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On the structure of some nonlinear maps in prime $*$-rings

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Abstract. The objective of this paper is to introduce the notion of skew Lie centralizers in $*$-rings, and to investigate the structure of skew Lie centralizers and strong skew commutativity preserving maps in prime $*$-rings without assuming the existence of a symmetric idempotent and the unital element. As an application, we shall characterize such maps in different operator algebras.

1. Introduction

Throughout this paper, unless otherwise stated, $\mathcal{A}$ represents a prime ring with centre $\mathcal{Z}(\mathcal{A})$. A ring $\mathcal{A}$ is called prime if for any $x, y \in \mathcal{A}$, whenever $x\mathcal{A}y = 0$ implies that either $x = 0$ or $y = 0$. A ring $\mathcal{A}$ is said to be n-torsion free if for any $x \in \mathcal{A}$, whenever $nx = 0$ implies $x = 0$. The maximal left ring of quotients of $\mathcal{A}$ is denoted by $Q_{ml}(\mathcal{A})$ and the maximal symmetric ring of quotients of $\mathcal{A}$ is denoted by $Q_{ms}(\mathcal{A})$. It is well known that $\mathcal{A} \subseteq Q_{ms}(\mathcal{A}) \subseteq Q_{ml}(\mathcal{A})$. The super rings $Q_{ms}(\mathcal{A})$ and $Q_{ml}(\mathcal{A})$ are also prime and they both share the same centre C, known as the extended centroid of $\mathcal{A}$. Moreover $C = \{\lambda \in Q_{ml}(\mathcal{A}) \mid \lambda a = a\lambda \text{ for all } a \in \mathcal{A}\}$ and $\mathcal{A}$ is prime if and only if C is field. For details one may refer to [3]. An involution '$*$' on $\mathcal{A}$ is an anti-automorphism of order 1 or 2. For $a, b \in \mathcal{A}$, the Lie product $ab - ba$ is denoted by $[a, b]$ and the skew Lie product $ab - ba^*$ by $[a, b]_*$.

It is well known that any anti-automorphism of $\mathcal{A}$ can be uniquely extended to an anti-automorphism of $Q_{ms}(\mathcal{A})$ and hence can also be viewed as an anti-automorphism of C. The anti-automorphism τ of $\mathcal{A}$ is said to be of the first kind if it acts as the identity map on C and of the second kind otherwise. For $x \in \mathcal{A}$, we write $\deg(x) = n$ if x is algebraic of minimal degree n over C and $\deg(x) = \infty$ otherwise. For a nonempty subset $\mathcal{M}$ of $\mathcal{A}$, $\deg(\mathcal{M}) = \sup\{deg(y) \mid y \in \mathcal{M}\}$.

An additive mapping $\psi : \mathcal{A} \to \mathcal{A}$ is called a Lie centralizer if $\psi([a, b]) = [\psi(a), b] = [a, \psi(b)]$ for all $a, b \in \mathcal{A}$. It can be easily seen that ψ is a Lie centralizer on $\mathcal{A}$ if and only if $\psi([a, b]) = [\psi(a), b]$ for all $a, b \in \mathcal{A}$ or $\psi([a, b]) = [a, \psi(b)]$ for any $a, b \in \mathcal{A}$. Lie centralizers on rings as well as algebras have been extensively investigated by many mathematicians (see [1, 8, 9, 12, 13, 18, 19] and references therein). Motivated by the concept of Lie centralizers on rings, we here introduce the definition of skew Lie centralizers as follows.

Definition 1.1. *Let $\mathcal{A}$ be a ring with an involution '$*$', and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map. Then Φ is called a skew Lie centralizer of $\mathcal{A}$ if*

$$\Phi([a, b]_*) = [\Phi(a), b]_* = [a, \Phi(b)]_*$$

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stands true for all $a, b \in \mathcal{A}$.

We remark that the conditions $\Phi([a,b]_*) = [\Phi(a), b]_*$ for all $a, b \in \mathcal{A}$ and $\Phi([a,b]_*) = [a, \Phi(b)]_*$ for all $a, b \in \mathcal{A}$ may not be equivalent. For the map $\Phi : \mathcal{A} \to \mathcal{A}$ given by $\Phi(a) = \zeta a$, where $\zeta^* \neq \zeta \in C$ satisfies $\Phi([a,b]_*) = [a, \Phi(b)]_*$ for all $a, b \in \mathcal{A}$ but does not satisfy $\Phi([a,b]_*) = [\Phi(a), b]_*$ for all $a, b \in \mathcal{A}$.

In the numerous recent papers, maps preserving skew Lie products or acting as derivations on skew Lie products have been extensively studied by various authors in the context of rings and operator algebras. We refer the readers to some recent papers [1, 2, 5, 14, 15, 22, 23, 25, 26] where further references can be found. Motivated by the above cited works, we will completely characterize skew Lie centralizers on prime rings (see Corollary 2.4).

If $\mathcal{A}$ is a $*$-ring, then a map $\Psi : \mathcal{A} \to \mathcal{A}$ is called $*$-linear if $\Psi(a^*) = \Psi(a)^*$ for all $a \in \mathcal{A}$. Moreover, Ψ is called a strong skew commutativity preserving map if $[\Psi(a), \Psi(b)]_* = [a, b]_*$ holds for all $a, b \in \mathcal{A}$.

