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Some new inequalities concerning the variable sum exdeg index/coindex of graphs

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Abstract. The variable sum exdeg index and coindex of a graph G are denoted by $SEI_a(G)$ and $\overline{SEI}_a(G)$, respectively, and they are defined as $SEI_a(G) = \sum_{i=1}^{n} d_i a^{d_i}$ and $\overline{SEI}_a(G) = \sum_{i=1}^{n}(n-1-d_i)a^{d_i}$, respectively, where '$a$' is a positive real number different from 1 and $(d_1, d_2, \cdots, d_n)$ is the vertex-degree sequence of G. The present paper gives several new inequalities involving the graph invariants SEI_a and/or $\overline{SEI}_a$. All graphs attaining the equality signs in the obtained inequalities are also characterized.

1. Introduction

Inequalities between different graph invariants are being studied since the early developments of the graph theory [18, 19], where a graph invariant I is a numerical quantity associated with a graph G satisfying the equality $I(G) = I(G')$ for every graph G' isomorphic to G. The current paper is a sequel of the recent two articles [2, 15], written by the present authors. These papers, and the present one, are mainly concerned with two graph invariants, namely the variable sum exdeg index and coindex. The variable sum exdeg index and coindex of a graph G are denoted by $SEI_a(G)$ and $\overline{SEI}_a(G)$, respectively, and they are defined as

$$SEI_a(G) = \sum_{i=1}^{n} d_i a^{d_i}$$

and

$$\overline{SEI}_a(G) = \sum_{i=1}^{n} (n-1-d_i) a^{d_i},$$

respectively, where 'a' is any positive real number different from 1 and $(d_1, d_2, \cdots, d_n)$ is the vertex-degree sequence of G. The variable sum exdeg index was put forwarded by Vukičević [21] about a decade ago,

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within a study of the bond additive modeling. Details about the bounds and extremal results concerning the variable sum exdeg index can be found in the papers [1, 4, 7, 8, 11, 13, 14, 20, 22] and related references listed therein. The coindex version of this graph invariant, that is the variable sum exdeg coindex, has recently been considered in [2, 15] where several mathematical inequalities involving both of these invariants are reported.

Throughout this paper, we assume that G is a graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$, edge set E and with the vertex–degree sequence $(d_1, d_2, \cdots, d_n)$ satisfying $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, where $n \geq 2$, $|E| = m$ and d_i is the degree of the vertex v_i for $i = 1, 2, \cdots, n$. Also, if v_i and v_j are adjacent in G, we write $i \sim j$, otherwise we write $i \nsim j$. The graph-theoretical notation and terminology used in the present paper but not described here, can be found in some standard graph-theoretical books, like [3, 5, 6].

The variable sum exdeg index and coindex of G can also be defined as

$$SEI_a(G) = \sum_{i \sim j} (a^{d_i} + a^{d_j})$$

and

$$\overline{SEI}_a(G) = \sum_{i \nsim j} (a^{d_i} + a^{d_j}),$$

respectively. The main purpose of the present paper is to establish several new inequalities involving the graph invariants SEI_a and/or $\overline{SEI}_a$. All graphs attaining the equality signs in the obtained inequalities are also characterized.

2. Preliminaries

Let t be a positive integer and x real number such that $x > -1$. In [10] (see also [17]) the following result was proven.

Lemma 2.1. [10] *Let*

$$F(k, t, x) = 1 + tx + \binom{t}{2}x^2 + \cdots + \binom{t}{k}x^k,$$

be the k-th partial sum of the binomial series for $(1 + x)^t$, where $x > -1$. Then if first omited term is

1. *positive, then $(1 + x)^t > F(k, t, x)$,*
2. *zero, then $(1 + x)^t = F(k, t, x)$,*
3. *negative, then $(1 + x)^t \leq F(k, t, x)$.*

For $k = 2$ and $k = 3$ and $x = a - 1$, by Lemma 2.1 the following result is obtained.

