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Isomorphism between L-valued tree transition systems and upper semilattice

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Abstract. This study aims to investigate the characterization of algebraic concepts of subsystem, retrievability and connectivity of an L-valued tree transition system based on its related layers. It also seeks to associate upper semilattices with L-valued tree transition systems. Further, a decomposition of an L-valued tree transition system is provided in terms of its layers. In addition, it proposes a construction of an L-valued tree transition system which corresponds to a given finite poset. Then an isomorphism is established between the poset of class of subsystem of an L-valued tree transition system and an upper semilattice.

1. Introduction

Tree automata, as it has been thoroughly recognized, play an important role in the field of computer Science. A tree automaton is regarded as a type of state machine which deals with tree structures, rather than the strings of more conventional state machines. The tree automata theory was first studied by Doner [8, 9] and Thatcher and Wright [30]. The definitions proposed for tree automata and tree languages have been on the basis of the algebraic techniques and it contains the heavy use of category theory and universal algebra. One of the most significant applications of tree automata has been related to decidability results in logic [7]. Lately, tree automata have been investigated and accordingly applied in abstract interpretation by researchers utilizing it in set constraints, rewriting, automated theorem proving and also program verification [12, 14]. Residuated lattice-valued logic has been proposed by Pavelka [21–23] using it as an algebraic structure for fuzzy logic. In addition, residuated lattices have been considered as significant algebras enclosing close connections with other main algebras [21, 24, 25, 27, 28].

Algebraic study of tree transition system plays a key role in human reasoning involving hierarchies. Some of its application areas, covering situations or systems (i) with precise natures are; formal concept analysis, category theory, logic, topology and logic, (ii) with imprecise or uncertain natures are; mathematical morphology, fuzzy transform, soft computing, and (iii) with vague natures are; data analysis, reasoning having incomplete information. Introducing complete residuated lattice valued logic; Ying [38] proposed L-fuzzifying topology. He further extended some of the reported results which were obtained in Ying [36, 37, 39]. Subsequently, a fundamental framework of automata theory was established by Qiu

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[24, 25, 27]. This framework was based on complete residuated lattice valued logic which accordingly generalized some of the findings in fuzzy finite automata investigated by Malik and Mordeson [18], Mordeson and Malik [20] and Qiu [26]. Furthermore, Qiu [25] examined homomorphisms relations which existed between two L-valued automata and continuous mappings and open mappings. Recently, lattice valued automata have been studied by several other researchers (see e.g. Abolpur and Zahedi [1–4], Lei and Li [15], Li and Pedrycz [16], Mockor [19], Qiu [27], Wu and Qiu [32], Xing et al. [33, 34] and Xing and Qiu [35]. Lu et al. [17] and Shang and Lu [29] also examined automata theory based on lattice-ordered semirings.

The concept of fuzzy tree automata has been studied by many authors and researchers in the field. In their investigation, Inagaki and Fukumura [11] examined fuzzy tree automaton as a particular case of weighted tree automaton at which formal tree series have been accepted over a complete semiring. On the other hand, Mordeson and Malik [20] introduced a fuzzy tree automaton as an acceptor of a fuzzy dendro-language. Fuzzy tree automata were also scrutinized with membership in a distributive lattice by Esik and Liu [10] Having defined fuzzy recognizable tree language, they derived a Kleene theorem for fuzzy tree languages. Recently, tree automata were studied based on complete residuated lattice valued logic by Ghorani and Zahedi [13] As their findings revealed, a pumping lemma was obtained for L-valued tree automata. In another study, they also examined the behavior of L-valued tree automata [13]. Further, they provided a minimization algorithm for lattice valued tree automata and subsequently analyzed its time complexity.

The present study is an attempt to introduce the new concept of "layer" for an L-valued tree transition system in order to reinforce the algebraic study of L-valued tree transition systems. In section 3, in specific, the characterization of some algebraic concepts such as subsystem, retrievability and connectivity of an L-valued tree transition system are introduced in terms of its layers. Further, this study demonstrates that the maximal layer of a cyclic L-valued tree transition system and minimal layer of a directable L-valued tree transition system are distinct and unique. Finally, a decomposition of an L-valued tree transition system is provided in terms of its layers. In section 4, the relationship between L-valued tree transition systems and upper semilattices is introduced and studied. It also offers an isomorphism between the partially ordered set (poset) of class of subsystem of an L-valued tree transition system and an upper semilattice.