We will also characterize strong skew commutativity preserving maps on prime rings without assuming the existence of a symmetric idempotent and the unity. The first characterization of such maps was obtained by Cui et al. [6] on factor von Neumann algebras. They proved that if ϕ is a nonlinear surjective strong skew commutativity preserving map on factor von Neumann algebra $\mathcal{N}$, then $\phi(a) = \chi(a) + f(a)I$ for all $a \in \mathcal{N}$, where $\chi : \mathcal{N} \to \mathcal{N}$ is a linear bijective map satisfying $[\chi(a), \chi(b)]_* = [a, b]_*$ for all $a, b \in \mathcal{N}$, f is a real functional on $\mathcal{N}$ with $f(0) = 0$ and I is the identity of $\mathcal{N}$. Qi et al. [21, Theorem 2.1] extended this result to unital prime ring $\mathcal{A}$ with involution containing a nontrivial symmetric idempotent and obtained that $\phi(a) = \lambda a + f(a)$ for all $a \in \mathcal{A}$, where $\lambda \in \{1, -1\}$ and f is a map from $\mathcal{A}$ to $\mathcal{Z}_{\mathcal{S}}(\mathcal{A}) = \{z \in \mathcal{Z}(\mathcal{A}) : z^* = z\}$. In [17] and [20] the authors generalized this result to unital $*$-ring $\mathcal{A}$ which contain a nontrivial symmetric idempotent e satisfying $x\mathcal{A}e = \{0\} \implies x = 0$ and $x\mathcal{A}(1-e) = \{0\} \implies x = 0$. Moreover, they obtained the same conclusion. Recently Hou and Wang [11, Theorem 2.1] improved this result by proving that the symmetric centre valued map f vanishes. All these characterizations of strong skew commutativity preserving maps were obtained with the help of Pierce decomposition and hence the symmetric idempotent and unity play an inevitable role. We shall completely characterize the surjective strong skew commutativity preserving maps on prime ring $\mathcal{A}$ by removing the assumptions $1 \in \mathcal{A}$ and the existence of nontrivial symmetric idempotent in $\mathcal{A}$, but getting the same conclusion (see Theorem 2.2).

2. Results

Before stating our results, it is worthwhile to mention the structure of involutions on matrix algebras. Let $\mathcal{M}_p(\mathbb{K})$ be the $p \times p$ matrix algebra over an algebraically closed field $\mathbb{K}$ with involution '$*$' of the first kind. It is known that in this case '$*$' is either the ordinary transpose or the symplectic involution (see [3, Theorem 4.6.12 and Corollary 4.6.13] and [10] for details). In case '$*$' is the symplectic involution, $p = 2q$ for some positive integer q and is given by: For $(M_{ij}) \in \mathcal{M}_p(\mathbb{K}) = \mathcal{M}_q(\mathcal{M}_2(\mathbb{K}))$, where $M_{ij} \in \mathcal{M}_2(\mathbb{K})$, we have $(M_{ij})^* = (N_{ij})$, where $N_{ij} = M_{ji}^{\sigma}$ and where

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}^{\sigma} = \begin{bmatrix} \alpha_4 & -\alpha_2 \\ -\alpha_3 & \alpha_1 \end{bmatrix}$$

for $\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \in \mathcal{M}_2(\mathbb{K})$.

We begin with the following result which plays a key role in the proof of our main results.

Lemma 2.1. *Let $\mathcal{A}$ be a prime ring with a non identity anti-automorphism τ. If $u \in Q_{ml}(\mathcal{A})$ is such that $ua^{\tau} = au$ for every $a \in \mathcal{A}$, then $u = 0$.*

Proof. If $\mathcal{A}$ is commutative, then it is easy to see that $u = 0$. So assume that $\mathcal{A}$ is non commutative. Now for every $a, b \in \mathcal{A}$, we have $abu = u(ab)^{\tau} = ub^{\tau}a^{\tau} = bua^{\tau} = bau$. Therefore $[a, b]u = 0$ for every $a, b \in \mathcal{A}$, forcing $u = 0$. $\square$

The following result gives a complete characterization of surjective strong skew commutativity preserving maps in prime rings without assuming the existence of a symmetric idempotent and the unital element.

Theorem 2.2. *Let $\mathcal{A}$ be a prime ring with a non identity involution '$*$'. Assume that $\Psi : \mathcal{A} \to \mathcal{A}$ is a surjective strong skew commutativity preserving map. Then $\Psi(a) = \lambda a$ for all $a \in \mathcal{A}$, where $\lambda \in \{1, -1\}$.*

Proof. First we prove some facts about Ψ.
Fact 1. Ψ is additive. Let $a, b, c \in \mathcal{A}$. Then we have

$$\begin{aligned}[\Psi(a), \Psi(b+c) - \Psi(b) - \Psi(c)]_* &= [\Psi(a), \Psi(b+c)]_* - [\Psi(a), \Psi(b)]_* - [\Psi(a), \Psi(c)]_* \\ &= [a, b+c]_* - [a,b]_* - [a,c]_* \\ &= 0.\end{aligned}$$

