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A nonlocal problem with multipoint conditions for partial differential equations of higher order

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Abstract. A nonlocal problem with multipoint conditions for the partial differential equations of higher order is considered. Algorithms for finding a solution to the nonlocal problem with multipoint conditions are constructed and their convergence is proved. Conditions for the unique solvability of the nonlocal problem with multipoint conditions for the partial differential equations of higher order are established in terms of the initial data.

1. Introduction

In recent decades, many authors have intensively studied nonlocal problems with multipoint conditions for partial differential equations of higher order (see the bibliography in [1-11]). The development of computing and information technologies requires the apply of constructive methods for the numerical analysis and approximate solution of nonlocal problems with multipoint conditions for partial differential equations of higher order. Earlier in the works of the authors, a number of problems with multipoint conditions were investigated and solved for systems of hyperbolic equations of the second order [12-14], for partial differential equations of third and fourth orders [15-18], as well as for impulsive partial differential equations of higher order [19] by Dzhumabaev's parametrization method [20].

In the present paper we propose the constructive approach for solve the nonlocal problem with multipoint conditions for partial differential equations of higher order based on method of introduction new functions.

Consider the nonlocal problem with multipoint conditions for the partial differential equations of higher order in $\Omega = [0, T] \times [0, \omega]$

$$\frac{\partial^{m+1} u}{\partial t \partial x^m} = \sum_{i=0}^{m} A_i(t, x) \frac{\partial^i u}{\partial x^i} + \sum_{j=0}^{m-1} B_j(t, x) \frac{\partial^{j+1} u}{\partial t \partial x^j} + f(t, x), \tag{1.1}$$

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$$\sum_{l=0}^{p}\sum_{i=0}^{m} K_{i,l}(x)\frac{\partial^i u(t_l,x)}{\partial x^i} = \varphi(x), \qquad x\in[0,\omega], \tag{1.2}$$

$$u(t,0)=\psi_0(t), \qquad \frac{\partial u(t,x)}{\partial x}\Big|_{x=0}=\psi_1(t),..., \qquad \frac{\partial^{m-1}u(t,x)}{\partial x^{m-1}}\Big|_{x=0}=\psi_{m-1}(t), \qquad t\in[0,T], \tag{1.3}$$

where $u(t,x)$ is unknown function, the functions $A_i(t,x)$, $i=\overline{0,m}$, $B_j(t,x)$, $j=\overline{0,m-1}$, and $f(t,x)$ are continuous on Ω, the functions $K_{i,l}(x)$ and $\varphi(x)$ are continuous on $[0,\omega]$, $0=t_0<t_1<...<t_{p-1}<t_p=T$, $i=\overline{0,m}$, $l=\overline{0,p}$, the functions $\psi_j(t)$, $j=\overline{0,m-1}$, are continuously differentiable on $[0,T]$.

A function $u(t,x)$ continuous on Ω, having continuous on Ω partial derivatives $\frac{\partial^{s+i}u}{\partial t^s\partial x^i}$, $s=0,1$, $i=\overline{0,m}$, satisfying Equation (1.1) for all $(t,x)\in\Omega$, multipoint and initial conditions (1.2), (1.3), is called the solution to the nonlocal problem with multipoint conditions (1.1)-(1.3).

Algorithms for finding a solution to the nonlocal problem with multipoint conditions (1.1)-(1.3) are constructed and their convergence is proved. Conditions for the unique solvability of the nonlocal problem with multipoint conditions (1.1)-(1.3) are established in terms of the initial data.

2. Reduction to an equivalent problem

Assume
$$v_k(t,x)=\frac{\partial^{m-k}u(t,x)}{\partial x^{m-k}}, \qquad k=\overline{1,m}.$$

Then pass from problem (1.1)-(1.3) to the next equivalent problem:

$$\frac{\partial^2 v_1}{\partial t\partial x}=A_m(t,x)\frac{\partial v_1}{\partial x}+B_{m-1}(t,x)\frac{\partial v_1}{\partial t}+A_{m-1}(t,x)v_1+\sum_{i=0}^{m-2}A_i(t,x)v_{m-i}(t,x)+\sum_{j=0}^{m-2}B_j(t,x)\frac{\partial v_{m-j}(t,x)}{\partial t}+f(t,x), \tag{2.1}$$

$$\sum_{l=0}^{p}K_{m,l}(x)\frac{\partial v_1(t_l,x)}{\partial x}+\sum_{l=0}^{p}K_{m-1,l}(x)v_1(t_l,x)+\sum_{l=0}^{p}\sum_{s=0}^{m-2}K_{s,l}(x)v_{m-s}(t_l,x)=\varphi(x), \qquad x\in[0,\omega], \tag{2.2}$$

$$v_1(t,0)=\psi_{m-1}(t), \qquad t\in[0,T], \tag{2.3}$$

$$v_r(t,x)=\psi_{m-r}(t)+\int_0^x v_{r-1}(t,\xi)d\xi, \qquad \frac{\partial v_r(t,x)}{\partial t}=\dot{\psi}_{m-r}(t)+\int_0^x\frac{\partial v_{r-1}(t,\xi)}{\partial t}d\xi, \qquad r=\overline{2,m}, \quad (t,x)\in\Omega. \tag{2.4}$$