2. Preliminaries

In this section, the concepts of L-valued tree transition systems and lattices are introduced. The understanding of related notions is therefore required in the subsequent sections.

Definition 2.1. ([6]) A ranked alphabet is a couple $(F, Arity)$ where F is a finite set and $Arity$ is a mapping from F into N (the set of positive integers). Whenever $Arity$ is clear from the context, we simply drop it. The arity of a symbol $f \in F$ is $Arity(f)$. The set of symbols of arity n is denoted by F_n. Elements of arity $0, 1, \ldots, n$ are respectively called constants, unary, ..., n-ary symbols. Here, we assume that F contains at least one constant. In the here, we use parenthesis and commas for a short declaration of symbols with arity. For instance, $f(,)$ is a short declaration for a binary symbol f.

Definition 2.2. ([6]) A tree transition system (TA) is a tuple $\mathcal{A} = (Q, F, \delta)$, where Q is a set of (unary) states, F is a ranked alphabet and δ is a set of transition rules of the following type:

$$f(q_1(x_1), \ldots, q_n(x_n)) \to q(f(x_1, \ldots, x_n)),$$

where $n \geq 0, f \in F_n, q, q_1, \ldots, q_n \in Q, x_1, \ldots, x_n \in X$, and X is a set of constants, called variables, which is separate from F_0.

Definition 2.3. ([13]) A triple $\mathcal{A} = (Q, F, \delta)$ is called an L-valued tree transition system, where

(i) Q is a set of states,

(ii) F is a ranked alphabet,

(iii) for each $n \geq 0$, δ_n is an L-valued set on $Q \times Q^n \times F_n$ i.e. a mapping from $Q \times Q^n \times F_n$ to L.

Remark 2.4. ([13]) The family of L-valued subsets $\delta = (\delta_n)_{n\geq 0}$ is called the transition. We will usually write δ for δ_n.

Definition 2.5. ([31]) Let $(P, \preceq)$ be a poset, $a, b \in P$ and $a \neq b$. Then a is called predecessor of b and b is called successor of a if $a \preceq c \preceq b$ and $c \in P$ imply $c = a$ or $c = b$. Also, $a \in P$ is called minimal if $b \preceq a$ and $b \in P$ imply $b = a$. Similarly, $b \in P$ is called maximal if $b \preceq a$ and $a \in P$ imply $a = b$.

Definition 2.6. ([31]) A poset $(P, \preceq)$ is called an upper semilattice if for all $a, b \in P$ there exists the least upper bound of a and b. An upper semilattice $(P, \preceq)$ is called a tree if for any two incomparable elements $b, c \in P$, there is no element $a \in P$ such that $a \preceq b$ and $a \preceq c$.

Definition 2.7. ([31]) An isomorphism from a poset $(P_1, \preceq_1)$ to a poset $(P_2, \preceq_2)$ is a bijective map $f : P_1 \to P_2$ such that for all $a, b \in P$, $a \preceq_1 b$ implies $f(a) \preceq_2 f(b)$. Throughout, $(P_1, \preceq_1) \cong (P_2, \preceq_2)$ denotes that the poset $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are isomorphic, i.e., there exists an isomorphism from poset $(P_1, \preceq_1)$ to poset $(P_2, \preceq_2)$.

Definition 2.8. ([31]) Let n be a positive integer. Then a finite upper semilattice $\mathcal{L}(n)$ is an upper semilattice such that $\mathcal{L}(n) \cong (P(\{1, 2, \dots, n\}), \subseteq)$, where $P(\{1, 2, \dots, n\})$ is the set of all subsets of $\{1, 2, \dots, n\}$ and $\subseteq$ is the inclusion relation on $P(\{1, 2, \dots, n\})$.