Therefore by surjectiveness of Ψ, we have $[a, \Psi(b+c) - \Psi(b) - \Psi(c)]_* = 0$ for all $a, b, c \in \mathcal{A}$. Invoking Lemma 2.1, we conclude that $\Psi(b+c) = \Psi(b) + \Psi(c)$ for all $b, c \in \mathcal{A}$. Thus Ψ is additive, as asserted.
Fact 2. Ψ is an additive isomorphism.
By the given hypothesis and Fact 1, Ψ is an additive epimorphism. Suppose that $\Psi(x) = 0$ for some $x \in \mathcal{A}$. Then $ax - xa^* = \Psi(a)\Psi(x) - \Psi(x)\Psi(a)^* = 0$ for all $a \in \mathcal{A}$. Hence by Lemma 2.1, we conclude that $x = 0$. Thus Ψ is an additive isomorphism, as asserted.
Fact 3. Ψ is $*$-linear.
For all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}[\Psi(a), \Psi(b)^*]_* &= -([\Psi(a), \Psi(b)]_*)^* \\ &= -([a,b]_*)^* \\ &= [a, b^*]_* \\ &= [\Psi(a), \Psi(b^*)]_*.\end{aligned}$$

Thus, we have

$$[\Psi(a), \Psi(b)^* - \Psi(b^*)]_* = 0$$

for all $a, b \in \mathcal{A}$. Now by Fact 2, Ψ is bijective. Hence applying Lemma 2.1, we infer that $\Psi(a)^* = \Psi(a^*)$ stands true for all $a \in \mathcal{A}$. Therefore Ψ is $*$-linear, as asserted and hence Ψ satisfies the relation

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a^*) = ab - ba^*, \tag{1}$$

for all $a, b \in \mathcal{A}$. Now we proceed by considering the following cases.
Case I. $\dim_C \mathcal{A}C > 4$.
By Fact 2, Ψ is bijective, therefore from (1) the functional identity

$$a\Psi(b) + b\Psi^{-1}(a^*) - \Psi(b)a^* - \Psi^{-1}(a)b = 0,$$

holds for all $a, b \in \mathcal{A}$. By [4, Theorem C.2], $\deg(\mathcal{A}) > 2$. Therefore by [4, Corollary 5.12], $\mathcal{A}$ is 3-free subring of $Q_{ml}(\mathcal{A})$ and hence by [4, Theorem 3.25], $\mathcal{A}$ is also $(*, 2)$-free subring of $Q_{ml}(\mathcal{A})$. Therefore there exists $q \in Q_{ml}(\mathcal{A})$ such that $\Psi(b) = bq$ for all $b \in \mathcal{A}$. Utilizing this in (1), we have

$$a(qbq - b) + b(a^* - qa^*q) = 0,$$

for all $a, b \in \mathcal{A}$. Again from $(*, 2)$-freeness of $\mathcal{A}$, we infer that $qbq = b$ for all $b \in \mathcal{A}$. Now by [3, Theorem 6.4.1], $\mathcal{A}$ and $Q_{ml}(\mathcal{A})$ satisfy the same GPIs. So setting $b = 1$ in the last relation, we find that $q^2 = 1$ and hence $bq = qb$ for all $b \in \mathcal{A}$. Thus $q \in C$, which yields $q = 1$ or $q = -1$. Therefore, either $\Psi(a) = a$ for all $a \in \mathcal{A}$ or $\Psi(a) = -a$ for all $a \in \mathcal{A}$.
Case II. $\dim_C \mathcal{A}C \leq 4$.
In this case $\mathcal{A}$ is a PI-ring and hence by [24, Theorem 2], $\mathcal{Z}(\mathcal{A}) \neq \{0\}$.
Subcase (i). $\alpha^* \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$.
Setting $a = \alpha$ in (1), we find that $(\alpha - \alpha^*)\Psi(b) = \Psi^{-1}(\alpha)b - b\Psi^{-1}(\alpha^*)$ for all $b \in \mathcal{A}$. Therefore $\Psi(b) = pb + bp^*$ for all $b \in \mathcal{A}$, where $p = \frac{\Psi^{-1}(\alpha)}{\alpha - \alpha^*} \in \mathcal{A}C$. Using this in (1), we have

$$(pa + ap^*)(pb + bp^*) - (pb + bp^*)(pa^* + a^*p^*) = ab - ba^*, \tag{2}$$

for all $a, b \in \mathcal{A}$. Replacing a by αa in the last relation, we have

$$\alpha(pa + ap^*)(pb + bp^*) - \alpha^*(pb + bp^*)(pa^* + a^*p^*) = \alpha ab - \alpha^* ba^*, \tag{3}$$

for all $a, b \in \mathcal{A}$. Multiplying (2) by α^* and then subtracting from (3), we get

$$(pa + ap^*)(pb + bp^*) = ab, \tag{4}$$

for all $a, b \in \mathcal{A}$. Now by [3, Theorem 6.4.1], $\mathcal{A}$ and $\mathcal{A}C$ satisfy the same GPIs. Hence (4) holds for all $a, b \in \mathcal{A}C$. Therefore putting $a = b = 1$ in the last expression, we see that $(p + p^*)^2 = 1$. Also on right multiplication in (4) by $p + p^*$ and putting $b = 1$, we find that $p \in C$. Now C is a field so either $p + p^* = 1$ or $p + p^* = -1$. Therefore $\Psi(a) = \lambda a$ for all $a \in \mathcal{A}$, where $\lambda \in \{-1, 1\}$.
Subcase (ii). $\alpha^* = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$.