Lemma 2.2. *Let t be a positive integer. Then, for any real $a > 1$ the following inequality is valid*

$$a^t \geq 1 + t(a - 1) + \frac{t(t - 1)}{2}(a - 1)^2. \tag{1}$$

When $0 < a < 1$, the opposite inequality is valid. Equality holds if and only if $t \in \{1, 2\}$.

Lemma 2.3. *Let t be a positive integer. For any real $a > 0$ the following inequality is valid*

$$a^t \geq 1 + t(a - 1) + \frac{t(t - 1)}{2}(a - 1)^2 + \frac{t(t - 1)(t - 2)}{6}(a - 1)^3. \tag{2}$$

Equality holds if and only if $t \in \{1, 2, 3\}$.

For $k = 2$ and $k = 3$ and $x = \frac{1}{a} - 1$, from Lemma 2.1 the following result is obtained.

Lemma 2.4. *Let t be a positive integer. Then, for any real $a > 1$ holds*

$$a^{-t} \leq 1 + t\left(\frac{1}{a} - 1\right) + \frac{t(t-1)}{2}\left(\frac{1}{a} - 1\right)^2 . \tag{3}$$

When $0 < a < 1$ the opposite inequality is valid. Equality holds if and oly if $t \in \{1, 2\}$.

Lemma 2.5. *Let t be a positive integer. Then, for any real $a > 1$ holds*

$$a^{-t} \geq 1 + t\left(\frac{1}{a} - 1\right) + \frac{t(t-1)}{2}\left(\frac{1}{a} - 1\right)^2 + \frac{t(t-1)(t-2)}{6}\left(\frac{1}{a} - 1\right)^3 . \tag{4}$$

Equality holds if and only if $t \in \{1, 2, 3, \}$.

3. Main results

In the next theorem we determine a relationship between $SEI_a(G)$ and $M_1(G)$ and $F(G)$.

Theorem 3.1. *Let G be a connected graph with $m \geq 1$ edges. Then, for any real $a > 1$ holds*

$$SEI_a(G) \geq 2m + M_1(G)(a-1) + \frac{1}{2}\left(F(G) - M_1(G)\right)(a-1)^2 . \tag{5}$$

When $0 < a < 1$, the opposite inequality is valid. Equality holds if and only if $G \cong C_n$ or $G \cong P_n$.

Proof. Let d_i be a degree of vertex v_i, $i = 1, 2, \ldots, n$. Then, by (1) we have that

$$a^{d_i} \geq 1 + d_i(a-1) + \frac{d_i(d_i-1)}{2}(a-1)^2 . \tag{6}$$

After multiplying the above inequality with d_i and then summing over i, $i = 1, 2, \ldots, n$, we obtain

$$\sum_{i=1}^{n} d_i a^{d_i} \geq \sum_{i=1}^{n}\left(d_i + d_i^2(a-1) + \frac{d_i^2(d_i-1)}{2}(a-1)^2\right) =$$
$$= 2m + M_1(G)(a-1) + \frac{1}{2}\left(F(G) - M_1(G)\right)(a-1)^2 ,$$

from which (5) is obtained. By a similar procedure it can be proved that when $0 < a < 1$, the opposite inequality is valid.

Equality in (6) holds if and only if $d_i \in \{1, 2\}$, for $i = 1, 2, \ldots, n$. Since G is connected, it follows that equality in (5) holds if and olny if $G \cong C_n$ or $G \cong P_n$. □

Corollary 3.2. *Let G be a connected graph with $m \geq 1$ edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq 2m - M_1(G)\frac{(a-1)(a-3)}{2} + M_2(G)(a-1)^2 . \tag{7}$$

Equality holds if and only if $G \cong C_n$ or $G \cong P_2$.

Proof. By the arithmetic–geometric mean inequality, AM–GM inequality, (see e.g. [17]) we have that

$$F(G) = \sum_{i\sim j}(d_i^2 + d_j^2) \geq 2\sum_{i\sim j} d_i d_j = 2M_2(G) . \tag{8}$$

From the above and (5) for $a > 1$, we arrive at (7).