Definition 2.9. ([31]) Let $\mathcal{P}_1 = (P_1, \preceq_1)$ and $\mathcal{P}_2 = (P_2, \preceq_2)$ be two finite posets with $P_1 \cap P_2 = \emptyset$. Also, let B be the set of all maximal elements of $\mathcal{P}_1$ and C be the set of all minimal elements of $\mathcal{P}_2$ such that for any $b \in B$ there exists a nonempty subset C_b of C with $\cup_{b\in B} C_b = C$. Then $\oplus$-composition of posets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a poset $\mathcal{P}_1 \oplus \mathcal{P}_2 = (P_1 \cup P_2, \preceq)$ such that

(i) for any $i = 1, 2$ and $a, b \in P_i$, $a \preceq b$ if $a \preceq_i b$, and

(ii) for any $b \in B$ and $c \in C_b$, $b \preceq c$.

3. Layers of L-valued tree transition systems

In this section, the concept of a layer of an L-valued tree transition system is introduced. Accordingly, it is shown that the layers play a very important role in the algebraic study of L-valued tree transition systems by characterizing the concepts of subsystems and separated subsystems of an L-valued tree transition system in terms of its layers. In addition, it is shown that for each cyclic L-valued tree transition system there is a unique maximal layer and for each directable L-valued tree transition system there is a distinctive and unique minimal layer. Finally, a decomposition of an L-valued tree transition system is provided.

In the present study, we let $L = [0, 1]$.

Remark 3.1. Let $(Q^n)^*$ be the free monoid generated by Q^n with identity element $Q^o = \emptyset \in (Q^n)^*$. Then the L-valued transition δ is extended to a map $\delta : Q \times (Q^n)^* \times F_n \to L$ such that $\forall (q_1, \dots, q_n), (p_1, \dots, p_n) \in Q^n$, $\forall \beta \in F_n$, and $\forall \gamma \in F_0$,

$$\delta(q, \phi, \gamma) = \delta(q, \gamma) = 1,$$
$$\delta(q, (q_1, \dots, q_n).(p_1, \dots, p_n), \beta) = \vee\{\delta(q, (q_1, \dots, q_n), \beta) \wedge \delta(p, (p_1, \dots, p_n), \beta) | p \in Q\}.$$

Definition 3.2. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and R be an equivalence relation on Q. We say that $R \in \mathcal{L}$, where $\mathcal{L}$ is an L-valued set of compatible relation, if:

$$\forall (p, q) \in R,\ \forall (q_1, \dots, q_n) \in Q^n, \forall \sigma \in F_n$$
$$\delta(p, (q_1, \dots, q_n), \sigma) > 0 \Rightarrow \exists (p_1, \dots, p_n) \in Q^n \ s.t.$$
$$\delta(q, (p_1, \dots, p_n), \sigma) > 0 \text{ and } (q_i, p_i) \in R\ \forall i.$$

Remark 3.3. From $R \in \mathcal{L}$ in the preceding definition, we can conclude $\exists (t_1, \dots, t_n) \in Q^n$ such that $\delta(p, (q_1, \dots, q_n), \sigma) \leq \delta(q, (t_1, \dots, t_n), \sigma)$, and $(q_i, t_i) \in R\ \forall i$.

Definition 3.4. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and $B \subseteq Q$. The predecessor and the successor of B are respectively the sets

$$\mathcal{P}_Q(B) = \{p \in Q | \exists R \in \mathcal{L}, q \in B, (p, q) \in R\}, and$$

$$\mathcal{S}_Q(B) = \{q \in Q | \exists R \in \mathcal{L}, p \in B, (p, q) \in R\}.$$

We shall frequently write $\mathcal{P}_Q(B)$ and $\mathcal{S}_Q(B)$ as just $\mathcal{P}(B)$ and $\mathcal{S}(B)$ and $\mathcal{P}(\{q\})$ and $\mathcal{S}(\{q\})$ as just $\mathcal{P}(q)$ and $\mathcal{S}(q)$.

Proposition 3.5. *Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and $B \subseteq Q$. Then $\mathcal{S}(Q - B) = Q - B$ if and only if $\mathcal{P}(B) = B$.*

Proof. It is clear by Definition 3.4. □

Definition 3.6. An L-valued tree transition system $\mathcal{B} = (Q', E, \lambda)$ is called a subsystem of an L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$ if $(Q' \subseteq Q)$, $\mathcal{S}(Q') = Q'$ and $\delta|_{Q' \times Q'^n \times E_n} = \lambda$. Further, this subsystem is called separated if $\mathcal{S}(Q - Q') \cap Q' = \emptyset$.