Since '$*$' is a non identity involution, we must have $\dim_C \mathcal{A}C = 4$. Let $\mathbb{F}$ be the algebraic closure of C. Then '$*$' can be extended uniquely to an involution on $\mathcal{A}C \otimes_C \mathbb{F}$, denoted by '$*$' also and is given by

$$\Big(\sum_i a_i \otimes \beta_i\Big)^* = \sum_i a_i^* \otimes \beta_i$$

for $a_i \in \mathcal{A}C$ and $\beta_i \in \mathbb{F}$. Now let $\alpha \in \mathcal{Z}(\mathcal{A})$. Then for $a, b \in \mathcal{A}$, we have

$$[\Psi(a), \Psi(\alpha a) - \alpha\Psi(a)]_* = [\Psi(a), \Psi(\alpha a)]_* - [\Psi(a), \alpha\Psi(a)]_* = 0.$$

Invoking Lemma 2.1, we infer that $\Psi(\alpha a) = \alpha\Psi(a)$ for all $a \in \mathcal{A}$ and $\alpha \in \mathcal{Z}(\mathcal{A})$, that is, Ψ is $\mathcal{Z}(\mathcal{A})$-linear. Now it is well known that if $\mathcal{A}$ is a prime PI-ring, then $\mathcal{A}C = Q_{ml}(\mathcal{A})$, $\mathcal{Z}(\mathcal{A}) \neq \{0\}$ and any element in $\mathcal{A}C$ is of the form $\dfrac{a}{\alpha}$, for some $a \in \mathcal{A}$ and some nonzero $\alpha \in \mathcal{Z}(\mathcal{A})$ (see 24, Corollary 1). Hence, Ψ can also be uniquely extended to a map $\Psi_1 : \mathcal{A}C \to \mathcal{A}C$, by defining $\Psi_1(\frac{a}{\alpha}) = \frac{\Psi(a)}{\alpha}$. A simple computation shows that Ψ_1 is a surjective strong skew commutativity preserving map on $\mathcal{A}C$. Now extend Ψ_1 to $\mathcal{A}C \otimes_C \mathbb{F}$, denoted by Ψ_2, by the rule

$$\Psi_2\Big(\sum_i a_i \otimes \alpha_i\Big) = \sum_i \Psi_1(a_i) \otimes \alpha_i.$$

for $a_i \in \mathcal{A}C$ and $\alpha \in \mathbb{F}$. Then, it can be easily verified that Ψ_2 is a $\mathbb{F}$-linear strong skew commutativity preserving map. Moreover, Ψ_2 is surjective. By using the same arguments as in the begining it can be seen that Ψ_2 is also a $*$-linear additive isomorphism and hence

$$\Psi_2(X)\Psi_2(Y) - \Psi_2(Y)\Psi_2(X^*) = XY - YX^*, \tag{5}$$

for all $X, Y \in \mathcal{A}C \otimes_C \mathbb{F}$. Now it is well known that $\mathcal{A}C \otimes_C \mathbb{F} \cong \mathcal{M}_k(\mathbb{F})$, where $k = \sqrt{\dim_C \mathcal{A}C} > 1$. Therefore, we can visualize Ψ as a map on $\mathcal{M}_2(\mathbb{F})$ to itself satisfying (5). Moreover, it can also be easily seen from (5) that $\Psi_2(\mathbb{F}) \subseteq \mathbb{F}$.

Now if '$*$' is the transpose involution, then obviously $\mathcal{M}_2(\mathbb{F})$ contains a non trivial rank 1 symmetric idempotent and hence by [11, Theorem 2.1], either $\Psi_2(X) = X$ for all $X \in \mathcal{M}_2(\mathbb{F})$ or $\Psi_2(X) = -X$ for all $X \in \mathcal{M}_2(\mathbb{F})$. Next assume that '$*$' is the symplectic involution. First we deal with the case when $\mathrm{char}(\mathbb{F}) = 2$. Setting $Y = I$ in (5), we have

$$\Psi_2(I)\Psi_2(X + X^*) = X + X^*$$

for all $X \in \mathcal{R}$. Let e_{ij} denote the matrix unit with 1 as $(i, j)^{th}$ entry and $0's$ elsewise. Then putting $X = e_{11}$ in the previous relation, we get $(\Psi_2(I))^2 = I$. Consequently, $\Psi_2(I) = I$.

Now, $\Psi_2(e_{11})^* = \Psi_2(e_{22})$. Hence we can write

$$\Psi_2(e_{11}) = \begin{pmatrix} u & v \\ w & k \end{pmatrix} \qquad \text{and} \quad \Psi_2(e_{22}) = \begin{pmatrix} k & v \\ w & u \end{pmatrix}.$$

Thus $I = \Psi_2(I) = \Psi_2(e_{11} + e_{22})$ yields $u + k = 1$ and hence

$$\Psi(e_{11}) = \begin{pmatrix} u & v \\ w & 1-u \end{pmatrix} \quad \text{and} \quad \Psi(e_{22}) = \begin{pmatrix} 1-u & v \\ w & u \end{pmatrix}.$$

Now taking $X = e_{22}$ and $Y = e_{11}$ in (5), we see that $u = 1$ and $v = w = 0$. Therefore, $\Psi_2(e_{11}) = e_{11}$ and $\Psi_2(e_{22}) = e_{22}$.

We also have, $\Psi_2(e_{12})^* = \Psi_2(e_{12})$. Therefore, we can write

$$\Psi(e_{12}) = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

Setting $X = e_{22}$ and $Y = e_{12}$ in (5), we find that $x = 0$. Also putting $X = e_{12}$ and $Y = e_{11}$ in (5), provide us $y = 1$ and $z = 0$. Thus $\Psi_2(e_{12}) = e_{12}$. Similarly, we can obtain $\Psi_2(e_{21}) = e_{21}$. Consequently, $\Psi_2(X) = X$ for all $X \in \mathcal{M}_2(\mathbb{F})$.