A system of functions $(v_1(t,x),v_2(t,x),...,v_m(t,x))$, where function $v_1(t,x)\in C(\Omega,\mathbb{R})$, has partial derivatives $\frac{\partial v_1(t,x)}{\partial x}\in C(\Omega,\mathbb{R})$, $\frac{\partial v_1(t,x)}{\partial t}\in C(\Omega,\mathbb{R})$, $\frac{\partial^2 v_1(t,x)}{\partial t\partial x}\in C(\Omega,\mathbb{R})$, and functions $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$ are related to $v_1(t,x)$ by integral relations (2.4), $r=\overline{2,m}$, which satisfies the equation (2.1) for all $(t,x)\in\Omega$ and conditions (2.2), (2.3), is a solution to problem (2.1)-(2.4).

For fixed $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$, $r=\overline{2,m}$, problem (2.1)-(2.4) is a nonlocal problem with multipoint conditions for the second-order hyperbolic equation. Questions of the unique, well-posed solvability of a nonlocal problem with multipoint conditions were studied in [13], [14]. We use results of [15]-[16] to solve problem (2.1)-(2.4).

We introduce a new functions $v(t,x)=\frac{\partial v_1(t,x)}{\partial x}$, $w(t,x)=\frac{\partial v_1(t,x)}{\partial t}$, and transfer problem (2.1)-(2.4) to the following family of multipoint problems for a differential equation with functional parameters and integral constraints

$$\frac{\partial v}{\partial t}=A_m(t,x)v+B_{m-1}(t,x)w(t,x)+A_{m-1}(t,x)v_1(t,x)+\sum_{i=0}^{m-2}A_i(t,x)v_{m-i}(t,x)+\sum_{j=0}^{m-2}B_j(t,x)\frac{\partial v_{m-j}(t,x)}{\partial t}+f(t,x), \tag{2.5}$$

$$\sum_{l=0}^{p} K_{m,l}(x)v(t_l,x) = \varphi(x) - \sum_{l=0}^{p} K_{m-1,l}(x)v_1(t_l,x) - \sum_{l=0}^{p}\sum_{s=0}^{m-2} K_{s,l}(x)v_{m-s}(t_l,x), \qquad x \in [0,\omega], \tag{2.6}$$

$$v_1(t,x) = \psi_{m-1}(t) + \int_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot{\psi}_{m-1}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \qquad (t,x) \in \Omega, \tag{2.7}$$

$$v_r(t,x) = \psi_{m-r}(t) + \int_0^x v_{r-1}(t,\xi)d\xi, \qquad \frac{\partial v_r(t,x)}{\partial t} = \dot{\psi}_{m-r}(t) + \int_0^x \frac{\partial v_{r-1}(t,\xi)}{\partial t}d\xi, \qquad r = \overline{2,m}, \quad (t,x) \in \Omega. \tag{2.8}$$

For fixed $w(t,x)$, $v_1(t,x)$, $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$, $r = \overline{2,m}$, problem (2.5), (2.6) is a family of multipoint problems for the differential equations with respect to $v(t,x)$. The unknown functions $w(t,x)$, $v_1(t,x)$, $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$, $r = \overline{2,m}$, are determined from integral constraints (2.7), (2.8).

A system of functions $(v(t,x), w(t,x), v_1(t,x), v_2(t,x), ..., v_m(t,x))$, where function $v(t,x) \in C(\Omega,\mathrm{R})$ has partial derivative $\frac{\partial v(t,x)}{\partial t} \in C(\Omega,\mathrm{R})$, functions $w(t,x)$, $v_1(t,x)$, $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$ are related to $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$, $v_1(t,x)$ and $\frac{\partial v_1(t,x)}{\partial t}$ by integral constraints (2.7), (2.8), respectively, $r = \overline{2,m}$, which satisfies the differential equation (2.5) for all $(t,x) \in \Omega$ and condition (2.6), integral constraints (2.7), (2.8) is a solution to problem (2.5)-(2.8).