Equality in (8) holds if and only if G is regular, which implies that equality in (7) holds if and only if $G \cong C_n$ or $G \cong P_2$. □

Corollary 3.3. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq 2ma + (M_1(G) - 2m)\left(1 + \frac{(M_1(G) - 2m)(a-1)}{2(2m-n)}\right)(a-1)\,. \tag{9}$$

Equality holds if and only if $G \cong C_n$ or $G \cong P_n$.

Proof. In [16] it was proven that

$$F(G) \geq M_1(G) + \frac{(M_1(G) - 2m)^2}{2m-n}\,. \tag{10}$$

From the above enad inequality (5) we obtain (9). Equality in (10) holds if and only if $d_i \in \{\Delta, 1\}$, for $i = 1, 2, \ldots, n$, which implies that equality in (9) holds if and only if $G \cong C_n$ or $G \cong P_n$. □

Corollary 3.4. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq 2m + M_1(G)(a-1) + \frac{2m^2(2m-n)}{n^2}(a-1)^2\,. \tag{11}$$

Equality holds if and only if $G \cong C_n$ or $G \cong P_2$.

Proof. In [16] it was proven that

$$F(G) - M_1(G) \geq \frac{4m^2(2m-n)}{n^2}\,. \tag{12}$$

From the above and (5) we arrive at (11). □

Corollary 3.5. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq 2m + M_1(G)(a-1) + \frac{1}{2}\frac{(2m-n)^3(a-1)^2}{(n - ID(G))^2}\,. \tag{13}$$

Equality holds if and only if $G \cong C_n$.

Proof. In [16] it was proven that

$$F(G) - M_1(G) \geq \frac{(2m-n)^3}{(n - ID(G))^2}\,. \tag{14}$$

From the above and (5) we obtain (13). □

Corollary 3.6. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq \frac{2m}{n}\left(na + (2m-n)\left(1 + \frac{m}{n}(a-1)\right)(a-1)\right)\,. \tag{15}$$

Equality holds if and only if $G \cong C_n$ or $G \cong P_2$.

Proof. In [9] it was proven that

$$M_1(G) \geq \frac{4m^2}{n}\,, \tag{16}$$

with equality if and only if G is regular. From the above inequality and (9) we obtain (15). □

Corollary 3.7. *Let U be a connected unicyclic graph with $n \geq 3$ vertices and m edges. Then, for any $a > 1$ hold*

$$SEI_a(U) \geq M_1(U)(a-1) + 2n(a-1)^2 + 2n\,, \tag{17}$$

$$SEI_a(U) \geq 2na^2\,. \tag{18}$$

Equalities hold if and only if $U \cong C_n$.

Denote by $D = \Delta a^{\Delta} - \Delta + 2m + (M_1(G) - \Delta^2)(a-1)$. Similarly as in the case of Theorem 3.1 the following result can be proved.

Theorem 3.8. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then, for any $a > 1$ holds*

$$SEI_a(G) \geq D + \frac{1}{2}\left(F(G) - M_1(G) - \Delta^2(\Delta-1)\right)(a-1)^2\,.$$

When $0 < a < 1$ the opposite inequality is valid. Equality holds if and only if either $G \cong K_{1,n-1}$, or $G \cong C_n$, or G is a connected graph with the property $\Delta = 2m - t - n + 2$, $d_2 = \cdots = d_t = 2$ and $d_{t+1} = \cdots = d_n = 1$, for some t, $2 \leq t \leq n-2$.

Theorem 3.9. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any real $a > 1$, holds*

$$\overline{SEI_a}(G) \geq 2\overline{m} + \overline{M_1}(G)(a-1) + \frac{1}{2}\left(\overline{F}(G) - \overline{M_1}(G)\right)(a-1)^2\,. \tag{19}$$

When $0 < a < 1$, the opposite inequality is valid. Equality holds if and only if either $G \cong K_n$, or $G \cong C_n$, or $G \cong P_n$, or $G \cong K_{1,n-1}$, or G is connected graph with the property $n-1 = d_1 = d_2$ and $d_3 = \cdots = d_n = 2$.