Next assume that char($\mathbb{F}$) $\neq 2$. Setting $Y = \alpha \in \mathbb{F}$ in (5), we infer that

$$\Psi_2(X - X^*) = \gamma(X - X^*)$$

for all $X \in \mathcal{M}_2(\mathbb{F})$, where $\gamma = \alpha(\Psi_2(\alpha))^{-1} \in \mathbb{F}$. Now let $\mathcal{K}_2(\mathbb{F}) = \{X \in \mathcal{M}_2(\mathbb{F}) : X^* = -X\}$. Then from the above relation, we see that $\Psi_2(k) = \gamma k$ for all $k \in \mathcal{K}_2(\mathbb{F})$. Thus from (5), we have $(\gamma^2 - 1)(k_1k_2 + k_2k_1) = 0$ for all $k_1, k_2 \in \mathcal{K}_2(\mathbb{F})$. Therefore, we have either $\gamma^2 = 1$ or $k_1k_2 + k_2k_1 = 0$ for all $k_1, k_2 \in \mathcal{K}_2(\mathbb{F})$. If the latter case prevails, then taking

$$k_1 = k_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we get the contradiction. Therefore, $\gamma \in \{1, -1\}$.

Now let $X = k_1 + s_1$ and $Y = k_2 + s_2$, where $k_1, k_2 \in \mathcal{K}_2(\mathbb{F})$ and $s_1, s_2 \in \mathbb{F}$. Then $\Psi_2(X) = \gamma k_1 + \Psi_2(s_1)$ and $\Psi_2(Y) = \gamma k_2 + \Psi_2(s_2)$ and hence from (5), we have

$$\begin{aligned}(\gamma k_1 + \Psi_2(s_1))(\gamma k_2 + \Psi_2(s_2)) \quad & -(\gamma k_2 + \Psi_2(s_2))(-\gamma k_1 + \Psi_2(s_1)) \\ & = (s_1 + k_1)(s_2 + k_2) - (s_2 + k_2)(s_1 - k_1).\end{aligned}$$

Thus $(\gamma\Psi_2(s_2) - s_2)k_1 = 0$. Hence we conclude that $\Psi_2(s) = \gamma s$ for all $s \in \mathbb{F}$. Now for any $X \in \mathcal{M}_2(\mathbb{F})$, we have $X = k + s$, where $k \in \mathcal{K}_2(\mathbb{F})$ and $s \in \mathbb{F}$. Hence $\Psi_2(X) = \Psi_2(s + k) = \gamma s + \gamma k = \gamma X$ for all $X \in \mathcal{M}_2(\mathbb{F})$.

Therefore, we have $\Psi_2(X) = \gamma X$ for all $X \in \mathcal{A}C \otimes \mathbb{F}$, where $\gamma \in \{1, -1\}$. Now it can be easily deduced that $\Psi_1(a) = \beta a$ for all $a \in \mathcal{A}C$ and for some $\beta \in \{-1, 1\}$. Consequently, $\Psi(a) = \lambda a$ for all $a \in \mathcal{A}$, where $\lambda \in \{-1, 1\}$. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{A}$ be a prime ring with a non identity involution '$*$' and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map satisfying*

$$[\Phi(a), b]_* = [a, \Phi(b)]_* \tag{6}$$

for all $a, b \in \mathcal{A}$. Then there exists $\lambda^* = \lambda \in C$ such that $\Phi(a) = \lambda a$ for all $a \in \mathcal{A}$.

Proof. First we prove Φ is additive. For every $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}[\Phi(a + c) - \Phi(a) - \Phi(c), b]_* &= [\Phi(a + c), b]_* - [\Phi(a), b]_* - [\Phi(c), b]_* \\ &= [a + c, \Phi(b)]_* - [a, \Phi(b)]_* - [c, \Phi(b)]_* \\ &= 0.\end{aligned}$$

Applying Lemma 2.1, we infer that $\Phi(a + b) = \Phi(a) + \Phi(b)$ for all $a, b \in \mathcal{A}$, that is, Φ is additive. Now (6), can be rewritten as

$$a\Phi(b) + b\Phi(a)^* - \Phi(a)b - \Phi(b)a^* = 0 \tag{7}$$

for all $a, b \in \mathcal{A}$. Now we proceed by considering the following two cases:-

Case 1. $\deg(\mathcal{A}) > 2$.

In this case by [4, Corollary 5.12 and Theorem 3.25], $\mathcal{A}$ is $(*, 2)$-free subring of $Q_{ml}(\mathcal{A})$. Thus, there exists $q \in Q_{ml}(\mathcal{A})$ such that $\Phi(b) = qb$ for all $b \in \mathcal{A}$. On the other hand there exists $q_1 \in Q_{ml}(\mathcal{A})$ such that $\Phi(b) = bq_1$ for all $b \in \mathcal{A}$. Thus $qb = bq_1$ for all $b \in \mathcal{A}$. By [3, Theorem 6.4.1], $\mathcal{A}$ and $Q_{ml}(\mathcal{A})$ satisfy the same GPIs. So putting $b = 1$ in the last relation, we see that $q_1 = q \in C$. Hence from (7), we get $(q^* - q)ba^* = 0$ for all $a, b \in \mathcal{A}$. This provides $q^* = q$.