Consider the following family of multipoint problems for the differential equation

$$\frac{\partial v}{\partial t} = A_m(t,x)v + F(t,x), \qquad (t,x) \in \Omega, \tag{2.9}$$

$$\sum_{l=0}^{p} K_{m,l}(x)v(t_l,x) = \Phi(x), \qquad x \in [0,\omega], \tag{2.10}$$

where $v(t,x)$ is an unknown function, the function $F(t,x) \in C(\Omega,\mathrm{R})$, the function $\Phi(x)$ is continuous on $[0,\omega]$.

A continuous function $v : \Omega \to \mathrm{R}$ that has a continuous derivative with respect to t on Ω is called a solution to the family of multipoint problems (2.9), (2.10), if it satisfies equation (2.9) for all $(t,x) \in \Omega$ and multipoint condition (2.10) for all $x \in [0,\omega]$. For fixed $x \in [0,\omega]$ problem (2.9), (2.10) is a linear multipoint problem for the ordinary differential equation. Suppose a variable x takes values on the interval $[0,\omega]$; then we obtain a family of multipoint problems for differential equation.

Various problems with multipoint condition for the differential equations (2.9), (2.10) have been studied by numerous authors (see [4, 21-23] and their bibliography).

The following theorem provides conditions for the unique solvability of the family of multipoint problems for differential equations (2.9), (2.10) in terms of the solution Cauchy problems.

Theorem 2.1. *Family of multipoint problems for the differential equations (2.9), (2.10) is uniquely solvable and for its solution $v^*(t,x)$ we have the estimate*

$$\max_{t\in[0,T]} ||v^*(t,x)|| \le C_0 \max\Big\{\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big\}$$

for all $x \in [0,\omega]$, for some $C_0 > 0$ independent of x, v^*, F and Φ, if the function $M(x) = \sum\limits_{l=0}^{p} K_{m,l}(x)U(t_l,x) \neq 0$ for every $x \in [0,\omega]$, where U is a solution to the family Cauchy problems

$$\frac{\partial U}{\partial t} = A_m(t,x)U, \qquad U(0,x) = 1.$$

Proof. Let U be a solution to the family of Cauchy problems

$$\frac{\partial U}{\partial t} = A_m(t,x)U, \qquad U(0,x) = 1.$$

By the Cauchy formula [24, p. 48], the function

$$v(t,x) = U(t,x)c(x) + U(t,x)\int_0^t U^{-1}(\tau,x)F(\tau,x)d\tau \tag{2.11}$$

is a solution to equation (2.9) for each $c(x) \in C([0,\omega],\mathbb{R})$. Conversely, for each solution of this equation there exists $c(x) \in C([0,\omega],\mathbb{R})$ such that representation (2.11) holds.

Substituting representation (2.11) into (2.10), we have:

$$\sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)c(x) + \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F(\tau,x)d\tau = \Phi(x), \qquad x \in [0,\omega].$$

This implies

$$M(x)c(x) = \Phi(x) - \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F(\tau,x)d\tau, \qquad x \in [0,\omega]. \tag{2.12}$$

If the function $M(x) \neq 0$ for all $x \in [0,\omega]$, then the functional equation (2.12) has a unique solution

$$c^*(x) = M^{-1}(x)\Big\{\Phi(x) - \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F(\tau,x)d\tau\Big\}, \qquad x \in [0,\omega]. \tag{2.13}$$

Replacing $c(x)$ by $c^*(x)$ in (2.11), we obtain the following representation of the unique solution to the family of problems (2.9), (2.10)

$$v^*(t,x) = U(t,x)M^{-1}(x)\Big\{\Phi(x) - \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F(\tau,x)d\tau\Big\} + U(t,x)\int_0^t U^{-1}(\tau,x)F(\tau,x)d\tau. \tag{2.14}$$

The solution v^* satisfies the following estimate

$$\max_{t\in[0,T]} \|v^*(t,x)\| \le C_0 \max\Big(\max_{t\in[0,T]} \|F(t,x)\|, \|\Phi(x)\|\Big), \tag{2.15}$$

where the constant C_0 does not depend on F, Φ and $x \in [0,\omega]$.

The following estimate is also valid:

$$\max\Big(\max_{t\in[0,T]}\Big\|\frac{\partial v^*(t,x)}{\partial t}\Big\|, \max_{t\in[0,T]} \|v^*(t,x)\|\Big) \le \max(\alpha_m C_0 + 1, C_0) \max\Big(\max_{t\in[0,T]} \|F(t,x)\|, \|\Phi(x)\|\Big), \tag{2.16}$$

where $\alpha_m = \max_{(t,x)\in\Omega} \|A_m(t,x)\|$. Theorem 2.1 is proved. □

Therefore, provided that the function $M(x) \neq 0$ for all $x \in [0,\omega]$, the family of problems (2.9), (2.10) is uniquely solvable and for its solution v^* the estimate (2.14) holds, i.e. problem (2.9), (2.10) is well-posed.

