in [2]. □

So jet like functors on $\mathcal{VB}_C$ are very often not commuting. By the following proposition, we have even an explicit example of two not commuting jet like functors on $\mathcal{VB}_m$.

Proposition 5.2. *Let $F = J^r$ be the r-jet prolongation functor on $\mathcal{VB}_m$ and F^1 be the vertical jet like functor on $\mathcal{VB}_m$ from Example 4.2. The functors F and F^1 do not commute.*

Proof. For $E = M \times \mathbf{R}$, we have $FF^1E = J^rTM$ and $F^1FE = J^r(M,\mathbf{R}) \otimes_{\mathbf{R}} TM$. Clearly, the bundles J^rTM and $J^r(M,\mathbf{R}) \otimes_{\mathbf{R}} TM$ are not $\mathcal{M}f_m$-natural isomorphic. Indeed, the first one is of (the minimal) order not less than $r+1$ and the second one is of order r. □

Clearly, the composition of jet like functors on $\mathcal{VB}_C$ of vertical type is again a jet like functor on $\mathcal{VB}_C$ of vertical type. For the iteration we have

Proposition 5.3. *Let F_1 and F_2 be jet like functors on $\mathcal{VB}_C$ of vertical type of (finite) order r_1 and r_2, respectively. Let G_1 and G_2 be the corresponding regular vector bundle functors on $\mathcal{C}$ of order r_1 and r_2, respectively. Then the vector bundle functor corresponding to $F_1 \circ F_2$ is $G_2 \otimes_{\mathbf{R}} G_1$. Consequently, any jet like functors on $\mathcal{VB}_C$ of vertical type of finite orders commute. In particular, $J_v^{r_1}$ and J_v^r commute.*

Proof. We have $F_1F_2E = E \otimes_{\mathbf{R}} G_2M \otimes_{\mathbf{R}} G_1M$ and $F_2F_1E = E \otimes_{\mathbf{R}} G_1M \otimes_{\mathbf{R}} G_2M$. Then the exchange isomorphism ends the proof. □

Corollary 5.4. *Let F_1 and F_2 be jet like functors on $\mathcal{VB}_C$ of vertical type of (finite) order r_1 and r_2, respectively. Then $F_1 \circ F_2$ is of order $\max(r_1, r_2)$. For example, $J_v^{r_1} J_v^{r_2}$ is of order $\max(r_1, r_2)$.*

Proof. It is clear because of the previous proposition. □

Remark 5.5. *(1) In [3] we obtained the similar property to the one of Proposition 5.3: All regular finite order fiber product preserving bundle functors of vertical type on $\mathcal{FM}_C$ (in particular for $\mathcal{C} = \mathcal{M}f_m$) commute.*

(2) By [1], not all fiber product preserving bundle functors on $\mathcal{FM}_m$ commute. Similar property holds for jet like functors on $\mathcal{VB}_m$. Namely, in Proposition 5.2 we presented explicitly a jet like functor on $\mathcal{VB}_m$ (even of vertical type) not commuting with J^r.

(3) By [1], any vertical Weil functor V^A corresponding to a Weil algebra A on $\mathcal{FM}_m$ commutes with J^r. Consequently, V^A (on $\mathcal{VB}_m$) commutes with J^r (on $\mathcal{VB}_m$).

(4) From Corollary 5.4 it follows the following surprising fact: There is no $\mathcal{VB}_m$-natural embedding $J_v^r \to J_v^1 \circ \dots \circ J_v^1$(r-times). Indeed, J_v^r is of order at last r and (by Corollary 5.4) $J_v^1 \circ \dots \circ J_v^1$ is of order 1.

6. The $J^r(-,\mathbf{R})$-mb-functors G on $\mathcal{M}f_m$ with $G = J^r(-,\mathbf{R})$ or $G = (J^r(-,\mathbf{R}))^*$ or $G = J^r(-,\mathbf{R})_0$ or $G = (J^r(-,\mathbf{R})_0)^*$

Let r be the positive integer and $\mathcal{C}$ be the fixed category.

Example 6.1. *Let $\alpha \in \mathbf{R}$. Let $G^{[\alpha]} = J^r(-,\mathbf{R})$ be the vector bundle functor on $\mathcal{C}$ (not necessarily $\mathcal{M}f_m$). It is a $J^r(-,\mathbf{R})$-mb-functor with respect to the multiplication $\star^\alpha : J^r_x(M,\mathbf{R}) \times G^{[\alpha]}_x M \to G^{[\alpha]}_x M$ given by*

$$j^r_x\gamma \star^\alpha j^r_x\rho = j^r_x(\gamma\rho) + \alpha\rho(x) j^r_x\gamma - \alpha\gamma(x)\rho(x) j^r_x 1$$

for any $\mathcal{C}$-object M, $j^r_x\gamma \in J^r_x(M,\mathbf{R})$, $j^r_x\rho \in G^{[\alpha]}_x M$ and $x \in M$.

Example 6.2. *Let $G^v = J^r(-,\mathbf{R})$ be the vector bundle functor on $\mathcal{C}$. It is a $J^r(-,\mathbf{R})$-mb-functor with respect to the multiplication $\odot : J^r_x(M,\mathbf{R}) \times G^v_x M \to G^v_x M$,*

$$j^r_x\gamma \odot j^r_x\rho = \gamma(x) j^r_x\rho$$

for any $\mathcal{C}$-object M, $j^r_x\gamma \in J^r_x(M,\mathbf{R})$, $j^r_x\rho \in G^v_x M$ and $x \in M$.

Example 6.3. *Let $\beta \in \mathbf{R}$ and $r = 2$. Let $G^{<\beta>} = J^2(-, \mathbf{R})$ be the vector bundle functor on $\mathcal{C}$. It is a $J^2(-, \mathbf{R})$-mb-functor with respect to the multiplication $*^\beta : J^2_x(M, \mathbf{R}) \times_M G^{<\beta>}_x M \to G^{<\beta>}_x M$ given by*

$$j^2_x\gamma *^\beta j^2_x\rho = (\beta + 1)j^2_x(\gamma\rho) - (\beta + 1)\rho(x)j^2_x\gamma - \beta\gamma(x)j^2_x\rho + (\beta + 1)\gamma(x)\rho(x)j^2_x 1$$

for any $\mathcal{C}$-object M, $j^2_x\gamma \in J^2_x(M, \mathbf{R})$, $j^2_x\rho \in G^{<\beta>}_x M$ and $x \in M$. To show that

$$j^2_x\gamma^1 *^\beta (j^2_x\gamma *^\beta j^2_x\rho) = j^2_x(\gamma^1\gamma) *^\beta j^2_x\rho\ ,$$

we can assume $x = 0$ and then it is sufficient to consider four cases: (a) $j^2_0\gamma^1 = j^2_0 1$, and (b) $j^2_0\gamma = j^2_0 1$, and (c) $j^2_0\rho^1 = j^2_0 1$, and (d) $\gamma^1(0) = \gamma(0) = \rho(0) = 0$. In the case (d) the above property holds (both sides are evidently equal to $j^2_0(0)$ because $r = 2$). In the remaining cases (a)-(c), we can verify the equality directly.

Proposition 6.4. *(i) Let $m \geq 2$ and $r \geq 3$ be integers. Any $J^r(-, \mathbf{R})$-mb-functor G on $\mathcal{C} = \mathcal{M}f_m$ with $G = J^r(-, \mathbf{R})$ (as vector bundle functors) is G^v or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$.*

(ii) Let $m \geq 2$ be an integer and $r = 2$. Any $J^2(-, \mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^2(-, \mathbf{R})$ (as vector bundle functors) is G^v or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$ or $G^{<\beta>}$ for some $\beta \in \mathbf{R} \setminus \{0, -1\}$. (For $\beta = 0$ we have $G^{<0>} = G^{[-1]}$. For $\beta = -1$ we have $G^{<-1>} = G^v$.)

(iii) Let $m \geq 2$ be an integer and $r = 1$. Any $J^1(-, \mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^1(-, \mathbf{R})$ (as vector bundle functors) is $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$ (For $\alpha = -1$ we have $G^{[-1]} = G^v$.)