Proof. Let $a > 1$. After multiplying (6) by $n-1-d_i$, $i = 1,2,\ldots,n$ and summation over i for $i = 1,2,\ldots,n$, we obtain

$$\begin{aligned}\sum_{i=1}^{n}(n-1-d_i)a^{d_i} \geq \sum_{i=1}^{n}\Big(&n-1-d_i + (n-1-d_i)d_i(a-1) + \\ &+ \frac{d_i(d_i-1)(n-1-d_i)}{2}(a-1)^2\Big)\,,\end{aligned} \tag{20}$$

from which (19).

Equality in (20) holds if and only if $d_1 = d_2 = \cdots = d_n = n-1$, or $d_i \in \{1,2\}$, for $i = 1,2,\ldots,n$. Since G is connected we conclude that equality in (19) holds if and only if if either $G \cong K_n$, or $G \cong C_n$, or $G \cong P_n$, or $G \cong K_{1,n-1}$, or G is connected graph with the property $n-1 = d_1 = d_2$ and $d_3 = \cdots = d_n = 2$. □

Theorem 3.10. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any real $a > 1$, holds*

$$SEI_a(\overline{G}) \leq a^{n-1}\left(2\overline{m} + \overline{M_1}(G)\left(\frac{1}{a}-1\right) + \frac{1}{2}\left(\overline{F}(G) - \overline{M_1}(G)\right)\left(\frac{1}{a}-1\right)^2\right) \tag{21}$$

When $0 < a < 1$ the opposite inequality is valid. Equality holds if and only if G is a connected graph with the property $d_i \in \{1,2,n-1\}$, for $i = 1,2,\ldots,n$.

Proof. Suppose $a > 1$. Then, based on (3) for any vertex degree d_i of vertex v_i we have that

$$a^{-d_i} \leq 1 + d_i\left(\frac{1}{a}-1\right) + \frac{d_i(d_i-1)}{2}\left(\frac{1}{a}-1\right)^2\,.$$

After multiplying the above inequality by $n-1-d_i$ and then summing over i, $i = 1,2,\ldots,n$, we obtain

$$\sum_{i=1}^{n}(n-1-d_i)a^{n-1-d_i} \leq \sum_{i=1}^{n}(n-1-d_i)a^{n-1}\left(1 + d_i\left(\frac{1}{a}-1\right) + \frac{d_i(d_i-1)}{2}\left(\frac{1}{a}-1\right)^2\right), \tag{22}$$

from which (21) is obtained. The case when $0 < a < 1$ can be proved in a similar way.

Equality in (22), and therefore in (21), holds if and only if G is a connected graph with the property $d_i \in \{1,2,n-1\}$, for $i = 1,2,\ldots,n$. □

Remark 3.11. *Equality in* (21) *is attained for a number of connected graphs with n vertices. Thus, for example, equality in* (21) *holds if $G \cong K_n$, or $G \cong C_n$, or $G \cong P_n$, or $G \cong K_{1,n-1}$, or when G is a connected graph with the properties $n-1 = d_1 = d_2$ and $d_3 = d_4 = \cdots = d_n = 2$.*

In what follows we will prove some inequalities of Nordhaus–Gaddum type (see [18]).

Theorem 3.12. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any real $a > 1$, holds*

$$SEI_a(G) + \overline{SEI_a}(G) \geq \frac{n-1}{2}\Big(2n + 4m(a-1) + (M_1(G) - 2m)(a-1)^2\Big). \tag{23}$$

When $0 < a < 1$, the opposite inequality is valid. Equality holds if and only if $G \cong C_n$ or $G \cong P_n$.

Proof. The following identity is valid

$$SEI_a(G) + \overline{SEI_a}(G) = (n-1)\sum_{i=1}^{n} a^{d_i}. \tag{24}$$

Let $a > 1$. After summation of (6) over i, for $i = 1, 2, \dots, n$, we obtain

$$\sum_{i=1}^{n} a^{d_i} \geq n + 2m(a-1) + \frac{1}{2}(M_1(G) - 2m)(a-1)^2. \tag{25}$$

From the above and (24) we obtain (23).