Note that we can use a function $\exp\Big\{\int_0^t A_m(\tau,x)d\tau\Big\}$ as a function $U(t,x)$.

3. Algorithm and unique solvability of problem (1.1)–(1.3)

In this section we propose an algorithm for finding solution to the original problem (1.1)-(1.3). The algorithm for finding solutions to the nonlocal problem with multipoint conditions for the partial differential equations of higher order (1.1)-(1.3) consists of seventh stages:

1st stage. Introduction of new unknown functions $v_1(t,x)$, $v_2(t,x)$, $\ldots$, $v_m(t,x)$, and transition to the equivalent problem (2.1)-(2.4).

2nd stage. Introduction of new unknown functions $v(t,x)$, $w(t,x)$, and reduction to the family of multipoint problems for the differential equation with functional parameters and integral constraints (2.5)-(2.8).

3rd stage. Solving the auxiliary family of multipoint problems for the differential equation (2.9), (2.10).

4th stage. For fixed $w(t,x)$, $v_1(t,x)$, $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$, $r=\overline{2,m}$, solving the family of multipoint problems for the differential equation (2.5), (2.6) by solution to the auxiliary family of multipoint problems for the differential equation (2.9), (2.10).

5th stage. Determination of functions $v_1(t,x)$, $w(t,x)$ from integral constraints (2.7) using $v(t,x)$, the solution to the family of problems (2.5), (2.6), and $\frac{\partial v(t,x)}{\partial t}$.

6th stage. Determination of functions $v_r(t,x)$ and $\frac{\partial v_r(t,x)}{\partial t}$, $r=\overline{2,m}$, from integral constraints (2.8) using $v_1(t,x)$ and $\frac{\partial v_1(t,x)}{\partial t}$.

7th stage. Definition of function $u(t,x)$, the solution to the original problem (1.1)-(1.3) from equality $u(t,x)=v_m(t,x)$ for all $(t,x)\in\Omega$.

The following theorem provides the conditions of unique solvability to problem (1.1)–(1.3) in terms of the solvability of family of multipoint problems (2.9), (2.10).

Theorem 3.1. *Suppose*

1) the functions $A_i(t,x)$, $i=\overline{0,m}$, $B_j(t,x)$, $j=\overline{0,m-1}$, and $f(t,x)$ are continuous on Ω;

2) the functions the functions $K_{i,l}(x)$ and $\varphi(x)$ are continuous on $[0,\omega]$, $i=\overline{0,m}$, $l=\overline{0,p}$;

3) the functions $\psi_j(t)$, $j=\overline{0,m-1}$, are continuously differentiable on $[0,T]$;

4) The family of multipoint problems for the differential equations (2.9), (2.10) is uniquely solvable.

Then the nonlocal problem with multipoint conditions for the partial differential equations of higher order (1.1)-(1.3) has a unique solution.

Proof. Let the assumptions 1)-3) of Theorem 2.1 be satisfied. From assumption 4) it follows that unique solvability to the family of multipoint problems for the differential equations (2.9), (2.10). From the nonlocal multipoint problem (1.1)-(1.3) we transfer to the equivalent family of multipoint problems for the differential equation with functional parameters and integral constraints (2.5)-(2.8).

To find a solution to the problem (2.5)-(2.8) we use an iterative method and the algorithm.

0 step. Suppose that $v_1(t,x)=\psi_{m-1}(t)$, $w(t,x)=\dot\psi_{m-1}(t)$, $v_r(t,x)=\psi_{m-r}(t)$, $\frac{\partial v_r(t,x)}{\partial t}=\dot\psi_{m-r}(t)$, $r=\overline{2,m}$, for all $(t,x)\in\Omega$ in the right-hand side of equation (2.5) and condition (2.6). Then, we get the following problem:

$$\frac{\partial v}{\partial t}=A_m(t,x)v+F^{(0)}(t,x),\qquad (t,x)\in\Omega, \tag{3.1}$$

$$\sum_{l=0}^{p}K_{m,l}(x)v(t_l,x)=\Phi^{(0)}(x),\qquad x\in[0,\omega], \tag{3.2}$$

where $F^{(0)}(t,x)=B_{m-1}(t,x)\dot\psi_{m-1}(t)+A_{m-1}(t,x)\psi_{m-1}(t)+\sum\limits_{i=0}^{m-2}A_i(t,x)\psi_i(t)+\sum\limits_{j=0}^{m-2}B_j(t,x)\dot\psi_j(t)+f(t,x)$,

$\Phi^{(0)}(x)=\varphi(x)-\sum\limits_{l=0}^{p}K_{m-1,l}(x)\psi_{m-1}(t_l)-\sum\limits_{l=0}^{p}\sum\limits_{s=0}^{m-2}K_{s,l}(x)\psi_s(t_l)$.