Case II. $\deg(\mathcal{A}) \leq 2$. In this case by [24, Theorem 2], $\mathcal{Z}(\mathcal{A}) \neq \{0\}$.

Subcase (i): $\alpha^* \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$

Setting $a = \alpha$ in (7), we find that $\Phi(b) = qb + bq^*$ for all $b \in \mathcal{A}$, where $q = \frac{\Phi(\alpha)}{\alpha - \alpha^*} \in \mathcal{A}C$. Utilizing this in (7), we get

$$aqb + abq^* + ba^*q^* + bqa^* = qab + aq^*b + qba^* + bq^*a^* \tag{8}$$

for all $a, b \in \mathcal{A}$. Replacing a by $\alpha^* a$ in (8), we get

$$\alpha^*aqb + \alpha^*abq^* + \alpha ba^*q^* + \alpha bqa^* = \alpha^*qab + \alpha^*aq^*b + \alpha qba^* + \alpha bq^*a^* \tag{9}$$

for all $a, b \in \mathcal{A}$. Also from (8), we have

$$\alpha aqb + \alpha abq^* + \alpha ba^*q^* + \alpha bqa^* = \alpha qab + \alpha aq^*b + \alpha qba^* + \alpha bq^*a^* \tag{10}$$

for all $a, b \in \mathcal{A}$. Subtracting (10) from (9), we find that

$$[a, q]b + a[b, q^*] = 0$$

for all $a, b \in \mathcal{A}$. Now by [3, Theorem 6.4.1], $\mathcal{A}$ and $Q_{ml}(\mathcal{A})$ satisfy the same GPIs. Therefore putting $b = 1$, in the last expression, it follows that $q \in C$. Hence $\Phi(a) = \lambda a$ for all $a \in \mathcal{A}$, where $\lambda = q + q^* \in C$.

Subcase (ii). $\alpha^* = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$. Since '$*$' is a non identity involution, we must have $\deg(\mathcal{A}) = 2$. Let $\alpha \in \mathcal{Z}(\mathcal{A})$ and $a, b \in \mathcal{A}$. Then, we have

$$\begin{aligned} [\Phi(\alpha a) - \alpha\Phi(a), b]_* &= [\Phi(\alpha a), b]_* - \alpha[\Phi(a), b]_* \\ &= [\alpha a, \Phi(b)]_* - \alpha[a, \Phi(b)]_* \\ &= 0. \end{aligned}$$

Therefore by Lemma 2.1, it follows that $\Phi(\alpha a) = \alpha\Phi(a)$. Thus Φ is a $\mathcal{Z}(\mathcal{A})$-linear map. Now, any element of $\mathcal{A}C$ is of the form $\frac{a}{\alpha}$, where $a \in \mathcal{A}$ and $0 \neq \alpha \in C$. So the map $\frac{a}{\alpha} \mapsto \frac{\Phi(a)}{\alpha}$ is an extension of Φ to $\mathcal{A}C$, which we again denote by Φ. Let $\mathbb{F}$ be the algebraic closure of C. Then Φ can be extended uniquely to $\mathcal{A}C \otimes \mathbb{F}$, again denoted by Φ, by defining

$$\Phi(\sum_i a_i \otimes \alpha_i) = \sum_i \Phi(a_i) \otimes \alpha_i$$

for $a_i \in \mathcal{A}C$ and $\alpha_i \in \mathbb{F}$. Moreover, '$*$' can also be extended to an involution on $\mathcal{A}C \otimes_C \mathbb{F}$, denoted by '$*$' also and is given by

$$\Big(\sum_i a_i \otimes \alpha_i\Big)^* = \sum_i a_i^* \otimes \alpha_i.$$

Hence for any $a_i, b_j \in \mathcal{A}C$ and $\alpha_i, \beta_j \in \mathbb{F}$, we have

$$\begin{aligned}
\Big[\Phi(\textstyle\sum_i a_i \otimes \alpha_i), \sum_j a_j \otimes \beta_j\Big]_* &= \Big[\sum_i \Phi(a_i) \otimes \alpha_i, \sum_j b_j \otimes \beta_j\Big]_* \\
&= \sum_i \sum_j \Big[\Phi(a_i), b_j\Big]_* \otimes \alpha_i\beta_j \\
&= \sum_i \sum_j \Big[a_i, \Phi(b_j)\Big]_* \otimes \alpha_i\beta_j \\
&= \Big[\sum_i a_i \otimes \alpha_i, \sum_j \Phi(b_j) \otimes \beta_j\Big]_* \\
&= \Big[\sum_i a_i \otimes \alpha_i, \Phi(\sum_j b_j \otimes \beta_j)\Big]_*.
\end{aligned}$$