Proof. Consider a $J^r(-, \mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^r(-, \mathbf{R})$ and let

$$\circ : J^r_x(M, \mathbf{R}) \times G_x M \to G_x M\ ,\ x \in M\ ,\ M \in \mathrm{Obj}(\mathcal{M}f_m)$$

be its multiplication. We have the family of values

$$j^r_0\gamma \circ j^r_0\eta \in G_0\mathbf{R}^m \text{ for all } j^r_0\gamma \in J^r_0(\mathbf{R}^m, \mathbf{R}) \text{ and } j^r_0\eta \in G_0\mathbf{R}^m \text{ with } \gamma(0) = 0\ .$$

Using the $\mathcal{M}f_m$-invariance of $\circ$, we can see that if two multiplications (in question) give the same family of values (in question), then they are equal. So, it sufficient to study the above family of values. Since $m \geq 2$, then by the rank theorem and the $\mathcal{M}f_m$-invariance of $\circ$, we can assume that

$$j^r_0\gamma = j^r_0 x^1 \text{ and } (j^r_0\eta = j^r_0 1 \ \text{ or } \ j^r_0\eta = j^r_0 x^2)\ .$$

By the invariance of $\circ$ with respect to $(t^1x^1, ..., t^mx^m)$ for $t^i \neq 0$ we deduce that

$$j^r_0 x^1 \circ j^r_0 x^2 = \sigma j^r_0(x^1x^2) \text{ and } j^r_0 x^1 \circ j^r_0 1 = \rho j^r_0 x^1$$

for some real numbers σ, ρ. Then by the $\mathcal{M}f_m$-invariance of $\circ$,

$$j^r_0\xi \circ j^r_0\eta = \sigma j^r_0(\xi\eta) \quad \text{and} \quad j^r_0\xi \circ j^r_0 1 = \rho j^r_0\xi$$

for all $j^r_0\xi \in J^r_0(\mathbf{R}^m, \mathbf{R})$, $j^r_0\eta \in G_0\mathbf{R}^m$ with $\xi(0) = \eta(0) = 0$. Then

$$\sigma j^r_0(x^1x^2x^2) = j^r_0(x^1x^2) \circ j^r_0 x^2 = j^r_0 x^1 \circ (j^r_0 x^2 \circ j^r_0 x^2) = \sigma^2 j^r_0(x^1x^2x^2)$$

and

$$\rho j^r_0(x^1x^2) = j^r_0(x^1x^2) \circ j^r_0 1 = j^r_0 x^1 \circ (j^r_0 x^2 \circ j^r_0 1) = \sigma\rho j^r_0(x^1x^2)\ .$$

Consider three cases.

(i) Assume $r \geq 3$. Then $\sigma^2 = \sigma$ and $\sigma\rho = \rho$, and then ($\sigma = 1$ and ρ is arbitrary) or ($\sigma = 0$ and $\rho = 0$). If $\sigma = \rho = 0$, then

$$j^r_0\gamma \circ j^r_0\eta = \gamma(0)j^r_0\eta\ ,$$

i.e. $G = G^v$. If $\sigma = 1$ and ρ is arbitrary, then

$$j^r_0\gamma \circ j^r_0\eta = j^r_0(\gamma\eta) + \alpha\eta(0)j^r_0\gamma - \alpha\gamma(0)\eta(0)j^r_0 1\,,$$

i.e. $G = G^{[\alpha]}$, where $\alpha = \rho - 1$.

(ii) Assume $r = 2$. Then σ is arbitrary and $\sigma\rho = \rho$, and then ($\sigma = 1$ and ρ is arbitrary) or ($\sigma \neq 1$ and $\rho = 0$). If $\sigma = 1$ and $\rho \in \mathbf{R}$, then (as above) $G = G^{[\alpha]}$ for $\alpha = \rho - 1$. If $\rho = 0$ then

$$j^2_0\gamma \circ j^2_0\eta = \sigma j^2_0(\gamma\eta) - \sigma\eta(0)j^2_0\gamma - (\sigma - 1)\gamma(0)j^2_0\eta + \sigma\gamma(0)\eta(0)j^2_0 1\,,$$

i.e. $G = G^{<\beta>}$ for $\beta = \sigma - 1$.

(iii) Assume $r = 1$. Then $j^1_0(x^1x^2) = 0$. Then

$$j^1_0\gamma \circ j^1_0\eta = j^1_0(\gamma\eta) + \alpha\eta(0)j^1_0\gamma - \alpha\gamma(0)\eta(0)j^1_0 1\,,$$

i.e. $G = G^{[\alpha]}$, where $\alpha = \rho - 1$.

□

We can see that Proposition 6.4 gives the full description of all $J^r(-,\mathbf{R})$-mb-functors G on $\mathcal{M}f_m$ with $G = J^r(-,\mathbf{R})$ (as vector bundle functors) for $r \geq 1$ and $m \geq 2$. All elements of the collection presented in Proposition 6.4 are different (except $G^{<-1>} = G^v$ and $G^{<0>} = G^{[-1]}$ (if $r = 2$) and $G^{[-1]} = G^v$ (if $r = 1$)). There is a question, whether the elements of the collection presented in Proposition 6.4 are mutually not isomorphic? We are going to answer to this question.

Lemma 6.5. *For any $\alpha \in \mathbf{R}$ we have a natural transformation $a : G^{[0]} \to G^{[\alpha]}$ of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{C} = \mathcal{M}f_m$ such that*

$$a_x(j^r_x\gamma) = (\alpha + 1)j^r_x\gamma - \alpha\gamma(x)j^r_x 1$$

for any m-manifold M, $j^r_x\gamma \in G^{[0]}_x M$ and $x \in M$. Any natural transformation $b : G^{[0]} \to G^{[\alpha]}$ of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$ is a constant multiple of a. Consequently, $G^{[\alpha]}$ is isomorphic to $G^{[0]}$ for any $\alpha \in \mathbf{R} \setminus \{-1\}$, and $G^{[0]}$ and $G^{[-1]}$ are not isomorphic.

Proof. By a direct verification, a is a natural transformation of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$. Further, by the invariance of b with respect to the homotheties,

$$b_0(j^r_0 1) = k j^r_0 1$$

for some $k \in \mathbf{R}$. Then $b_x(j^r_x 1) = k j^r_x 1$ for any m-manifold M and $x \in M$. Then

$$b_x(j^r_x\gamma) = b_x(j^r_x\gamma \star^0 j^r_x 1) = j^r_x\gamma \star^\alpha b_x(j^r_x 1) = k(j^r_x\gamma \star^\alpha j^r_x 1) = k(j^r_x\gamma + \alpha j^r_x\gamma - \alpha\gamma(x)j^r_x 1) = k a_x(j^r_x\gamma).$$

□

Lemma 6.6. *We have a natural transformation $c : G^{[0]} \to G^v$ of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$ given by*

$$c_x(j^r_x\gamma) = \gamma(x)j^r_x 1$$

for any m-manifold M, $j^r_x\gamma \in G^{[0]}_x M$ and $x \in M$. Any natural transformation $d : G^{[0]} \to G^v$ of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$ is a constant multiple of c. Consequently, G^v and $G^{[0]}$ are not isomorphic.

Proof. The proof is quite similar to the one of Lemma 6.5. More detailed, $d_x(j^r_x 1) = k j^r_x 1$ for some $k \in \mathbf{R}$. Then

$$d_x(j^r_x\gamma) = d_x(j^r_x\gamma \star^0 j^r_x 1) = j^r_x\gamma \odot d_x(j^r_x 1) = k j^r_x\gamma \odot j^r_x 1 = k\gamma(x)j^r_x 1 = k c_x(j^r_x\gamma).$$

□

Lemma 6.7. *Let $\mathcal{C} = \mathcal{M}f_m$. If $r \geq 2$ then G^v and $G^{[-1]}$ are not isomorphic.*

Proof. For any natural transformation $e : G^{[-1]} \to G^v$ of $J^r(-,\mathbf{R})$-mb-functors we have

$$e_0(j_0^r((x^1)^2)) = e_0(j_0^r x^1 \star^{-1} j_0^r x^1) = j_0^r x^1 \odot e_0(j_0^r x^1) = x^1(0) e_0(j_0^r x^1) = 0\,.$$

□

Lemma 6.8. *Let $C = \mathcal{M}f_m$ and $r = 2$. If $\beta \neq \beta_1$, then $G^{<\beta>}$ and $G^{<\beta_1>}$ are not isomorphic. In particular, if $\beta \in \mathbf{R} \setminus \{0, -1\}$ then $G^{<\beta>}$ and $G^{[-1]}$ are not isomorphic (as $G^{[-1]} = G^{<0>}$) and $G^{<\beta>}$ and G^v are not isomorphic (as $G^v = G^{<-1>}$).*

Proof. Let $h : J^2(-,\mathbf{R}) \to J^2(-,\mathbf{R})$ be a (fiber linear) natural transformation of vector bundle functors on $\mathcal{M}f_m$. By the invariance of h with respect to $(\tau^1 x^1, ..., \tau^m x^m)$ we derive

$$h_0(j_0^2 1) = k j_0^2 1 \text{ and } h_0(j_0^2 x^1) = l j_0^2 x^1$$

for some $k, l \in \mathbf{R}$. Then (by the invariance of h)

$$h_x(j_x^2 \gamma) = l j_x^2 \gamma + (k - l)\gamma(x) j_x^2 1$$

for any m-manifold M, $j_x^2 \gamma \in J_x^2(M, \mathbf{R})$ and $x \in M$. Suppose that $h : G^{<\beta>} \to G^{<\beta_1>}$ is a natural isomorphism of $J^2(-,\mathbf{R})$-mb-functors. Then $l \neq 0$ and

$$\begin{aligned} &l(\beta+1) j_0^2((x^1)^2) = h_0((\beta+1) j_0^2((x^1)^2)) = h_0(j_0^2 x^1 *^{\beta} j_0^2 x^1) \\ &= j_0^2 x^1 *^{\beta_1} h(j_0^2 x^1) = l j_0^2 x^1 *^{\beta_1} j_0^2 x^1 = l(\beta_1 + 1) j_0^2((x^1)^2)\,. \end{aligned}$$