Using assumption 4) we obtain of the unique solvability to the problem (2.9), (2.10) with $F(t,x)=F^{(0)}(t,x)$, $\Phi(x)=\Phi^{(0)}(x)$.

From assertion of Theorem 2.1 we have the following representation of the unique solution to the family of problems (3.1), (3.2):

$$v^{(0)}(t,x) = U(t,x)M^{-1}(x)\Big\{\Phi^{(0)}(x) - \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau\Big\}$$

$$+U(t,x)\int_0^t U^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau, \qquad (t,x)\in\Omega. \tag{3.3}$$

Here $U(t,x) = \exp\Big\{\int\limits_0^t A_m(\tau,x)d\tau\Big\}$.

The solution $v^{(0)}(t,x)$ satisfies the following estimate

$$\max_{t\in[0,T]} \|v^{(0)}(t,x)\| \le C_0 \max\Big(\max_{t\in[0,T]} \|F^{(0)}(t,x)\|, \|\Phi^{(0)}(x)\|\Big), \tag{3.4}$$

where the constant C_0 does not depend on $F^{(0)}$, $\Phi^{(0)}$ and $x\in[0,\omega]$.

Moreover, we can find its expression:

$$C_0 = e^{\alpha_m T}\max_{x\in[0,\omega]}\Big[\sum_{l=0}^{p}|K_{m,l}(x)|e^{\int_0^{t_l} A_m(\tau,x)d\tau}\Big]^{-1}\Big(1+\sum_{l=0}^{p}\max_{x\in[0,\omega]}|K_{m,l}(x)|t_l e^{\alpha_m t_l}\Big) + Te^{\alpha_m T}.$$

The following estimate is also valid:

$$\max\Big(\max_{t\in[0,T]}\Big\|\frac{\partial v^{(0)}(t,x)}{\partial t}\Big\|, \max_{t\in[0,T]}\|v^{(0)}(t,x)\|\Big) \le \max(\alpha_m C_0 + 1, C_0)\max\Big(\max_{t\in[0,T]}\|F^{(0)}(t,x)\|, \|\Phi^{(0)}(x)\|\Big). \tag{3.5}$$

Further, we assume that $v(t,\xi) = v^{(0)}(t,\xi)$, $\frac{\partial v(t,\xi)}{\partial t} = \frac{\partial v^{(0)}(t,\xi)}{\partial t}$, for all $(t,\xi)\in\Omega$ in integral relations (2.7) and determine $v_1^{(0)}(t,x)$ and $w^{(0)}(t,x)$:

$$v_1^{(0)}(t,x) = \psi_{m-1}(t) + \int_0^x v^{(0)}(t,\xi)d\xi, \qquad w^{(0)}(t,x) = \dot{\psi}_{m-1}(t) + \int_0^x \frac{\partial v^{(0)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega. \tag{3.6}$$

At the next stage, using $v_1^{(0)}(t,x)$ and $\frac{\partial v_1^{(0)}(t,x)}{\partial t}$ we sequentially find the functions $v_r^{(0)}(t,x)$ and $\frac{\partial v_r^{(0)}(t,x)}{\partial t}$, $r=\overline{2,m}$, from integral constraints (2.8):

$$v_2^{(0)}(t,x) = \psi_{m-2}(t) + \int_0^x v_1^{(0)}(t,\xi)d\xi, \qquad \frac{\partial v_2^{(0)}(t,x)}{\partial t} = \dot{\psi}_{m-2}(t) + \int_0^x \frac{\partial v_1^{(0)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.7}$$

$$v_3^{(0)}(t,x) = \psi_{m-3}(t) + \int_0^x v_2^{(0)}(t,\xi)d\xi, \qquad \frac{\partial v_3^{(0)}(t,x)}{\partial t} = \dot{\psi}_{m-3}(t) + \int_0^x \frac{\partial v_2^{(0)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.8}$$

...

$$v_m^{(0)}(t,x) = \psi_0(t) + \int_0^x v_{m-1}^{(0)}(t,\xi)d\xi, \qquad \frac{\partial v_m^{(0)}(t,x)}{\partial t} = \dot{\psi}_0(t) + \int_0^x \frac{\partial v_{m-1}^{(0)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega. \tag{3.9}$$

Finally, we define a function $u^{(0)}(t,x)$ by the following equality:

$$u^{(0)}(t,x) = v_m^{(0)}(t,x), \qquad (t,x) \in \Omega, \tag{3.10}$$

According to the algorithm above, the function $u^{(0)}(t,x)$ is an initial approximation of solution to the original problem (1.1)-(1.3).