Definition 3.7. An L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$ is called:

(i) strongly connected if $\forall p, q \in Q, p \in \mathcal{S}(q)$, and

(ii) retrievable if $\delta(q, (q_1, \ldots, q_n), \sigma) > 0$ for some $(q, (q_1, \ldots, q_n), \sigma) \in Q \times Q^n \times F_n \Rightarrow \exists R \in \mathcal{L}, p \in Q$ s.t. $(q, p) \in R$, $\delta(p, (p_1, \ldots, p_n), \sigma) > 0$ and $(q_i, p_i) \in R \forall i$.

Definition 3.8. An L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$ is called cyclic if for all $p \in Q$, there exists $R \in \mathcal{L}, q \in Q$ such that $(q, p) \in R$.

Definition 3.9. A homomorphism from an L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$ to an L-valued tree transition system $\mathcal{B} = (Q', E, \lambda)$ is a pair (f, g) of maps, where $f : Q \to Q'$ and $g : F \to E$ are functions such that $\forall (q, (q_1, \ldots, q_n), \sigma) \in Q \times Q^n \times F_n$, $\delta(q, (q_1, \ldots, q_n), \sigma) \leq \lambda(f(q), (f(q_1), \ldots, f(q_n)), g(\sigma))$.

Remark 3.10. In the above mentioned definition, if $F = E$ and g is the identity map on F, then we say that f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Definition 3.11. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and $R \in \mathcal{L}$. For $p \in Q$, we call the set $L_p = \{q \in Q | q \in \mathcal{S}(p)\}$ a layer of $\mathcal{A}$.

For two layers L_p and L_q of Q, define $L_p \leq L_q$ if $\delta(q, (q_1, \ldots, q_n), \sigma) \leq \delta(p, (t_1, \ldots, t_n), \sigma)$ for some $(q_1, \ldots, q_n), (t_1, \ldots, t_n) \in Q^n$ and $\sigma \in F_n$. It is easy to see that $\leq$ is a partial order. By E, we mean $(\{L_p : p \in Q\}, \leq)$, which is obviously a poset.

Proposition 3.12. *Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system. Then:*

(i) if $\mathcal{A}$ is retrievable, then for all $q \in Q$, $\mathcal{S}(q)$ is a layer of $\mathcal{A}$, and

(ii) if $\mathcal{A}$ is strongly connected, then Q itself is a layer of $\mathcal{A}$.

Proof. It follows from the definition of retrievable and strongly connected L-valued tree transition systems. □

Proposition 3.13. *Let $E = \{L_p : p \in Q\}$ be the set of all layers of an L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$. Then $\mathcal{B} = (Q', E, \lambda)$ is a subsystem of $\mathcal{A}$ if and only if*

(i) $\exists L_{p_1}, \ldots, L_{p_r} \in E$ such that $Q' = \{q \in Q | L_q \leq L_{p_i}$, for some $i \in \{1, 2, \ldots, r\}\}$, and

(ii) $\lambda(q, (q_1, \ldots, q_n), \sigma) = \delta(q, (q_1, \ldots, q_n), \sigma), \forall q \in Q', (q_1, \ldots, q_n) \in Q'^n$ and $\sigma \in F_n$.

Proof. Let $\mathcal{B} = (Q', E, \lambda)$ be a subsystem of $\mathcal{A}$. Then $Q' \subseteq Q$, $\mathcal{S}(Q') = Q'$ and $\lambda = \delta|_{Q' \times Q'^n \times E_n}$. Now, $\mathcal{S}(Q') = Q' \Rightarrow Q' = \{q \in Q | \exists R \in \mathcal{L}, p \in Q', (p, q) \in R\}$, or that $\exists L_{p_i} \in E = \{L_p | p \in Q\}$ such that $Q' = \{q \in Q | L_p \leq L_{p_i}\}$, i.e., $\exists L_{p_1}, L_{p_2}, \ldots, L_{p_r} \in E$ such that $Q' = \{q \in Q | L_q \leq L_{p_i}$, for some $i \in \{1, 2, \ldots, r\}\}$. Also, as $\lambda = \delta|_{Q' \times Q'^n \times E_n}$, (ii) follows obviously.