Then $\beta = \beta_1$ and the proof is complete. □

Lemma 6.9. *Let $C = \mathcal{M}f_m$ and $r = 2$. If $\beta \in \mathbf{R} \setminus \{0\}$ then $G^{<\beta>}$ and $G^{[0]}$ are not isomorphic.*

Proof. Let h be as in the previous lemma. Suppose that $h : G^{<\beta>} \to G^{[0]}$ is a natural isomorphism of $J^2(-,\mathbf{R})$-mb-functors. Then $l \neq 0$ and

$$\begin{aligned} &l(\beta+1) j_0^2((x^1)^2) = h_0((\beta+1) j_0^2((x^1)^2)) = h_0(j_0^2 x^1 *^{\beta} j_0^2 x^1) \\ &= j_0^2 x^1 \star^0 h_0(j_0(j_0^2 x^1)) = l j_0^2 x^1 \star^0 j_0^2 x^1 = l j_0^2((x^1)^2)\,. \end{aligned}$$

Then $\beta = 0$. Contradiction. □

Summing up, we have proved

Proposition 6.10. *(i) Let $m \geq 2$ and $r \geq 3$. Then any $J^r(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^r(-,\mathbf{R})$ (as vector bundle functors) is isomorphic to $G^{[-1]}$ or $G^{[0]}$ or G^v. Moreover, $G^{[-1]}$, $G^{[0]}$ and G^v represent different (isomorphism) classes.*

(ii) Let $m \geq 2$ and $r = 2$. Then any $J^2(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^2(-,\mathbf{R})$ (as vector bundle functors) is isomorphic to $G^{[-1]}$ or $G^{[0]}$ or G^v or $G^{<\beta>}$ for some $\beta \in \mathbf{R} \setminus \{0, -1\}$. Moreover, $G^{[-1]}$, $G^{[0]}$, G^v and $G^{<\beta>}$ for all different $\beta \in \mathbf{R} \setminus \{0, -1\}$ represent different classes.

(iii) Let $m \geq 2$ and $r = 1$. Then any $J^1(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = J^1(-,\mathbf{R})$ (as vector bundle functors) is isomorphic to $G^{[0]}$ or G^v. Moreover, $G^{[0]}$ and G^v represent different classes.

Example 6.11. *In general, if G is a $J^r(-,\mathbf{R})$-mb-functor on C with $G = J$ (as vector bundle functors), where J is a vector bundle functor on C (not necessarily $\mathcal{M}f_m$) of order r, then we have the corresponding (dual) $J^r(-,\mathbf{R})$-mb-functor G^* on C with $G^* = J^*$ (as vector bundle functors), where J^* is the vector bundle functors dual to J (i.e. $J_x^* M = (J_x M)^*$ and $J_x^* \varphi = ((J_x \varphi)^*)^{-1}$ for any C-object M and any C-map $\varphi : M \to M_1$ and any $x \in M$). The multiplication $\circ^* : J_x^r(M,\mathbf{R}) \times G_x^* M \to G_x^* M$ is given by*

$$(j_x^r \gamma \circ^* \omega)(v) := \omega(j_x^r \gamma \circ v)\,,$$

$j_x^r \gamma \in J_x^r(M,\mathbf{R})$, $\omega \in G_x^* M$, $v \in G_x M$, $x \in M$, where $\circ$ is the multiplication for G.

Proposition 6.12. *Let J be a vector bundle functor of order r on $\mathcal{C}$. The correspondence $G \to G^*$ is one to one (between the $J^r(-,\mathbf{R})$-mb-functors G with $G = J$ and the $J^r(-,\mathbf{R})$-mb-functors G^* with $G^* = J^*$). The inverse correspondence is $G^* \to (G^*)^* = G$ (modulo the standard canonical isomorphism).*

Proof. The proposition is clear. □

From Propositions 6.10 and 6.12 it follows directly

Proposition 6.13. *(i) Let $m \geq 2$ and $r \geq 3$. Then any $J^r(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = (J^r(-,\mathbf{R}))^*$ (as vector bundle functors) is isomorphic to $(G^{[-1]})^*$ or $(G^{[0]})^*$ or $(G^v)^*$. Moreover, $(G^{[-1]})^*$, $(G^{[0]})^*$ and $(G^v)^*$ represent different isomorphism classes.*

(ii) Let $m \geq 2$ and $r = 2$. Then any $J^2(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = (J^2(-,\mathbf{R}))^*$ (as vector bundle functors) is isomorphic to $(G^{[-1]})^*$ or $(G^{[0]})^*$ or $(G^v)^*$ or $(G^{<\beta>})^*$ for some $\beta \in \mathbf{R} \setminus \{0,-1\}$. Moreover, $(G^{[-1]})^*$, $(G^{[0]})^*$, $(G^v)^*$ and $(G^{<\beta>})^*$ for all different $\beta \in \mathbf{R} \setminus \{-1,0\}$ represent different isomorphism classes.

(iii) Let $m \geq 2$ and $r = 1$. Then any $J^1(-,\mathbf{R})$-mb-functor G on $\mathcal{M}f_m$ with $G = (J^1(-,\mathbf{R}))^*$ (as vector bundle functors) is isomorphic to $(G^{[0]})^*$ or $(G^v)^*$. Moreover, $(G^{[0]})^*$ and $(G^v)^*$ represent different isomorphism classes.

Example 6.14. *Let $H^{[]} = J^r(-,\mathbf{R})_0$ be the vector bundle functor on $\mathcal{C}$. It is a $J^r(-,\mathbf{R})$-mb-functor with respect to the multiplication $\star : J^r_x(M,\mathbf{R}) \times H^{[]}_x M \to H^{[]}_x M$,*

$$j^r_x\gamma \star j^r_x\rho = j^r_x(\gamma\rho)$$

for any $\mathcal{C}$-object M, $j^r_x\gamma \in J^r_x(M,\mathbf{R})$, $j^r_x\rho \in H^{[]}_x M$ and $x \in M$.

Example 6.15. *Let $H^v = J^r(-,\mathbf{R})_0$ be the vector bundle functor on $\mathcal{C}$. It is a $J^r(-,\mathbf{R})$-mb-functor with respect to the multiplication $\odot : J^r_x(M,\mathbf{R}) \times H^v_x M \to H^v_x M$,*

$$j^r_x\gamma \odot j^r_x\rho = \gamma(x) j^r_x\rho$$

for any $\mathcal{C}$-object M, $j^r_x\gamma \in J^r_x(M,\mathbf{R})$, $j^r_x\rho \in H^v_x M$ and $x \in M$.

Example 6.16. *Let $\beta \in \mathbf{R}$ and $r = 2$. Let $H^{<\beta>} = J^2(-,\mathbf{R})_0$ be the vector bundle functor on $\mathcal{C}$. It is a $J^2(-,\mathbf{R})$-mb-functor with respect to the multiplication $*^\beta : J^2_x(M,\mathbf{R}) \times_M H^{<\beta>}_x M \to H^{<\beta>}_x M$,*

$$j^2_x\gamma *^\beta j^2_x\rho = (\beta+1) j^2_x(\gamma\rho) - \beta\gamma(x) j^2_x\rho$$

for any $\mathcal{C}$-object M, $j^2_x\gamma \in J^2_x(M,\mathbf{R})$, $j^2_x\rho \in H^{<\beta>}_x M$ and $x \in M$.

Proposition 6.17. *(i) Let $m \geq 2$ and $r \geq 3$ be integers. Any $J^r(-,\mathbf{R})$-mb-functor H on $\mathcal{C} = \mathcal{M}f_m$ with $H = J^r(-,\mathbf{R})_0$ (as vector bundle functors) is H^v or $H^{[]}$.*

(ii) Let $m \geq 2$ be an integer and $r = 2$. Any $J^2(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^2(-,\mathbf{R})_0$ (as vector bundle functors) is $H^{<\beta>}$ for some $\beta \in \mathbf{R}$. (For $\beta = 0$ we have $H^{<0>} = H^{[]}$. For $\beta = -1$ we have $H^{<-1>} = H^v$.)

(iii) Let $m \geq 2$ be an integer and $r = 1$. Any $J^1(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^1(-,\mathbf{R})_0$ (as vector bundle functors) is $H^{[]}$. (If $r = 1$, we have $H^{[]} = H^v$.)