On the first step of the iterative method, we suppose that $v_1(t,x) = v_1^{(0)}(t,x)$, $w(t,x) = w^{(0)}(t,x)$, $v_r(t,x) = v_r^{(0)}(t,x)$, $\frac{\partial v_r(t,x)}{\partial t} = \frac{\partial v_r^{(0)}(t,x)}{\partial t}$, $r = \overline{2,m}$, for all $(t,x) \in \Omega$ in the right-hand side of equation (2.5) and condition (2.6). Then, we get the following problem:

$$\frac{\partial v}{\partial t} = A_m(t,x)v + F^{(1)}(t,x), \qquad (t,x) \in \Omega, \tag{3.11}$$

$$\sum_{l=0}^{p} K_{m,l}(x)v(t_l,x) = \Phi^{(1)}(x), \qquad x \in [0,\omega], \tag{3.12}$$

where

$$F^{(1)}(t,x) = B_{m-1}(t,x)w^{(0)}(t,x) + A_{m-1}(t,x)v_1^{(0)}(t,x) + \sum_{i=0}^{m-2} A_i(t,x)v_{m-i}^{(0)}(t,x) + \sum_{j=0}^{m-2} B_j(t,x)\frac{\partial v_{m-j}^{(0)}(t,x)}{\partial t} + f(t,x),$$

$$\Phi^{(1)}(x) = \varphi(x) - \sum_{l=0}^{p} K_{m-1,l}(x)v_1^{(0)}(t_l,x) - \sum_{l=0}^{p}\sum_{s=0}^{m-2} K_{s,l}(x)v_{m-s}^{(0)}(t_l,x).$$

Using assumption 4) again, we obtain of the unique solvability to the problem (2.9), (2.10) with $F(t,x) = F^{(1)}(t,x)$, $\Phi(x) = \Phi^{(1)}(x)$.

From assertion of Theorem 2.1 we have the following representation of the unique solution to the family of problems (3.11), (3.12):

$$v^{(1)}(t,x) = U(t,x)M^{-1}(x)\Big\{\Phi^{(1)}(x) - \sum_{l=0}^{p} K_{m,l}(x)U(t_l,x)\int_0^{t_l} U^{-1}(\tau,x)F^{(1)}(\tau,x)d\tau\Big\}$$

$$+U(t,x)\int_0^{t} U^{-1}(\tau,x)F^{(1)}(\tau,x)d\tau, \qquad (t,x) \in \Omega. \tag{3.13}$$

The solution $v^{1)}(t,x)$ satisfies the following estimate

$$\max_{t\in[0,T]} \|v^{(1)}(t,x)\| \le C_0 \max\Big(\max_{t\in[0,T]} \|F^{(1)}(t,x)\|, \|\Phi^{(1)}(x)\|\Big). \tag{3.14}$$

The following estimate is also valid:

$$\max\Big(\max_{t\in[0,T]}\Big\|\frac{\partial v^{(1)}(t,x)}{\partial t}\Big\|, \max_{t\in[0,T]} \|v^{(1)}(t,x)\|\Big) \le \max(\alpha_m C_0 + 1, C_0)\max\Big(\max_{t\in[0,T]} \|F^{(1)}(t,x)\|, \|\Phi^{(1)}(x)\|\Big). \tag{3.15}$$

Further, we assume that $v(t,\xi) = v^{(1)}(t,\xi)$, $\frac{\partial v(t,\xi)}{\partial t} = \frac{\partial v^{(1)}(t,\xi)}{\partial t}$, for all $(t,\xi) \in \Omega$ in integral relations (2.7) and determine $v_1^{(1)}(t,x)$ and $w^{(1)}(t,x)$:

$$v_1^{(1)}(t,x) = \psi_{m-1}(t) + \int_0^x v^{(1)}(t,\xi)d\xi, \qquad w^{(1)}(t,x) = \dot{\psi}_{m-1}(t) + \int_0^x \frac{\partial v^{(1)}(t,\xi)}{\partial t}d\xi, \qquad (t,x) \in \Omega. \tag{3.16}$$