Conversely, let condition (i) and (ii) be held. To show that $\mathcal{B}$ is a subsystem of $\mathcal{A}$, it is sufficient to show that $\mathcal{S}(Q') \subseteq Q'$. For this, let $q \in \mathcal{S}(Q')$. Then there exist $p \in Q'$ and $R \in \mathcal{L}$ such that $(p,q) \in R$. Now, $p \in Q'$ implies that $L_p \leq L_{p_i}$, for some $i \in \{1,2,\ldots,r\}$, i.e., $\delta(p_i,(q_1,\ldots,q_n),\sigma) \leq \delta(p,(t_1,\ldots,t_n),\sigma)$ for some $(q_1,\ldots,q_n),(t_1,\ldots,t_n) \in Q^n$ and $\sigma \in F_n$. Also, $(p,q) \in R$ implies that $\delta(p,(p_1,\ldots,p_n),\gamma) \leq \delta(q,(r_1,\ldots,r_n),\gamma)$, i.e., $L_q \leq L_p$, then $L_q \leq L_{p_i}$, or that $q \in Q'$. Thus, $\mathcal{S}(Q') \subseteq Q'$. □

Proposition 3.14. *Let $E = \{L_p : p \in Q\}$ be the set of all layers of an L-valued tree transition system $\mathcal{A} = (Q,F,\delta)$. Then $\mathcal{B} = (Q',E,\lambda)$ is a separated subsystem of $\mathcal{A}$ if and only if*

(i) $\exists L_{p_1},\ldots,L_{p_r} \in E$ such that $Q' = \{q \in Q | L_q \leq L_{p_i}$, and $L_{p_j} \leq L_q$ for some $i,j \in \{1,2,\ldots,r\}\}$, and

(ii) $\lambda(q,(q_1,\ldots,q_n),\sigma) = \delta(q,(q_1,\ldots,q_n),\sigma), \forall q \in Q', (q_1,\ldots,q_n) \in Q'^n$ and $\sigma \in E_n$.

Proof. In view of Definition 3.6 and Proposition 3.5 and 3.13, we only need to show that $\mathcal{P}(Q') = Q'$ if only if $q \in Q'$ such that $L_{p_j} \leq L_q$, for some $j \in \{1,2,\ldots,r\}$. For this, let $\mathcal{P}(Q') = Q'$. Then $Q' = \{p \in Q | \exists R \in \mathcal{L}, q \in Q', (p,q) \in R\}$, or that $\exists L_{p_j} \in E = \{L_p : p \in Q'\}$ such that $q \in Q'$ and $L_{p_j} \leq L_q$, i.e., $q \in Q'$ such that $L_{P_j} \leq L_q$, for some $j \in \{1,2,\ldots,r\}$.

Conversely, let $q \in Q'$ such that $L_{p_j} \leq L_q$, for some $j \in \{1,2,\ldots,r\}$. Also, let $p \in \mathcal{P}(Q')$. Then there exist $t \in Q'$ and $R \in \mathcal{L}$ such that $(p,t) \in R$. Now, $t \in Q'$ implies that $L_{p_j} \leq L_t$, for some $j \in \{1,2,\ldots,r\}$. Also, $(p,t) \in R$ implies that $L_t \leq L_p$, then $L_{p_j} \leq L_p$, or that $p \in Q'$. Thus, $\mathcal{P}(Q') \subseteq Q'$ which, together with $Q' \subseteq \mathcal{P}(Q')$, shows that $\mathcal{P}(Q') = Q'$. □

Proposition 3.15. *Every L-valued tree transition system has at least one strongly connected subsystem.*