Equality in (25), and hence in (23), holds if and only if $G \cong C_n$ or $G \cong P_n$. □

Corollary 3.13. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any real $a > 1$, holds*

$$SEI_a(G) + \overline{SEI_a}(G) \geq (n-1)\left(n + 2m(a-1) + \frac{m}{n}(2m-n)(a-1)^2\right).$$

Equality holds if and only if $G \cong C_n$ or $G \cong P_2$.

Corollary 3.14. *Let U be a connected unicyclic graph with $n \geq 3$ vertices. Then, for any real $a > 1$, holds*

$$SEI_a(U) + \overline{SEI_a}(U) \geq n(n-1)a^2. \tag{26}$$

Equality holds if and only if $U \cong C_n$.

Remark 3.15. *The inequality* (26) *was proven in* [2].

Corollary 3.16. *Let T be a tree with $n \geq 2$ vertices. Then, for any real $a > 1$, holds*

$$SEI_a(T) + \overline{SEI_a}(T) \geq (n-1)(2a + (n-2)a^2). \tag{27}$$

Equality holds if and only if $T \cong P_n$.

Proof. In [12] it was proven that

$$M_1(T) \geq 4n - 6 = M_1(P_n).$$

For the above and inequality (24) for $m = n-1$, we obtain (27). □

Remark 3.17. *The inequality* (27) *was proven in* [15].

Let $A = n + 2m(a-1) + \frac{1}{2}(M_1(G) - 2m)(a-1)^2$. Similarly as in the case of Theorem 3.12, based on (2), the following result can be proven.

Theorem 3.18. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then, for any real $a > 0$ holds*

$$SEI_a(G) + \overline{SEI_a}(G) \geq (n-1)\left(A + \frac{1}{6}(F(G) - 3M_1(G) + 4m)(a-1)^3\right). \tag{28}$$

Equality holds if and only if G is a connected graph with the property $d_i \in \{1,2,3\}$, for $i = 1,2,\ldots,n$.

Theorem 3.19. *Let G be a connected graph of order $n \geq 2$ and size m. Then, for any real $a > 1$, holds*

$$SEI_a(\overline{G}) + \overline{SEI_a}(\overline{G}) \leq (n-1)a^{n-1}\left(n + 2m\left(\frac{1}{a} - 1\right) + \frac{1}{2}(M_1(G) - 2m)\left(\frac{1}{a} - 1\right)^2\right). \tag{29}$$

When $0 < a < 1$ the sense of the above inequality reverses. Equality holds if and only if $G \cong C_n$ or $G \cong P_n$.

Proof. Let $a > 1$. According to the (3), for any vertex degree d_i of vertex v_i, holds

$$a^{-d_i} \leq 1 + d_i\left(\frac{1}{a} - 1\right) + \frac{d_i(d_i-1)}{2}\left(\frac{1}{a} - 1\right).$$

After summation of the above inequality over i, for $i = 1,2,\ldots,n$, we obtain

$$\sum_{i=1}^{n} a^{-d_i} \leq n + 2m\left(\frac{1}{a} - 1\right) + \frac{1}{2}(M_1(G) - 2m)\left(\frac{1}{a} - 1\right)^2. \tag{30}$$

On the other hand, we have that

$$\begin{aligned} SEI_a(\overline{G}) + \overline{SEI_a}(\overline{G}) &= \sum_{i=1}^{n}(n-1-d_i)a^{n-1-d_i} + \sum_{i=1}^{n} d_i a^{n-1-d_i} = \\ &= (n-1)a^{n-1}\sum_{i=1}^{n} a^{-d_i}. \end{aligned} \tag{31}$$

Now, from the above and inequality (30) we arrive at (29).

Equality in (30) holds if and only if $d_i \in \{1,2\}$, for $i = 1,2,\ldots,n$. Since G is connected, it follows that equality in (29) holds if and only if $G \cong C_n$ or $G \cong P_n$. □

Let $B = n + 2m\left(\frac{1}{a} - 1\right) + \frac{1}{2}(M_1(G) - 2m)\left(\frac{1}{a} - 1\right)^2$. By a similar procedure as in Theorem 3.19, from (4) following theorem can be proved.