At the next stage, using $v_1^{(1)}(t,x)$ and $\frac{\partial v_1^{(1)}(t,x)}{\partial t}$ we sequentially find the functions $v_r^{(1)}(t,x)$ and $\frac{\partial v_r^{(1)}(t,x)}{\partial t}$, $r=\overline{2,m}$, from integral constraints (2.8):

$$v_2^{(1)}(t,x)=\psi_{m-2}(t)+\int\limits_0^x v_1^{(1)}(t,\xi)d\xi, \qquad \frac{\partial v_2^{(1)}(t,x)}{\partial t}=\dot{\psi}_{m-2}(t)+\int\limits_0^x \frac{\partial v_1^{(1)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.17}$$

$$v_3^{(1)}(t,x)=\psi_{m-3}(t)+\int\limits_0^x v_2^{(1)}(t,\xi)d\xi, \qquad \frac{\partial v_3^{(1)}(t,x)}{\partial t}=\dot{\psi}_{m-3}(t)+\int\limits_0^x \frac{\partial v_2^{(1)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.18}$$

...

$$v_m^{(1)}(t,x)=\psi_0(t)+\int\limits_0^x v_{m-1}^{(1)}(t,\xi)d\xi, \qquad \frac{\partial v_m^{(1)}(t,x)}{\partial t}=\dot{\psi}_0(t)+\int\limits_0^x \frac{\partial v_{m-1}^{(1)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega. \tag{3.19}$$

Finally, we define a function $u^{(1)}(t,x)$ by the following equality:

$$u^{(1)}(t,x)=v_m^{(1)}(t,x), \qquad (t,x)\in\Omega, \tag{3.20}$$

According to the algorithm above, the function $u^{(1)}(t,x)$ is a first approximation of solution to the original problem (1.1)-(1.3).

And so on.

On the kth step of the iterative method, we suppose that $v_1(t,x)=v_1^{(k-1)}(t,x)$, $w(t,x)=w^{(k-1)}(t,x)$, $v_r(t,x)=v_r^{(k-1)}(t,x)$, $\frac{\partial v_r(t,x)}{\partial t}=\frac{\partial v_r^{(k-1)}(t,x)}{\partial t}$, $r=\overline{2,m}$, for all $(t,x)\in\Omega$ in the right-hand side of equation (2.5) and condition (2.6). Then, we get the following problem:

$$\frac{\partial v}{\partial t}=A_m(t,x)v+F^{(k)}(t,x), \qquad (t,x)\in\Omega, \tag{3.21}$$

$$\sum_{l=0}^{p}K_{m,l}(x)v(t_l,x)=\Phi^{(k)}(x), \qquad x\in[0,\omega], \tag{3.22}$$

where

$$F^{(k)}(t,x)=B_{m-1}(t,x)w^{(k-1)}(t,x)+A_{m-1}(t,x)v_1^{(k-1)}(t,x)+\sum_{i=0}^{m-2}A_i(t,x)v_{m-i}^{(k-1)}(t,x)+\sum_{j=0}^{m-2}B_j(t,x)\frac{\partial v_{m-j}^{(k-1)}(t,x)}{\partial t}+f(t,x),$$

$$\Phi^{(k)}(x)=\varphi(x)-\sum_{l=0}^{p}K_{m-1,l}(x)v_1^{(k-1)}(t_l,x)-\sum_{l=0}^{p}\sum_{s=0}^{m-2}K_{s,l}(x)v_{m-s}^{(k-1)}(t_l,x).$$

Using assumption 4) again, we obtain of the unique solvability to the problem (2.9), (2.10) with $F(t,x)=F^{(k)}(t,x)$, $\Phi(x)=\Phi^{(k)}(x)$.

We have the representation of the unique solution to the family of problems (3.21), (3.22) in the following form:

$$v^{(k)}(t,x)=U(t,x)M^{-1}(x)\Big\{\Phi^{(k)}(x)-\sum_{l=0}^{p}K_{m,l}(x)U(t_l,x)\int\limits_0^{t_l}U^{-1}(\tau,x)F^{(k)}(\tau,x)d\tau\Big\}$$

$$+U(t,x)\int\limits_0^{t}U^{-1}(\tau,x)F^{(k)}(\tau,x)d\tau, \qquad (t,x)\in\Omega. \tag{3.23}$$

The solution $v^{k)}(t,x)$ satisfies the following estimate

$$\max_{t\in[0,T]}\|v^{(k)}(t,x)\| \le C_0 \max\Big(\max_{t\in[0,T]}\|F^{(k)}(t,x)\|, \|\Phi^{(k)}(x)\|\Big). \tag{3.24}$$

The next estimate is also valid:

$$\max\Big(\max_{t\in[0,T]}\Big\|\frac{\partial v^{(k)}(t,x)}{\partial t}\Big\|, \max_{t\in[0,T]}\|v^{(k)}(t,x)\|\Big) \le \max(\alpha_m C_0+1, C_0)\max\Big(\max_{t\in[0,T]}\|F^{(k)}(t,x)\|, \|\Phi^{(k)}(x)\|\Big). \tag{3.25}$$