Proof. Let $\mathcal{A} = (Q,F,\delta)$ be an L-valued tree transition system, $p \in Q$ and $L_p \in E$ be a minimal layer (with respect to the partial order $\leq$). Then for $q \in \mathcal{S}(L_p)$, there exist $R \in \mathcal{L}$ and $t \in L_p$ such that $(t,q) \in R$. Now, $t \in L_p$ implies that $(p,t) \in R$. According to (p,t) and $(t,q) \in R$, we have $(p,q) \in R$, which shows that $q \in L_p$. Thus, for all $q \in \mathcal{S}(L_p)$, $q \in L_p$, or that $(L_p,F,\delta|_{L_p \times (L_p)^n \times F_n})$ is a subsystem of $\mathcal{A}$. Further, let $q,r \in L_p$. Then there exist $R \in \mathcal{L}$ such $(p,q) \in R$ and $(p,r) \in R$. Then $(r,q) \in R$, i.e., $q \in \mathcal{S}(r)$, whereby the subsystem $(L_p,F,\delta|_{L_p \times (L_p)^n \times F_n})$ is strongly connected. Hence, every L-valued tree transition system has at least one strongly connected subsystem. □

Proposition 3.16. *Let $\mathcal{A} = (Q,F,\delta)$ be a cyclic L-valued tree transition system. Then $\mathcal{A}$ has a unique maximal layer which is maximum in E.*

Proof. Let $\mathcal{A} = (Q,F,\delta)$ be a cyclic L-valued tree transition system and L_p be a maximal layer in E. Then there exists $R \in \mathcal{L}$ and $q \in Q$ such that $(q,p) \in R$; and therefore, $L_p \leq L_q$. Also $L_p = L_q$, because $L_p \neq L_q$ implies that $L_p < L_q$, which contradicts the maximality of L_p. Hence, $L_q \in E$ is a unique maximal layer. □

Definition 3.17. An L-valued tree transition system $\mathcal{A} = (Q,F,\delta)$ is called directable if for all $p,q \in Q$ there exist $R \in \mathcal{L}$ and $r \in Q$ such that $(p,r) \in R$ and $(q,r) \in R$.

Proposition 3.18. *Every directable L-valued tree transition system has a unique minimal layer.*

Proof. Let $\mathcal{A} = (Q,F,\delta)$ be a directable L-valued tree transition system. Also, let L_p, L_q be two distinct layers of $\mathcal{A}$, where $p,q \in Q$. Then there does not exist any $r \in Q$ and $R \in \mathcal{L}$ such that $(p,r) \in R$ and $(q,r) \in R$ (as $L_p \cap L_q = \emptyset$), and therefore a contradiction. Hence, every directable L-valued tree transition system has a unique minimal layer. □

The following part is towards the construction of an L-valued tree transition system, having singleton as a unique minimal layer from a given L-valued tree transition system with a unique minimal layer. Interestingly, the obtained L-valued tree transition system is a homomorphic image of the original L-valued tree transition system.

Let $\mathcal{A} = (Q,F,\delta)$ be an L-valued tree transition system having unique minimal layer L_p. Construct an L-valued tree transition system $\mathcal{A}' = (((Q \backslash L_P) \cup \{r\}),E,\lambda)$, where r is a new state and $\lambda : ((Q \backslash L_p) \cup \{r\}) \times ((Q \backslash L_p) \cup \{r\})^n \times E_n \to L$

$$\lambda(q,(q_1,\ldots,q_n),\alpha) = \begin{cases} \delta(q,(q_1,\ldots,q_n),\alpha), & if p,q \in Q \backslash L_p \\ 1, & otherwise. \end{cases}$$

Proposition 3.19. *The L-valued tree transition system $\mathcal{A}'$ is a homomorphic image of $\mathcal{A}$.*

Proof. Let $f : \mathcal{A} \to \mathcal{A}'$ be a map such that $\forall q \in Q$,

$$f(q) = \begin{cases} q, & if\ q \in (Q \backslash L_p) \\ r, & otherwise. \end{cases}$$

Then four cases will arise:

Case 1. If $(q,(q_1,\ldots,q_n)) \in (Q\backslash L_p) \times (Q\backslash L_p)^n$, then

$$\lambda(f(q),(f(q_1),\ldots,f(q_n)),\sigma) = \delta(q,(q_1,\ldots,q_n),\sigma).$$

Case 2. If $(q,(q_1,\ldots,q_n)) \in L_p \times (L_p)^n$, then

$$\lambda(f(q),(f(q_1),\ldots,f(q_n)),\sigma) = \lambda(r,(r,\ldots,r),\sigma) = 1 \geq \delta(q,(q_1,\ldots,q_n),\sigma).$$