Proof. Consider a $J^r(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^r(-,\mathbf{R})_0$ (as vector bundle functors) and let

$$\circ : J^r_x(M,\mathbf{R}) \times H_x M \to H_x M\ ,\ x \in M\ ,\ M \in \mathrm{Obj}(\mathcal{M}f_m)$$

be its multiplication. Similarly as in the proof of Proposition 6.4, it is sufficient to study the family of values $j^r_0\gamma \circ j^r_0\eta \in H_0\mathbf{R}^m$ for all $j^r_0\gamma \in J^r_0(\mathbf{R}^m,\mathbf{R})$, $j^r_0\eta \in H_0\mathbf{R}^m$ with $\gamma(0) = 0$. Since $m \geq 2$, then by the rank theorem and the $\mathcal{M}f_m$-invariance of $\circ$, we can assume that

$$j^r_0\gamma = j^r_0 x^1 \text{ and } j^r_0\eta = j^r_0 x^2\ .$$

By the invariance of $\circ$ with respect to $(t^1x^1, ..., t^mx^m)$ for $t^i \neq 0$ we deduce that

$$j_0^r x^1 \circ j_0^r x^2 = \sigma j_0^r (x^1 x^2)$$

for some real number σ. Then by the $\mathcal{M}f_m$-invariance of $\circ$,

$$j_0^r \xi \circ j_0^r \eta = \sigma j_0^r (\xi\eta)$$

for all $j_0^r\xi \in J_0^r(\mathbf{R}^m, \mathbf{R})$, $j_0^r\eta \in H_0\mathbf{R}^m$ with $\xi(0) = 0$. Then

$$\sigma j^r(x^1x^2x^2) = j_0^r(x^1x^2) \circ j_0^r x^2 = j_0^r x^1 \circ (j_0^r x^2 \circ j_0^r x^2) = \sigma^2 j_0^r (x^1x^2x^2)\ .$$

(i) if $r \geq 3$, $\sigma^2 = \sigma$, and then $\sigma = 1$ or $\sigma = 0$. If $\sigma = 0$, then

$$j_0^r \gamma \circ j_0^r \eta = \gamma(0) j_0^r \eta\ ,$$

i.e. $H = H^v$. If $\sigma = 1$, then

$$j_0^r \gamma \circ j_0^r \eta = j_0^r(\gamma\eta)$$

i.e. $H = H^{[]}$.

(ii) If $r = 2$, then

$$j_0^r \gamma \circ j_0^r \eta = (\beta + 1) j_0^2(\gamma\eta) - \beta\gamma(0) j_0^2 \eta\ ,$$

i.e. $H = H^{<\beta>}$, where $\beta = \sigma - 1$.

(iii) If $r = 1$, then

$$j_0^1 \gamma \circ j_0^1 \eta = \gamma(0) j_0^1 \eta = j_0^1(\gamma\eta)\ ,$$

i.e. $H = H^v = H^{[]}$. The proof is complete. □

Lemma 6.18. *Let $C = \mathcal{M}f_m$. If $r \geq 2$, then H^v and $H^{[]}$ are not isomorphic as $J^r(-, \mathbf{R})$-mb-functors.*

Proof. Suppose $c : H^{[]} \to H^v$ is an isomorphism. This leads to the contradiction

$$0 \neq c_0(j_0^r((x^1)^2)) = c_0(j_0^r x^1 \star j_0^r x^1) = j_0^r x^1 \odot c_0(j_0^r x^1) = 0\ .$$

□

Lemma 6.19. *Let $C = \mathcal{M}f_m$. Let $r = 2$. If $\beta \neq \beta_1$, then $H^{<\beta>}$ and $H^{<\beta_1>}$ are not isomorphic as $J^2(-, \mathbf{R})$-mb-functors.*

Proof. Suppose $h : G^{<\beta>} \to G^{<\beta_1>}$ is an isomorphism. By the invariance of h with respect to $(\tau^1x^1, ..., \tau^mx^m)$ we derive that $h_0(j_0^2 x^1) = l j_0^2 x^1$ for some $l \in \mathbf{R}$. Then

$$h_x(j_x^2\gamma) = l j_x^2 \gamma$$

for any m-manifold M, $j_x^2\gamma \in G_x^{<\beta>}M$ and $x \in M$. Then $l \neq 0$ and

$$\begin{aligned} & l(\beta + 1) j_0^2((x^1)^2) = h_0((\beta + 1) j_0^2((x^1)^2)) = h_0(j_0^2 x^1 *^{\beta} j_0^2 x^1) \\ & = j_0^2 x^1 *^{\beta_1} h(j_0^2 x^1) = l j_0^2 x^1 *^{\beta_1} j_0^2 x^1 = l(\beta_1 + 1) j_0^2((x^1)^2)\ . \end{aligned}$$

Then $\beta = \beta_1$ and the proof is complete. □

Summing up, we have proved

Proposition 6.20. *(i) Let $m \geq 2$ and $r \geq 3$. Then any $J^r(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^r(-,\mathbf{R})_0$ (as vector bundle functors) is $H^{[]}$ or H^v. Moreover, $H^{[]}$ and H^v are not isomorphic.*

(ii) Let $m \geq 2$ and $r = 2$. Then any $J^2(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^2(-,\mathbf{R})_0$ (as vector bundle functors) is $H^{<\beta>}$ for some $\beta \in \mathbf{R}$. If $\beta \neq \beta_1$, then $H^{<\beta>}$ and $H^{<\beta_1>}$ are not isomorphic. Moreover, $H^{<0>} = H^{[]}$ and $H^{<-1>} = H^v$.

(iii) Let $m \geq 2$ and $r = 1$. Then any $J^1(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = J^1(-,\mathbf{R})_0$ (as vector bundle functors) is $H^{[]}$. Moreover, $H^{[]} = H^v$.

Proposition 6.20 yields directly

Proposition 6.21. *(i) Let $m \geq 2$ and $r \geq 3$. Then any $J^r(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = (J^r(-,\mathbf{R})_0)^*$ (as vector bundle functors) is $(H^{[]})^*$ or $(H^v)^*$. Moreover, $(H^{[]})^*$ and $(H^v)^*$ are not isomorphic.*

(ii) Let $m \geq 2$ and $r = 2$. Then any $J^2(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = (J^2(-,\mathbf{R})_0)^*$ (as vector bundle functors) is $(H^{<\beta>})^*$ for some $\beta \in \mathbf{R}$. If $\beta \neq \beta_1$, then $(H^{<\beta>})^*$ and $(H^{<\beta_1>})^*$ are not isomorphic. Moreover, $(H^{<0>})^* = (H^{[]})^*$ and $(H^{<-1>})^* = (H^v)^*$.

(iii) Let $m \geq 2$ and $r = 1$. Then any $J^1(-,\mathbf{R})$-mb-functor H on $\mathcal{M}f_m$ with $H = (J^1(-,\mathbf{R})_0)^*$ (as vector bundle functors) is $(H^{[]})^*$. Moreover, $(H^{[]})^* = (H^v)^*$.

7. Classification theorems for jet-like functors of some type on $\mathcal{VB}_m$

Let $G^{[\alpha]}$, G^v, $G^{<\beta>}$, $H^{[]}$, H^v and $H^{<\beta>}$ be vector bundle functors on $\mathcal{C}$ from Example 6.1, 6.2, 6.3, 6.14, 6.15 and 6.16 respectively. By Proposition 6.10 we have

Theorem 7.1. *(i) If $r \geq 3$ and $m \geq 2$, there are (up to natural isomorphism) only three (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order r such that $G^F = J^r(-,\mathbf{R})$ (as vector bundle functors), where G^F is the corresponding (to F) $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{M}f_m$. Namely, the r-jet prolongation functor J^r (if $G^F = G^{[0]}$), the vertical r-jet prolongation functor J^r_v (if $G^F = G^v$) and the so called "exotic" r-jet prolongation functor J^r_{ex} (if $G^F = G^{[-1]}$).*

(ii) If $r = 2$ and $m \geq 2$, there are not-countable many mutually not-isomorphic jet like functors F on $\mathcal{VB}_m$ of order 2 such that $G^F = J^2(-,\mathbf{R})$ (as vector bundle functors). Namely, the 2-jet prolongation functor J^2 (if $G^F = G^{[0]}$), the vertical 2-jet prolongation functor J^2_v (if $G^F = G^v$) and the (so called) "$<\beta>$-exotic" 2-jet prolongation functors $J^2_{\beta-ex}$ (if $G^F = G^{<\beta>}$) for all $\beta \in \mathbf{R} \setminus \{-1\}$, only. More precisely, any such F in question is naturally isomorphic to a exactly one functor of the collection from the previous sentence.

(iii) If $r = 1$ and $m \geq 2$, there are (up to natural isomorphism) only two (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order 1 such that $G^F = J^1(-,\mathbf{R})$ (as vector bundle functors). Namely, the 1-jet prolongation functor J^1 (if $G^F = G^{[0]}$) and the vertical 1-jet prolongation functor J^1_v (if $G^F = G^v$).

We can see that the collection of jet like functors F on $\mathcal{VB}_m$ of order r such that $G^F = J^r(-,\mathbf{R})$ (as vector bundle functors), where G^F is the corresponding (to F) $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{M}f_m$, is very rich: beside the classical jet functors J^r and J^r_v we have also some "exotic" jet functors J^r_{ex} and $J^2_{\beta-ex}$.