Therefore, we have $[\Phi(a), b]_* = [a, \Phi(b)]_*$ for all $a, b \in \mathcal{A}C \otimes \mathbb{F}$. Since $\mathcal{A}C \otimes \mathbb{F} \cong \mathcal{M}_n(\mathbb{F})$, where n =deg($\mathcal{A}$). Thus, we have

$$[\Phi(X), Y]_* = [X, \Phi(Y)]_* \tag{11}$$

and

$$\Phi(X)(X + X^*) = X(\Phi(X) + \Phi(X)^*) \tag{12}$$

for all $X, Y \in \mathcal{M}_2(\mathbb{F})$. Firstly, assume that '$*$' is the symplectic involution. Then, putting $X = e_{11}$ in (12), we find that $\Phi(e_{11}) = \eta e_{11}$ for some $\eta \in \mathbb{F}$. Similarly, it can be seen that $\Phi(e_{22}) = \beta e_{22}$ for some $\beta \in \mathbb{F}$. Now, putting $X = e_{11}$ and $Y = e_{22}$ in (11), we get $\eta = \beta$. Also taking $X = e_{12}$ in (12), we find that

$$\Phi(e_{12}) = \begin{pmatrix} u & v \\ w & -u \end{pmatrix}.$$

Next setting $X = e_{11}$ and $Y = e_{12}$ in (11), we get $u = 0$. Finally, putting $X = e_{12}$ and $Y = e_{22}$ in (11), give $v = \eta$ and $w = 0$. Similarly, it can also be checked that $\Phi(e_{21}) = \eta e_{21}$. Thus $\Phi(X) = \eta X$ for all $X \in \mathcal{M}_2(\mathbb{F})$, where $\eta \in \mathbb{F}$.

Secondly, assume that '$*$' is the transpose involution. Then taking $X = e_{11}$ and $Y = e_{12}$ in (11), we find that

$$\Phi(e_{11}) = \begin{pmatrix} \phi & 0 \\ 0 & \nu \end{pmatrix} \text{ and } \Phi(e_{12}) = \begin{pmatrix} \zeta & \theta \\ 0 & \delta \end{pmatrix}$$

with $\theta + \nu = \phi$. Now setting $X = e_{12}$ in (12), we see that $\zeta = \delta = 0$. Also putting $X = e_{12}$ and $Y = e_{11}$ in (11), we get $\nu = 0$. Hence $\Phi(e_{11}) = \theta e_{11}$ and $\Phi(e_{12}) = \theta e_{12}$.

Similarly, it can also be verified that $\Phi(e_{22}) = \zeta e_{22}$ and $\Phi(e_{21}) = \zeta e_{21}$ for some $\zeta \in \mathbb{F}$. Finally, taking $X = e_{12}$ and $Y = e_{21}$ in (11), we find that $\theta = \zeta$. Consequently, $\Phi(X) = \zeta X$ for all $X \in \mathcal{M}_2(\mathbb{F})$.

Therefore from the above discussions, we have $\Phi(X) = \eta X$ for all $X \in \mathcal{A}C \otimes \mathbb{F}$, where $\eta \in \mathbb{F}$. Choose a basis $\nu_1, \nu_2, ...$ for $\mathbb{F}$ over C with $\nu_1 = 1$. Write $\eta = \lambda_1\nu_1 + \lambda_2\nu_2 + \cdots$ for some $\lambda_1, \lambda_2, ... \in C$. Then for every $a \in \mathcal{A}$, we have $\Phi(a) \otimes 1 = \lambda_1 a \otimes \nu_1 + \lambda_2 a \otimes \nu_2 + \cdots$. Hence $(\Phi(a) - \lambda_1 a) \otimes \nu_1 - \lambda_2 a \otimes \nu_2 + \cdots = 0$. Therefore, we have $\Phi(a) = \lambda a$ for all $a \in \mathcal{A}$, where $\lambda = \lambda_1 \in C$ and this completes the proof.

□

As a corollary of the above result we have the following characterization of skew Lie centralizers on prime rings.

Corollary 2.4. *Let $\mathcal{A}$ be a prime ring with a non identity involution '$*$' and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a skew Lie centralizer. Then there exists $\lambda^* = \lambda \in C$ such that $\Phi(a) = \lambda a$ for all $a \in \mathcal{A}$.*

3. Applications to Some Operator Algebras

As an application of the results in the previous section, we will characterize strong skew commutativity preserving maps and skew Lie centralizers on the standard operator algebras acting on Hilbert spaces and factor von Neumann algebras. Throughout this section all algebras and vector spaces will be over the complex field $\mathbb{C}$.

Standard operator algebras: Let $\mathcal{H}$ be a real or complex Banach space and let $\mathcal{L}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ and the ideal of all finite rank operators in $\mathcal{L}(\mathcal{H})$ respectively. Recall that a standard operator algebra is any subalgebra $\mathcal{A}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}(\mathcal{H})$ contains the identity operator and $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{A}(\mathcal{H})$. It is clear that $\mathcal{L}(\mathcal{H})$ is a standard operator algebra. Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. We denote by A^* the adjoint operator of $A \in \mathcal{L}(\mathcal{H})$. According to the results obtained in the previous section, we have the following corollaries.

Corollary 3.1. *Let $\mathcal{S}$ be a self-adjoint standard operator algebra in a Hilbert space $\mathcal{H}$. Suppose that $\chi : \mathcal{S} \to \mathcal{S}$ is a surjective map. Then χ is strong skew commutativity preserving map if and only if $\chi(a) = \lambda a$ for all $a \in \mathcal{S}$, where $\lambda \in \{1, -1\}$.*

Corollary 3.2. *Let $\mathcal{S}$ be a self-adjoint standard operator algebra in a Hilbert space $\mathcal{X}$. Then a map $\chi : \mathcal{S} \to \mathcal{S}$ is a skew Lie centralizer if and only if $\chi(a) = \lambda a$ for all $a \in \mathcal{S}$, where $\lambda \in \mathbb{R}$.*

Factor von Neumann algebras: Let $\mathcal{H}$ be a Hilbert space. Recall that a von Neumann algebra $\mathcal{M}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which satisfies the double commutant property, that is, $\mathcal{M}'' = \mathcal{M}$ where $\mathcal{M}' = \{\mathcal{T} \in \mathcal{B}(H) : \mathcal{T}\mathcal{A} = \mathcal{A}\mathcal{T}$ for all $\mathcal{A} \in \mathcal{M}\}$ and $\mathcal{M}'' = (\mathcal{M}')'$. It is clear that a von Neumann algebra is unital. A von Neumann algebra $\mathcal{M}$ is called a factor von Neumann algebra if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$. Factor von Neumann algebras are unital prime algebras.