Theorem 3.20. *Let G be a connected graph of order $n \geq 2$ and size m. Then, for any real $a > 0$, holds*

$$SEI_a(\overline{G}) + \overline{SEI_a}(\overline{G}) \geq (n-1)a^{n-1}\left(B + \frac{1}{6}(F(G) - 3M_1(G) + 4m)\left(\frac{1}{a} - 1\right)^3\right).$$

Equality holds if and only if G is a connected graph with the property $d_i \in \{1,2,3\}$, for $i = 1,2,\ldots,n$.

References

[1] A. Ali, D. Dimitrov, On the extremal graphs with respect to bound incident degree indices, *Discr. Appl. Math.* **238** (2018) 32–40.
[2] A. Ali, E. Milovanović, M. Matejić, I. Milovanović, On the variable sum exdeg index/coindex of graphs, *MATCH Commun. Math. Comput. Chem.*, in press.
[3] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, London, 2008.
[4] W. Carballosa, J. M. Rodríguez, J. M. Sigarreta, Extremal problems on the variable sum exdeg index, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 753–772.
[5] G. Chartrand, L. Lesniak, P. Zhang, *Graphs & Digraphs*, Sixth Edition, CRC Press, Boca Raton, 2016.
[6] R. Diestel, Graph Theory, Third Edition, Springer, New York, 2005.
[7] D. Dimitrov, A. Ali, On the extremal graphs with respect to the variable sum exdeg index, *Discr. Math. Lett.* **1** (2019) 42–48.
[8] J. Du, X. Sun, On the graph connectivity and the variable sum exdeg index, *AIMS Math.* **6** (2020) 607–622.

[9] C. S. Edwards, The largest vertex degree sum for a triangle in a graph, *Bull. London Math. Soc.* **9** (1977) 203–208.
[10] L. Gerber, An extension of Bernouli's inequality, *Amer. Math. Monthly* **75** (1968) 875–876.
[11] A. Ghalavand, A. R. Ashrafi, Extremal graphs with respect to variable sum exdeg index via majorization, *Appl. Math. Comput.* **303** (2017) 19–23.
[12] I. Gutman, K. Ch. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.
[13] M. Javaid, A. Ali, I. Milovanović, E. Milovanović, On the extremal cactus graphs for variable sum exdeg index with a fixed number of cycles, *AKCE Int. J. Graphs Combin.* **17** (2020) 920–923.
[14] S. Khalid, A. Ali, On the zeroth-order general Randić index, variable sum exdeg index and trees having vertices with prescribed degree, *Discr. Math. Algorithm. Appl.* **10** (2018) #1850015.
[15] M. Matejić, E. Milovanović, I. Milovanović, A. Ali, A note on the variable sum exdeg index/coindex of trees, *Contrib. Math.* **2** (2020) 42–46.
[16] P. Milošević, I. Milovanović, E. Milovanović, M. Matejić, Some inequalities for zeroth–order Randić index, *Filomat* **33** (2019) 5249–5258.
[17] D. S. Mitrinović, P. M. Vasić, *Analytic Inequalities*, Springer Verlag, Berlin–Heidelberg–New York, 1970.
[18] E. A. Nordhaus, J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956) 175–177.
[19] O. Ore, *Theory of Graphs*, American Mathematical Society, Providence, 1962.
[20] X. Sun, J. Du, On variable sum exdeg indices of quasi-tree graphs and unicyclic graphs, *Discr. Dynam. Natur. Soc.* **2020** (2020) #1317295.
[21] B. Vukičević, Bond additive modeling 4. QSPR and QSAR studies of the variable Adriatic indices, *Croat. Chem. Acta* **84** (2011) 87–91.
[22] D. Vukičević, Bond additive modeling 5. Mathematical properties of the variable sum exdeg index, *Croat. Chem. Acta* **84** (2011) 93–101.