Case 3. If $q \in Q\backslash L_P, (q_1,\ldots,q_n) \in L_p$, then

$$\lambda(f(q),(f(q_1),\ldots,f(q_n)),\sigma) = \lambda(q,(r,\ldots,r),\sigma) = 1 \geq \delta(q,(q_1,\ldots,q_n),\sigma).$$

Case 4. If $q \in L_p$, $(q_1,\ldots,q_n) \in (Q\backslash L_p)^n$, then

$$\lambda(f(q),(f(q_1),\ldots,f(q_n)),\sigma) = \lambda(r,(q_1,\ldots,q_n),\sigma) = 1 \geq \delta(q,(q_1,\ldots,q_n),\sigma).$$

Thus, $\forall(q,(q_1,\ldots,q_n),\sigma) \in Q \times Q^n \times F_n$,

$$\delta(q,(q_1,\ldots,q_n),\sigma) \leq \lambda(f(q),(f(q_1),\ldots,f(q_n)),\sigma).$$

Also, it is clear from the definition of f, demonstrating that f is onto. Hence, $\mathcal{A}'$ is a homomorphic image of $\mathcal{A}$. □

Definition 3.20. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system having unique minimal layer L_p. A decomposition of $\mathcal{A}$ is a pair of L-valued tree transition systems $\{\mathcal{A}_1, \mathcal{A}_2\}$, where $\mathcal{A}_1 = (L_p, F, \delta|_{L_p \times (L_p)^n \times F_n})$ and $\mathcal{A}_2 = (((Q\backslash L_p) \cup \{r\}), F, \lambda)$, here r is a new state and $\lambda : ((Q\backslash L_p) \cup \{r\}) \times ((Q\backslash L_p) \cup \{r\})^n \times F_n \to L$ is a map such that $\forall(q,(q_1,\ldots,q_n),\sigma) \in ((Q\backslash L_p) \cup \{r\}) \times ((Q\backslash L_p) \cup \{r\})^n \times F_n$

$$\lambda(q,(q_1,\ldots,q_n),\sigma) = \begin{cases} \delta(q,(q_1,\ldots,q_n),\sigma), & if q \in Q\backslash L_p, (q_1,\ldots,q_n) \in (Q\backslash L_p)^n \\ 1, & otherwise. \end{cases}$$

Proposition 3.21. *Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system having a unique minimal layer L_p and let $\{\mathcal{A}_1, \mathcal{A}_2\}$ be its decomposition. Then $\mathcal{A}$ is directable if and only if $\mathcal{A}_2$ is directable.*

Proof. Let $\mathcal{A}$ be a directable L-valued tree transition system and $q, t \in L_p$. Then there exist $R \in \mathcal{L}$ and $r \in Q$ such that $(t, r) \in R$ and $(q, r) \in R$. According to $q, t \in L_p$, we have $(p, q) \in R$ and $(p, t) \in R$. Then $(p, r) \in R$ and hence $r \in L_p$. Thus, $\mathcal{A}_2$ is directable.

Conversely, let $\mathcal{A}_2$ be a directable L-valued tree transition system and L_p be a unique minimal layer of L-valued tree transition system. Also, let $q, t \in Q$. Then there exist $q', t' \in Q$ such that $q \in L_{q'}$ and $t \in L_{t'}$, i.e., there exists $R \in \mathcal{L}$ such that $(q', q) \in R$ and $(t', t) \in R$. Also, since L_p is a unique minimal layer of $\mathcal{A}$, there exists $q'', t'' \in L_p$ such that $(q'', q') \in R$ and $(t'', t') \in R$. Now, $(q'', q) \in R$ and $(t'', t) \in R$. Also, as $\mathcal{A}_2$ is directable and $q'', t'' \in L_p$, there exists $p' \in L_p$ such that $(q'', p') \in R$ and $(t'', p') \in R$. Thus, $(q, p') \in R$ and $(t, p') \in R$, i.e., for all $q, t \in Q$ there exist $R \in \mathcal{L}$ and $p' \in Q$ such that $(q, p') \in R$ and $(t, p') \in R$. Hence, $\mathcal{A}$ is directable. □

4. Semilattices and L-valued tree transition systems

In this section, the relationship between L-valued tree transition systems and upper semilattices is introduced and studied. In this regard, the discussion is submitted by providing a construction of an L-valued tree transition system for a given finite poset demonstrating that there is an isomorphism between the post of class of subsystem of an L-valued tree transition system and upper semilattice.