Example 7.2. *Let F be a jet like functor on $\mathcal{VB}_C$ of order r. Then we have the so called "dual" F jet like functor*

$$F^{(*)} := F^{G^*}$$

on $\mathcal{VB}_C$ of order r, where $G := G^F$ is the corresponding (to F) $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{C}$ and where G^* is the $J^r(-,\mathbf{R})$-mb-functor dual to G (as in Example 6.11) and where F^{G^*} is the jet like functor corresponding to G^*.

From Proposition 6.13 it follows

Theorem 7.3. *(i) If $r \geq 3$ and $m \geq 2$, there are (up to natural isomorphism) only three (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order r such that $G^F = (J^r(-,\mathbf{R}))^*$ (as vector bundle functors). Namely, the so called "dual" r-jet prolongation functor $(J^r)^{(*)}$ (if $G^F = (G^{[0]})^*$), the so called "dual" vertical r-jet prolongation functor $(J^r_v)^{(*)}$ (if $G^F = (G^v)^*$) and the so called "dual" "exotic" r-jet prolongation functor $(J^r_{ex})^{(*)}$ (if $G^F = (G^{[-1]})^*$).*

(ii) If $r = 2$ and $m \geq 2$, there are not-countable many mutually not-isomorphic jet like functors F on $\mathcal{VB}_m$ of order 2 such that $G^F = (J^2(-,\mathbf{R}))^*$ (as vector bundle functors). Namely, the "dual" 2-jet prolongation functor $(J^2)^{(*)}$ (if $G^F = (G^{[0]})^*$), the "dual" vertical 2-jet prolongation functor $(J^2_v)^{(*)}$ (if $G^F = (G^v)^*$) and the functors $(J^2_{\beta-ex})^{(*)}$ (if $G^F = (G^{<\beta>})^*$) for all $\beta \in \mathbf{R} \setminus \{-1\}$, only. More precisely, any such F in question is naturally isomorphic to a exactly one functor of the collection from the previous sentence.

(iii) If $r = 1$ and $m \geq 2$, there are (up to natural isomorphism) only two (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order 1 such that $G^F = (J^1(-,\mathbf{R}))^*$ (as vector bundle functors). Namely, the "dual" 1-jet prolongation functor $(J^1)^{(*)}$ (if $G^F = (G^{[0]})^*$) and the "dual" vertical 1-jet prolongation functor $(J^1_v)^{(*)}$ (if $G^F = (G^v)^*$).

Example 7.4. *Given a jet like functor F on $\mathcal{VB}_C$ of order r we have the jet like functor F^o on $\mathcal{VB}_C$ of order r such that*

$$F^o_x E = \mathrm{m}_x \cdot F_x E$$

for any $\mathcal{VB}_C$-object E with basis M and any $x \in M$, where $\mathrm{m}_x = J^r_x(M,\mathbf{R})_0$ is the unique maximal ideal in $J^r_x(M,\mathbf{R})$. Clearly, $F^o f : F^o E \to F^o E'$ is (by definition) the restriction of Ff for any $\mathcal{VB}_C$-map $f : E \to E'$.

By Proposition 6.20 we have

Theorem 7.5. *(i) If $r \geq 3$ and $m \geq 2$, there are (up to natural isomorphism) only two (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order r such that $G^F = J^r(-,\mathbf{R})_0$ (as vector bundle functors). Namely, $(J^r)^o$ (if $G^F = H^{[]}$) and $(J^r_v)^o$ (if $G^F = H^v$). (If $r \geq 3$, $(J^r)^o$ is naturally isomorphic to $(J^r_{ex})^o$.)*

(ii) If $r = 2$ and $m \geq 2$, there are not-countable many mutually not-isomorphic jet like functors F on $\mathcal{VB}_m$ of order 2 such that $G^F = J^2(-,\mathbf{R})_0$ (as vector bundle functors). Namely, $(J^2)^o$ (if $G^F = H^{[]} = H^{<0>}$), $(J^2_v)^o$ (if $G^F = H^v = H^{<-1>}$) and the functors $(J^2_{\beta-ex})^o$ for all $\beta \in \mathbf{R} \setminus \{0,-1\}$, only. More precisely, any such F in question is naturally isomorphic to a exactly one functor of the collection from the previous sentence.

(iii) If $r = 1$ and $m \geq 2$, there is (up to natural isomorphism) only one jet like functor F on $\mathcal{VB}_m$ of order 1 such that $G^F = J^1(-,\mathbf{R})_0$ (as vector bundle functors). Namely, $(J^1)^o$ (if $G^F = H^{[]} = H^v$). (If $r = 1$, $(J^1)^o$ is naturally isomorphic to $(J^1_v)^o$.)

Finally, from Proposition 6.21 it follows

Theorem 7.6. *(i) If $r \geq 3$ and $m \geq 2$, there are (up to natural isomorphism) only two (mutually not-isomorphic) jet like functors F on $\mathcal{VB}_m$ of order r such that $G^F = (J^r(-,\mathbf{R})_0)^*$ (as vector bundle functors). Namely, $((J^r)^o)^{(*)}$ (if $G^F = (H^{[]})^*$) and $((J^r_v)^o)^{(*)}$ (if $G^F = (H^v)^*$). (If $r \geq 3$, $((J^r)^o)^{(*)}$ is naturally isomorphic to $((J^r_{ex})^o)^{(*)}$.)*

(ii) If $r = 2$ and $m \geq 2$, there are not-countable many mutually not-isomorphic jet like functors F on $\mathcal{VB}_m$ of order 2 such that $G^F = (J^2(-,\mathbf{R})_0)^*$ (as vector bundle functors). Namely, $((J^2)^o)^{(*)}$ (if $G^F = (H^{[]})^* = (H^{<0>})^*$), $((J^2_v)^o)^{(*)}$ (if $G^F = (H^v)^* = (H^{<-1>})^*$) and the functors $((J^2_{\beta-ex})^o)^{(*)}$ for all $\beta \in \mathbf{R} \setminus \{0,-1\}$, only. More precisely, any such F in question is naturally isomorphic to a exactly one functor of the collection from the previous sentence.

(iii) If $r = 1$ and $m \geq 2$, there is (up to natural isomorphism) only one jet like functor F on $\mathcal{VB}_m$ of order 1 such that $G^F = (J^1(-,\mathbf{R})_0)^*$ (as vector bundle functors). Namely, $((J^1)^o)^{(*)}$ (if $G^F = (H^{[]})^* = (H^v)^*$). (If $r = 1$, $((J^1)^o)^{(*)}$ is naturally isomorphic to $((J^1_v)^o)^{(*)}$.)

8. The $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structures on J^r and J^r_v

In this section we determine all natural $J^r(M,\mathbf{R})$-module bundle structures on vector bundles $J^rE \to M$ and $J^r_vE \to M$, where $E \to M$ is a $\mathcal{VB}_m$-object and $m \geq 2$.

Definition 8.1. *Let F be a gauge vector bundle functor (not necessarily jet like functor) of order r on $\mathcal{VB}_C$. For example $F = J^r$, the r-jet prolongation functor. A $\mathcal{VB}_C$-natural $J^r(-, \mathbf{R})$-module bundle structure on F is a $\mathcal{VB}_C$-invariant system of base preserving fibre $\mathbf{R}$-bilinear maps*

$$\circ : J^r(M, \mathbf{R}) \times_M FE \to FE$$

for all $\mathcal{VB}_C$-objects E with bases M such that F_xE is a $J^r_x(M, \mathbf{R})$-module with $\circ_x : J^r_x(M, \mathbf{R}) \times F_xE \to F_xE$ as the multiplication (and with the sum map being the one in the vector space F_xE) for any $\mathcal{VB}_C$-object E with base M and $x \in M$. The $\mathcal{VB}_C$-invariance of $\circ$ means that $F_xf : F_xE \to F_{\underline{f}(x)}E'$ is a module map over $J^r_x(\underline{f}, \mathrm{id}_{\mathbf{R}})$ for any $\mathcal{VB}_C$-morphism $f : E \to E'$ with base map $\underline{f} : M \to M'$ and $x \in M$.

Definition 8.2. *If F' is an another gauge vector bundle functor of order r on $\mathcal{VB}_C$ and $\circ'$ is a $\mathcal{VB}_C$-natural $J^r(-, \mathbf{R})$-module bundle structure on F', then a morphism $\circ \to \circ'$ is a natural transformation $h : F \to F'$ of gauge bundle functors such that $h_x(\eta \circ_x v) = \eta \circ'_x h_x(v)$ for any $\mathcal{VB}_C$-object E with basis M and any $x \in M$ and any $\eta \in J^r_x(M, \mathbf{R})$ and any $v \in F_xE$.*

From now on, for the simplicity of notation, we will often omit x on o_x.

Example 8.3. *We have the usual $\mathcal{VB}_C$-natural $J^r(-, \mathbf{R})$-module bundle structures $\circ^1$ and $\circ^2$ on J^r defined by*

$$j^r_x\gamma \circ^1 j^r_x\sigma := j^r_x(\gamma\sigma) \text{ and } j^r_x\gamma \circ^2 j^r_x\sigma := \gamma(x) j^r_x\sigma \,,$$

where E is a $\mathcal{VB}_C$-object and $j^r_x\gamma \in J^r_x(M, \mathbf{R})$ and $j^r_x\sigma \in J^r_xE$ and $x \in M$.