Further, we assume that $v(t,\xi) = v^{(k)}(t,\xi)$, $\frac{\partial v(t,\xi)}{\partial t} = \frac{\partial v^{(k)}(t,\xi)}{\partial t}$, for all $(t,\xi)\in\Omega$ in integral relations (2.7) and determine $v_1^{(k)}(t,x)$ and $w^{(k)}(t,x)$:

$$v_1^{(k)}(t,x) = \psi_{m-1}(t) + \int_0^x v^{(k)}(t,\xi)d\xi, \qquad w^{(k)}(t,x) = \dot\psi_{m-1}(t) + \int_0^x \frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega. \tag{3.26}$$

Now, using $v_1^{(k)}(t,x)$ and $\frac{\partial v_k^{(1)}(t,x)}{\partial t}$ we sequentially find the functions $v_r^{(k)}(t,x)$ and $\frac{\partial v_r^{(k)}(t,x)}{\partial t}$, $r=\overline{2,m}$, from integral constraints (2.8):

$$v_2^{(k)}(t,x) = \psi_{m-2}(t) + \int_0^x v_1^{(k)}(t,\xi)d\xi, \qquad \frac{\partial v_2^{(k)}(t,x)}{\partial t} = \dot\psi_{m-2}(t) + \int_0^x \frac{\partial v_1^{(k)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.27}$$

$$v_3^{(k)}(t,x) = \psi_{m-3}(t) + \int_0^x v_2^{(k)}(t,\xi)d\xi, \qquad \frac{\partial v_3^{(k)}(t,x)}{\partial t} = \dot\psi_{m-3}(t) + \int_0^x \frac{\partial v_2^{(k)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega, \tag{3.28}$$

$$\dots$$

$$v_m^{(k)}(t,x) = \psi_0(t) + \int_0^x v_{m-1}^{(k)}(t,\xi)d\xi, \qquad \frac{\partial v_m^{(k)}(t,x)}{\partial t} = \dot\psi_0(t) + \int_0^x \frac{\partial v_{m-1}^{(k)}(t,\xi)}{\partial t}d\xi, \qquad (t,x)\in\Omega. \tag{3.29}$$

Finally, we define a function $u^{(k)}(t,x)$ by the following equality:

$$u^{(k)}(t,x) = v_m^{(k)}(t,x), \qquad (t,x)\in\Omega, \tag{3.30}$$

According to the algorithm above, the function $u^{(k)}(t,x)$ is a kth approximation of solution to the original problem (1.1)-(1.3), $k=1,2,\dots$.

Convergence of the functional sequences $\{v^{(k)}(t,x)\}$, $\{\frac{\partial v^{(k)}(t,\xi)}{\partial t}\}$, $\{v_1^{(k)}(t,x)\}$, $\{w^{(k)}(t,x)\}$, $\{v_r^{(k)}(t,x)\}$, $\{\frac{\partial v_r^{(k)}(t,x)}{\partial t}\}$, $r=\overline{2,m}$, are established similarly in [AssTok].

Therefore, the sequence $\{u^{(k)}(t,x)\}$ converges to $u^*(t,x)$ as $k\to\infty$ for all $(t,x)\in\Omega$. In this case, the limit function $u^*(t,x)$ are continuous on Ω. Moreover, there exist partial derivatives $\frac{\partial^{s+i}u^*(t,x)}{\partial t^s\partial x^i}$, $s=0,1$, $i=\overline{0,m}$, are continuous on Ω. So, we found a solution to the problem (1.1)-(1.3).

The uniqueness of the solution to problem (1.1)-(1.3) is established by the method of contradiction. The theorem is proved. $\square$

We can see the sufficient conditions for the unique solvability of nonlocal multipoint problem (1.1)–(1.3) are established in terms of the initial data.

Conclusion. In the paper is investigated the nonlocal problem with multipoint conditions for the partial differential equations of higher order (1.1)-(1.3). Algorithms for finding a solution to the nonlocal problem with multipoint conditions are constructed and their convergence is proved. Conditions for the unique solvability of the nonlocal problem with multipoint conditions for the partial differential equations of higher order are established in the terms of unique solvability to the family of multipoint problems for the differential equations (2.9), (2.10). In the future, using fundamental matrix of family problems for the system of differential equations, the results of the paper will be developed to the nonlocal problem with multipoint conditions for the system of partial differential equations of higher order. The questions of the existence of new general solutions [25, 26] of the above classes of problems will be investigated.

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