Corollary 3.3. *Let $\mathcal{N}$ be a factor von Neumann algebra and let $\chi : \mathcal{N} \to \mathcal{N}$ be a surjective map. Then χ is strong skew commutativity preserving map if and only if $\chi(a) = \lambda a$ for all $a \in \mathcal{N}$, where $\lambda \in \{1, -1\}$.*

Corollary 3.4. *Let $\mathcal{N}$ be a factor von Neumann algebra. Then a map $\psi : \mathcal{N} \to \mathcal{N}$ is a skew Lie centralizer if and only if $\psi(a) = \lambda a$ for all $a \in \mathcal{N}$, where $\lambda \in \mathbb{R}$.*

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References

[1] A. Abbasi, C. Abdioglu, S. Ali, M. R. Mozumder, *A characterization of Jordan left $*$- centralizers via skew Lie and Jordan products*, Bull. Iranian Math. Soc. **48**(5) (2022), 2765-2778.
[2] Z. Bai, S. Du, *Maps preserving products $xy - yx^*$ on von Neumann algebras*, J. Math. Anal. Appl. **386**(1) (2012), 103–109.
[3] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities*, Pure and Applied Math., Dekker, New York, 1996.
[4] M. Bresar, M. A. Chebotar, W. S. Martindale III, *Functional Identities*, Frontiers in Mathematics (Birkhauser-Verlag, Basel), 2007.
[5] J. Cui, C. K. Li, *Maps preserving product $xy - yx^*$ on factor von Neumann algebras*, Linear Algebra Appl. **431**(5-7) (2009),833–842
[6] J. L. Cui, C. Park, *Maps preserving strong skew Lie product on factor von Neumann algebras*, Acta Math. Sci. Ser. B **32** (2012), 531–538.
[7] Q. Deng, M. Ashraf, *On strong commutativity preserving maps*, Results Math. **30** (1996), 259–263.
[8] B. Fadaee, A. Fošner, H. Ghahramani, *Centralizers of Lie structure of triangular algebras*, Results Math. **77**(6) (2022), 1-16.
[9] A. Fosner, W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory **4** (2019), 342–350
[10] I. N. Herstein, *Rings with Involution*, Chicago Lectures in Mathematics, Chicago London: The University of Chicago Press, 1976.
[11] J. Hou, W. Wang, *Strong 2-skew commutativity preserving maps on prime rings with involution*, Bull. Malays. Math. Sci. Soc. **42** (2016), 33–49.
[12] A. Jabeen, *Lie (Jordan) centralizers on generalized matrix algebras*, Comm. Algebra **49**(1) (2021), 278-291.
[13] W. Jing, *Additivity of Lie centralizers on triangular rings*, Math and computer sience working papers (2011) 1–10.
[14] L. Kong, L. Chao, *Nonlinear mixed Lie triple derivations on prime $*$-rings*, ScienceAsia **48**(3) (2022), 335-339.
[15] L. Kong, J. Zhang, *Nonlinear skew Lie derivations on prime $*$-rings*, Indian J. Pure Applied Math. **30** 2022, 1-10.
[16] T. K. Lee, T. L. Wong, *Nonadditive strong commutativity preserving maps*, Comm. Algebra **40**(6) (2012), 2213–2218.

[17] C. J. Li, Q. Y. Chen, *Strong skew commutativity preserving maps on rings with involution*, Acta Math. Sin. (Ser. B Engl. Ed.), **32**(6) 2016, 745–752.
[18] L. Liu, K. Gao, *Characterizations of Lie centralizers of triangular algebras*, Linear Multilinear Algebra, (2022), 1-17.
[19] L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear Multilinear Algebra, **70**(4) (2022), 2693-2705.
[20] L. Liu, *Strong skew commutativity preserving maps on rings*, J. Aust. Math. Soc. **100**(1) (2016), 78–85.
[21] X. Qi, J. Hou, *Strong skew commutativity preserving maps on von Neumann algebras*, J. Math. Anal. Appl. **397**(1) (2013), 362–370.
[22] X. Qi, Y. Zhang, *k-skew Lie products on prime rings with involution*, Comm. Algebra **46**(3) (2018), 1001-1010.
[23] N. Rehman, J. Nisar, M. Nazim, *A note on nonlinear mixed Jordan triple derivation on $*$-algebras*, Comm. Algebra **51**(4) (2022), 1334-1343.
[24] L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. **79** (1973), 219-223.
[25] Z. Yang, J. Zhang, *Nonlinear maps preserving the mixed skew Lie triple product on factor von Neumann algebras*, Ann. Funct. Anal. **10** (2019), 325-336.
[26] Z, Yang, J. Zhang, *Nonlinear maps preserving the second mixed Lie triple products on factor von Neumann algebras*, Linear Multilinear Algebra **68**(2) (2020), 377-390.