Example 8.4. *Let $r = 1$. For any $\alpha \in \mathbf{R}$, we have a $\mathcal{VB}_C$-natural $J^1(-, \mathbf{R})$-module bundle structure $\circ^{(\alpha)}$ on J^1 defined by*

$$j^1_x\gamma \circ^{(\alpha)} j^1_x\sigma := j^1_x(\gamma\sigma) + \alpha j^1_x((\gamma - \gamma(x))\sigma) \,,$$

where E is a $\mathcal{VB}_C$-object with basis M and $j^1_x\gamma \in J^1_x(M, \mathbf{R})$ and $j^1_x\sigma \in J^1_xE$ and $x \in M$. If $\alpha = 0$, we get $\circ^1$ (as above). If $\alpha = -1$, we get $\circ^2$.

Lemma 8.5. *Any $\mathcal{VB}_m$-natural transformation $J^r \to J^r$ is the constant multiple of the identity one.*

Proof. Clearly, $G^{[0]}$ is the $J^r(-, \mathbf{R})$-mb-functor corresponding to J^r. By Lemma 6.5, any morphism of $J^r(-, \mathbf{R})$-mb-functors $G^{[0]} \to G^{[0]}$ is the constant multiple of the identity. Finally we can use Proposition 3.6. □

Lemma 8.6. *Let $C = \mathcal{M}f_m$. The structures $\circ^1$ and $\circ^2$ are not isomorphic.*

Proof. By Lemma 8.5, any natural transformation $h : J^rE \to J^rE$ is $h = k\mathrm{id}_{J^rE}$ for some $k \in \mathbf{R}$. Suppose h is a morphism $\circ^1 \to \circ^2$. Then $h_x(j^r_x\gamma \circ^1 j^r_x\sigma) = j^r_x\gamma \circ^2 h_x(j^r_x\sigma)$. Then $kj^r_x(\gamma\sigma) = k\gamma(x)j^r_x\sigma$. In particular, $kj^r_0(x^1) = 0$ (for $\gamma = x^1 : \mathbf{R}^m \to \mathbf{R}$ and $\sigma = (\mathrm{id}_{\mathbf{R}^m}, 1) : \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}$). Then $k = 0$. So, $\circ^1$ and $\circ^2$ are not isomorphic. □

Lemma 8.7. *Let $C = \mathcal{M}f_m$ and $r = 1$. The structures $\circ^{(\alpha)}$ are mutually not isomorphic.*

Proof. We proceed quite similarly as in the proof of the previous lemma. Suppose $h = k\mathrm{id} : J^1 \to J^1$ is an isomorphism $\circ^{(\alpha)} \to \circ^{(\alpha')}$. Then $k \neq 0$ and $h_x(j^1_x\gamma \circ^{(\alpha)} j^1_x\sigma) = j^r_x\gamma \circ^{(\alpha')} h_x(j^1_x\sigma)$. Then $kj^1_x(\gamma\sigma) + k\alpha j^1_x((\gamma - \gamma(x))\sigma) = kj^1_x(\gamma\sigma) + k\alpha' j^1_x((\gamma - \gamma(x))\sigma)$. In particular, $k\alpha j^1_0(x^1) = k\alpha' j^1_0(x^1)$. Then $\alpha = \alpha'$. □

Theorem 8.8. *(i) If $m \geq 2$ and $r = 1$, then any $\mathcal{VB}_m$-natural $J^1(-, \mathbf{R})$-module bundle structure $\circ$ on J^1 is $\circ^{(\alpha)}$ for a (uniquely determined by $\circ$) number $\alpha \in \mathbf{R}$. The structures $\circ^{(\alpha)}$ for all $\alpha \in \mathbf{R}$ are mutually not isomorphic.*

(ii) If $m \geq 2$ and $r \geq 2$, there exist only two $\mathcal{VB}_m$-natural $J^r(-, \mathbf{R})$-module bundle structures $\circ$ on J^r. Namely, $\circ^1$ and $\circ^2$. The structures $\circ^1$ and $\circ^2$ are not isomorphic.

Proof. That $\circ^1$ and $\circ^2$ are not isomorphic and that the $\circ^{(\alpha)}$ are mutually not isomorphic was observed in the above lemmas.

To prove the rest of the theorem, consider a $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ$ on J^r. Then we have the family of maps

$$\circ : J^r_x(M,\mathbf{R}) \times G_xM \to G_xM \text{ for all } \mathcal{M}f_m\text{-objects } M \text{ and all } x \in M\,,$$

where $GM = J^r(M\times\mathbf{R}) = J^r(M,\mathbf{R})$ and $G\varphi = J^r(\varphi\times \mathrm{id}_{\mathbf{R}}) = J^r(\varphi, \mathrm{id}_{\mathbf{R}})$ for all $\mathcal{M}f_m$-objects M and all $\mathcal{M}f_m$-maps φ. (Here and below we identify $j^r_x\sigma \in J^r_x(M,\mathbf{R})$ with $j^r_x(\mathrm{id}_M,\sigma) \in J^r_x(M\times\mathbf{R})$.)

Using the $\mathcal{VB}_m$-invariance of $\circ$ and J^r is fiber product preserving, one can see that if two $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structures on J^r give the same family of maps $J^r_x(M,\mathbf{R})\times G_xM \to G_xM$ in question, then they are equal. Further, we can see that G is a $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{M}f_m$ with the multiplication defined by

$$\circ : J^r_x(M,\mathbf{R}) \times G_xM \to G_xM \text{ for all } \mathcal{M}f_m\text{-objects } M \text{ and all } x \in M\,.$$

Moreover, we have $G = J^r(-,\mathbf{R})$ (as vector bundle functors).

Let us consider three cases:

(1) Let $r \geq 3$. By Proposition 6.4(i), $G = G^v$ or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$. If $G = G^v$, then $j^r_x\gamma \circ j^r_x\sigma = \gamma(x)j^r_x\sigma$, and then $\circ = \circ^2$. So, we may assume $G = G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$. Then

$$j^r_0x^1 \circ j^r_0\sigma = j^r_0(x^1\sigma) + \alpha\sigma(0)j^r_0x^1$$

for any $\sigma : \mathbf{R}^m \to \mathbf{R}$. If $\sigma = 1$, we get

$$j^r_0x^1 \circ j^r_0 1 = (\alpha+1)j^r_0x^1\,.$$

By the invariance of $\circ$ with respect to $\tilde{\sigma} : M\times\mathbf{R} \to M\times\mathbf{R}$, where $\tilde{\sigma}(x,v) := (x,\sigma(x)v)$, we have

$$j^r_0x^1 \circ j^r_0\sigma = (\alpha+1)j^r_0(x^1\sigma)\,.$$

Then $\alpha\sigma(0)j^r_0x^1 = \alpha j^r_0(x^1\sigma)$. If $\sigma = x^1$, $\alpha j^r_0((x^1)^2) = 0$, i.e. $\alpha = 0$ (if $r \geq 2$). Then $\circ = \circ^1$.

(2) Let $r = 2$. By Proposition 6.4(ii), $G = G^v$ or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$ or $G = G^{<\beta>}$ for some $\beta \in \mathbf{R}\setminus\{-1,0\}$. If $G = G^v$, then $\circ = \circ^2$. If $G = G^{[\alpha]}$, then $\circ = \circ^1$ (by case (1)). So, we may assume $G = G^{<\beta>}$ for some $\beta \in \mathbf{R}\setminus\{-1,0\}$. Then

$$j^2_0x^1 \circ j^2_0\sigma = (\beta+1)j^2_0(x^1\sigma) - (\beta+1)\sigma(0)j^2_0(x^1)$$

for any $\sigma : \mathbf{R}^m \to \mathbf{R}$. If $\sigma = 1$, we get $j^2_0x^1 \circ j^2_0 1 = 0$. Using the invariance of $\circ$ with respect to $\tilde{\sigma}$ (as above) we get $j^2_0x^1 \circ j^2_0\sigma = 0$. Then

$$(\beta+1)j^2_0(x^1\sigma) - (\beta+1)\sigma(0)j^2_0(x^1) = 0\,.$$

If $\sigma = x^1$, $(\beta+1)j^2_0((x^1)^2) = 0$. Then $\beta+1 = 0$. Then $\beta = -1$. Contradiction.

(3) Let $r = 1$. Using Proposition 6.4(iii), we easily complete the part (i) of the theorem. □

Proposition 8.9. *Let $m \geq 2$ and $r \geq 1$. Let $\circ$ be a $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structure on J^r. Then*

$$\mathrm{Aut}_{\mathrm{nat}}(\circ) = \{k\mathrm{id} \mid k \in \mathbf{R}\setminus\{0\}\}\,,$$

where $\mathrm{Aut}_{\mathrm{nat}}(\circ)$ is the group of natural automorphisms of $\circ$.

Proof. By Theorem 8.8, any such $\circ$ is $\circ^1$ or $\circ^2$ or $\circ^{(\alpha)}$ for some $\alpha \in \mathbf{R}$. It is obvious that $k$id (with $k \neq 0$) is a natural automorphism of any such $\circ$. On the other hand, by Lemma 8.5, any natural automorphism (then natural isomorphism of J^r) is $k$id for some $k \in \mathbf{R}\setminus\{0\}$. □

Using Proposition 6.4, we determine all natural $J^r(M,\mathbf{R})$-module bundle structures on the vector bundle $J^r_vE \to M$, where J^r_v is the vertical r-jet prolongation functor.

Example 8.10. *We have the trivial $\mathcal{VB}_C$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ^{[]}$ on J^r_v defined by*

$$j^r_x\gamma \circ^{[]} j^r_x\sigma := \gamma(x) j^r_x\sigma \,,$$

where E is a $\mathcal{VB}_C$-object and $j^r_x\gamma \in J^r_x(M,\mathbf{R})$ and $j^r_x\sigma \in (J^r_v)_x E$ and $x \in M$.

Example 8.11. *For any $\alpha \in \mathbf{R}$, we have a $\mathcal{VB}_C$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ^{[\alpha]}$ on J^r_v defined by*

$$j^r_x\gamma \circ^{[\alpha]} j^r_x\sigma := j^r_x(\gamma\sigma) + \alpha j^r_x((\gamma - \gamma(x))\sigma(x)) \,,$$

where E is a $\mathcal{VB}_C$-object with basis M and $j^r_x\gamma \in J^r_x(M,\mathbf{R})$ and $j^r_x\sigma \in (J^r_v)_x E$ and $x \in M$.

Example 8.12. *Let $r = 2$. For any $\beta \in \mathbf{R}$, we have a $\mathcal{VB}_C$-natural $J^2(-,\mathbf{R})$-module bundle structure $\circ^{<\beta>}$ on J^2_v defined by*

$$j^2_x\gamma \circ^{<\beta>} j^2_x\sigma := j^2_x(\gamma\sigma) + \beta j^2_x((\gamma - \gamma(x))\sigma) - (\beta + 1) j^2_x((\gamma - \gamma(x))\sigma(x)) \,,$$

where E is a $\mathcal{VB}_C$-object with basis M and $j^2_x\gamma \in J^2_x(M,\mathbf{R})$ and $j^2_x\sigma \in (J^2_v)_x E$ and $x \in M$.

Proposition 8.13. *(i) If $m \geq 2$ and $r = 1$, then any $\mathcal{VB}_m$-natural $J^1(-,\mathbf{R})$-module bundle structure $\circ$ on J^1_v is $\circ^{[\alpha]}$ for a (uniquely determined by $\circ$) number $\alpha \in \mathbf{R}$. (For $\alpha = -1$, we get $\circ^{[-1]} = \circ^{[]}$.)*

(ii) If $m \geq 2$ and $r = 2$, then any $\mathcal{VB}_m$-natural $J^2(-,\mathbf{R})$-module bundle structure $\circ$ on J^2_v is $\circ^{[]}$ or $\circ^{[\alpha]}$ for a (uniquely determined by $\circ$) number $\alpha \in \mathbf{R}$ or $\circ^{<\beta>}$ for a (uniquely determined by $\circ$) number $\beta \in \mathbf{R} \setminus \{-1, 0\}$. (If $\beta = -1$, we get $\circ^{<-1>} = \circ^{[]}$. If $\beta = 0$, we get $\circ^{<0>} = \circ^{[-1]}$.)

(iii) If $m \geq 2$ and $r \geq 3$, then any $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ$ on J^r_v is $\circ^{[]}$ or $\circ^{[\alpha]}$ for a (uniquely determined by $\circ$) number $\alpha \in \mathbf{R}$.

Proof. Consider a $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ$ on J^r_v. Similarly as in the proof of Theorem 8.8, it is sufficient to consider the family of maps

$$\circ : J^r_x(M,\mathbf{R}) \times G_x M \to G_x M$$

for all $\mathcal{M}f_m$-objects M and $x \in M$, where $GM = J^r_v(M \times \mathbf{R}) = J^r(M,\mathbf{R})$ and $G\varphi = J^r_v(\varphi \times \mathrm{id}_{\mathbf{R}}) = J^r(\varphi, \mathrm{id}_{\mathbf{R}})$ for all $\mathcal{M}f_m$-objects M and all $\mathcal{M}f_m$-maps φ. (Here and below we identify $j^r_x\sigma \in J^r_x(M,\mathbf{R})$ with $j^r_x(\sigma) \in (J^r_v)_x(M \times \mathbf{R}) = J^r_x(M, \{x\} \times \mathbf{R})$.)

Using the $\mathcal{VB}_m$-invariance of $\circ$ and the fact that J^r_v is fiber product preserving, one can easily see that if two $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structures on J^r_v determine the same family of maps $J^r_x(M,\mathbf{R}) \times G_x M \to G_x M$ in question, then they are equal. Further, we can see that G is a $J^r(-,\mathbf{R})$-mb-functor on $\mathcal{M}f_m$ with the multiplication defined by $\circ : J^r_x(M,\mathbf{R}) \times G_x M \to G_x M$ for all $\mathcal{M}f_m$-objects M and $x \in M$. Moreover, we have $G = J^r(-,\mathbf{R})$ (as vector bundle functors). Let us consider three cases.

(1) Let $r \geq 3$. By Proposition 6.4(i), $G = G^v$ or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$. If $G = G^v$, then $\circ = \circ^{[]}$. If $G = G^{[\alpha]}$, then $\circ = \circ^{[\alpha]}$.

(2) Let $r = 2$. By Proposition 6.4(ii), $G = G^v$ or $G^{[\alpha]}$ for some $\alpha \in \mathbf{R}$ or $G = G^{<\beta>}$ for some $\beta \in \mathbf{R} \setminus \{-1, 0\}$. If $G = G^v$, then $\circ = \circ^{[]}$. If $G = G^{[\alpha]}$, then $\circ = \circ^{[\alpha]}$. If $G = G^{<\beta>}$, then $\circ = \circ^{<\beta>}$.

(3) Let $r = 1$. Using Proposition 6.4(iii), we easily end the proof. □

Of course, some of the structures $\circ^{[]}$, $\circ^{[\alpha]}$, $\circ^{<\beta>}$ can be isomorphic or not isomorphic. We have

Lemma 8.14. *Let C be the category. If $\alpha \neq -1$, then $\circ^{[\alpha]}$ is isomorphic to $\circ^{[0]}$.*

Proof. We have natural transformation $a : J^r_v \to J^r_v$ of jet like functors given by

$$a_x(j^r_x\sigma) = (\alpha + 1) j^r_x\sigma - \alpha j^r_x(\sigma(x)) \,,$$

$j^r_x\sigma \in (J^r_v)_x E$, $x \in M$. If $\alpha \neq -1$, a is a natural isomorphism. One can easily verify directly that $a : \circ^{[0]} \to \circ^{[\alpha]}$ is a morphism of $\mathcal{VB}_C$-natural $J^r(-,\mathbf{R})$-module bundle structures. □

Lemma 8.15. *Let $C = \mathcal{M}f_m$. Any $\mathcal{VB}_m$-natural transformation $J^r_v \to J^r_v$ is of the form*

$$j^r_x\sigma \mapsto k j^r_x\sigma + l j^r_x(\sigma(x))$$

for some $k, l \in \mathbf{R}$.

Proof. The $J^r(-,\mathbf{R})$-mb-functor corresponding to J^r_v is G^v (from Example 6.2). Moreover, any morphism $G^v \to G^v$ of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$ is of the form

$$j^r_x\gamma \mapsto k j^r_x\gamma + l j^r_x(\gamma(x))$$

for some $k, l \in \mathbf{R}$. Indeed, let $c : G^v \to G^v$ be a morphism of $J^r(-,\mathbf{R})$-mb-functors on $\mathcal{M}f_m$. By the invariance of c with respect to the homotheties $(t^1x^1, ..., t^mx^m)$ for $t^1 > 0, ..., t^m > 0$ we derive that

$$c(j^r_0 1) = l' j^r_0 1 \text{ and } c(j^r_0 x^1) = k' j^r_0 x^1$$

for some $k', l' \in \mathbf{R}$. So, by the $\mathcal{M}f_m$-invariance of c and the rank theorem and the fibre linearity of c, one can get that $c(j^r_x\gamma) = k j^r_x\gamma + l j^r_x(\gamma(x))$ for any $\mathcal{M}f_m$-object M and any $x \in M$ and any $j^r_x(\gamma) \in G^v_x M$, where for $k = k'$ and $l = l' - k'$, as well. Now, using Proposition 3.6, we complete the proof of the lemma. □

Lemma 8.16. *Let $C = \mathcal{M}f_m$. Then $\circ^{[0]}$ and $\circ^{[-1]}$ are not isomorphic.*

Proof. Suppose $a : \circ^{[0]} \to \circ^{[-1]}$ is a morphism of the structures on J^r_v. Then a is a $\mathcal{VB}_m$-natural transformation $J^r_v \to J^r_v$. So, by Lemma 8.15, a is of the form

$$a_x(j^r_x\sigma) = k j^r_x\sigma + l j^r_x(\sigma(x)) \,,$$

$j^r_x\sigma \in (J^r_v)_x E = J^r_x(M, E_x)$, $x \in M$, where $k, l \in \mathbf{R}$. Next, since a is a morphism of the structures, we have

$$a_x(j^r_x\gamma \circ^{[0]} j^r_x\sigma) = j^r_x\gamma \circ^{[-1]} a_x(j^r_x\sigma) \,.$$

Then

$$k j^r_x(\gamma\sigma) + l j^r_x(\gamma(x)\sigma(x)) = k j^r_x(\gamma\sigma) - k j^r_x((\gamma - \gamma(x))\sigma(x)) + l j^r_x(\gamma\sigma(x)) - l j^r_x((\gamma - \gamma(x))\sigma(x)).$$

Then $k j^r_x((\gamma - \gamma(x))\sigma(x)) = 0$, which yields $k = 0$. But if $k = 0$, then a is not invertible. □

Quite similarly we prove Lemmas 8.17, 8.18 and 8.19 below.

Lemma 8.17. *Let $C = \mathcal{M}f_m$. Then $\circ^{[0]}$ and $\circ^{[]}$ are not isomorphic.*

Lemma 8.18. *Let $C = \mathcal{M}f_m$ and $r \geq 2$. Then $\circ^{[-1]}$ and $\circ^{[]}$ are not isomorphic.*

Lemma 8.19. *Let $C = \mathcal{M}f_m$ and $r = 2$. Then we have:*
(a) $\circ^{<\beta>}$ for all $\beta \in \mathbf{R} \setminus \{-1, 0\}$ are mutually not isomorphic,
(b) $\circ^{<\beta>}$ is not isomorphic with $\circ^{[0]}$ for all $\beta \in \mathbf{R} \setminus \{-1, 0\}$,
(c) $\circ^{<\beta>}$ is not isomorphic with $\circ^{[-1]}$ for all $\beta \in \mathbf{R} \setminus \{-1, 0\}$,
(d) $\circ^{<\beta>}$ is not isomorphic with $\circ^{[]}$ for all $\beta \in \mathbf{R} \setminus \{-1, 0\}$.

Summing up we have

Theorem 8.20. *(i) If $m \geq 2$ and $r = 1$, then any $\mathcal{VB}_m$-natural $J^1(-,\mathbf{R})$-module bundle structure $\circ$ on J^1_v is isomorphic to $\circ^{[]}$ or $\circ^{[0]}$. The structures $\circ^{[]}$ and $\circ^{[0]}$ are not isomorphic.*

(ii) If $m \geq 2$ and $r = 2$, then any $\mathcal{VB}_m$-natural $J^2(-,\mathbf{R})$-module bundle structure $\circ$ on J^2_v is isomorphic to $\circ^{[]}$ or $\circ^{[0]}$ or $\circ^{[-1]}$ or $\circ^{<\beta>}$ for a (uniquely determined by $\circ$) number $\beta \in \mathbf{R} \setminus \{-1, 0\}$. The structures $\circ^{[]}$ and $\circ^{[0]}$ and $\circ^{[-1]}$ and $\circ^{<\beta>}$ for all $\beta \in \mathbf{R} \setminus \{0, -1\}$ are mutually not isomorphic.

(iii) If $m \geq 2$ and $r \geq 3$, then any $\mathcal{VB}_m$-natural $J^r(-,\mathbf{R})$-module bundle structure $\circ$ on J^r_v is isomorphic to $\circ^{[]}$ or $\circ^{[0]}$ or $\circ^{[-1]}$. The structures $\circ^{[]}$ and $\circ^{[0]}$ and $\circ^{[-1]}$ are mutually not isomorphic.

Proposition 8.21. *Let $m \geq 2$ and $r \geq 1$. Let $\circ$ be a $\mathcal{VB}_m$-natural $J^r(-, \mathbf{R})$-module bundle structure on J^r_v. Then all possibilities of $\circ$ are:*

(a) If $\circ = \circ^{[\alpha]}$ with $\alpha \neq -1$, then

$$\mathrm{Aut}_{\mathrm{nat}}(\circ) = \{k\mathrm{id} \mid k \in \mathbf{R} \setminus \{0\}\},$$

(b) if $r = 2$ and $\circ = \circ^{<\beta>}$ with $\beta \in \mathbf{R} \setminus \{-1, 0\}$, then

$$\mathrm{Aut}_{\mathrm{nat}}(\circ) = \{k\mathrm{id} + l\mathrm{O} \mid k \in \mathbf{R} \setminus \{0\},\ l \in \mathbf{R}\},$$

(c) if $r \geq 2$ and $\circ = \circ^{[-1]}$ then

$$\mathrm{Aut}_{\mathrm{nat}}(\circ) = \{k\mathrm{id} + l\mathrm{O} \mid k \in \mathbf{R} \setminus \{0\},\ l \in \mathbf{R}\},$$

(d) if $\circ = o^{[]}$, then

$$\mathrm{Aut}_{\mathrm{nat}}(\circ) = \{k\mathrm{id} + l\mathrm{O} \mid k \in \mathbf{R} \setminus \{0\}, l \in \mathbf{R}\}\ ,$$

where $\mathrm{O} : J^r_v \to J^r_v$, $\mathrm{O}(j^r_x\sigma) = j^r_x(\sigma(x))$, $j^r_x\sigma \in (J^r_v)_xE$, $x \in M$.

Proof. In the text of the proposition, there are considered all possibilities of $\circ$ because of Proposition 8.13. Further, any natural transformation $\nu : J^r_v \to J^r_v$ is $\nu = k\mathrm{id} + l\mathrm{O}$ for some (arbitrary) $k, l \in \mathbf{R}$ (because of Lemma 8.15). It is a natural isomorphism if $k \neq 0$. Else, for such ν it must be

$$\nu(j^r_x\gamma \circ j^r_x\sigma) = j^r_x\gamma \circ \nu(j^r_x\sigma)$$

for any $j^r_x\gamma \in J^r_x(M, \mathbf{R})$ and any $j^r_x\sigma \in (J^r_v)_xE$ and any $\mathcal{VB}_m$ object $E \to M$ and any $x \in M$. Then, putting $E = \mathbf{R}^m \times \mathbf{R}^n$, $M = \mathbf{R}^m$, $x = 0 \in \mathbf{R}^m$ and $\gamma = x^1 : \mathbf{R}^m \to \mathbf{R}$, we obtain (in any presented case of $\circ$) the respective conditions on k and l. Next one can directly verify (in any case) that such obtained ν is a natural automorphism of $\circ$. We left the details for the reader. □

References

[1] M. Doupovec, I. Kolář, *Iteration of fiber product preserving bundle functors*, Monatshefte Math. **134** (2001), 39-50.
[2] M. Doupovec, J. Kurek, W. M. Mikulski, *On jet like bundles of vector bundles*, Math. Vesnik **73(4)** (2021), 244-255.
[3] M. Doupovec, J. Kurek, W. M. Mikulski, *Vertical functors on fibered manifolds*, Differ. Geom. Appl. **54** (2017), 100-110.
[4] C. Ehresmann, *Extension du calcul des jets non holonomes*, C. R. A. S. Paris **239** (1954), 1762–1764.
[5] I. Kolář, *Weil bundles as generalized jet spaces*, In: Handbook of Global Analysis. Amsterdam, Elsevier, 2008, pp. 625-664.
[6] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer Verlag, Berlin 1993.
[7] I. Kolář, W. M. Mikulski, *On the fiber product preserving bundle functors*, Differ. Geom. Appl. **11** (1999), 105-115.
[8] I. Kolář, W. M. Mikulski, *Fiber product preserving bundle functors on all morphisms of fibered manifolds*, Archivum Mathematicum (Brno) **42(3)** (2006), 265-271.
[9] J. Kurek, W. M. Mikulski, *Fiber product preserving bundle functors of vertical type*, Differ. Geom. Appl. **35(S1)** (2014), 150-155.
[10] J. Kurek, W. M. Mikulski, *Higher order jet prolongation type gauge natural bundles over vector bundles*, Ann. Acad. Paed. Cracoviensis **23(IV)** (2004), 111-122.
[11] P. Libermann, *Introduction to the theory of semi-holonomic jets*, Archivum Mathematicum (Brno) **33** (1997), 173–189.
[12] W. M. Mikulski, *On the fiber product preserving gauge bundle functors on vector bundles*, Annales Polonici Mathematici **82** (2003), 251-264.
[13] W. M. Mikulski, *Fiber product preserving bundle functors as modified vertical Weil functors*, Czechoslovak Math. J. **65(2)** (2015), 517-528.
[14] W. M. Mikulski, *Fiber product preserving bundle functors on fibered-fibered manifolds*, Turkish J. Math. **39(5)** (2015), 719-732.