Proposition 4.1. *Let $(\mathcal{P}, \preceq)$ be a finite poset. Then there exists an L-valued tree transition system $\mathcal{A}$ such that $E \cong (\mathcal{P}, \preceq)$.*

Proof. Let $(\mathcal{P}, \preceq)$ be a finite poset. Also, for $p \in \mathcal{P}$, let $q_1, q_2, \ldots, q_k$ be the predecessor of $\mathcal{P}$. Now, define an L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$, where $Q = \mathcal{P}$, $F = F_n$ (here $n = |\mathcal{P}|$) and $\delta : Q \times Q^n \times F_n \to [0, 1]$ is a map such that $\forall p \in Q, (p_1, \ldots, p_n) \in Q^n$ and $\forall \sigma \in F_n$

$$\delta(p, (p_1, \ldots, p_n), \sigma) = \begin{cases} t \in (0, 1], & if\;\; p_i = q_j, 1 \leq i \leq k \\ 0, & if\;\; p_i = q_j, k+1 \leq i \leq n \\ 1, & if\;\; p_i = p, k+1 \leq i \leq n \\ 0, & if\;\; p_i = p, 1 \leq i \leq n. \end{cases}$$

Obviously, $L_p = \{p\}, \forall p \in Q$. Also, let $f : (\mathcal{P}, \preceq) \to E$ such that $f(p) = L_p$, $\forall p \in \mathcal{P}$. Then f is a bijective map and for all $i = 1, 2, \ldots, k, q_i \leq p$ if and only if $L_{q_i} \leq L_p$, i.e., $f(q_i) \leq f(p)$. Hence, $E \cong (\mathcal{P}, \preceq)$.

Now, Let $\mathcal{T}(\mathcal{A})$ be the class of all subsystems of an L-valued tree transition system. For $\mathcal{B}, \mathcal{B}' \in \mathcal{T}(\mathcal{A})$, by $\mathcal{B} \sqsubseteq \mathcal{B}'$, we mean that $\mathcal{B}$ is a subsystem of $\mathcal{B}'$. It is easy to see that $\sqsubseteq$ is a partial order on $\mathcal{T}(\mathcal{A})$, and therefore $(\mathcal{T}(\mathcal{A}), \sqsubseteq)$ is a poset. Even, $(\mathcal{T}(\mathcal{A}), \sqsubseteq)$ is a finite upper semilattice, which is shown below. □

Proposition 4.2. *Let $\mathcal{A}$ be an L-valued tree transition system. Then $(\mathcal{T}(\mathcal{A}), \sqsubseteq)$ is a finite upper semilattice.*

Proof. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and $A_1 = (Q', F, \delta|_{Q' \times (Q')^n \times F_n})$, $A_2 = (Q'', F, \delta|_{Q'' \times (Q'')^n \times F_n}) \in \mathcal{T}(\mathcal{A})$. Then $(Q' \cup Q'', F, \delta|_{Q' \cup Q'' \times (Q' \cup Q'')^n \times F_n}) \in \mathcal{T}(\mathcal{A})$ and also, it is a unique least upper bound of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to $\sqsubseteq$. Hence $(\mathcal{T}(\mathcal{A}), \sqsubseteq)$ is a finite upper semilattice. □

The following shows that the existence of an L-valued tree transition system $\mathcal{A}$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq)$ is isomorphic to a given tree depends on the number of minimal elements in tree.

Proposition 4.3. *Let ℓ be a tree. If the number of minimal elements of ℓ is greater than two, then there does not exist any L-valued tree transition system $\mathcal{A}$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell$.*

Proof. Let the number of minimal elements of the tree $\ell = (\mathcal{P}, \preceq)$ be greater than two. Also, if possible, let there exists an L-valued tree transition system $\mathcal{A}$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell$. Then the number of minimal layers of $\mathcal{A}$ is greater that two. Now, Let L_{q_1}, L_{q_2} and L_{q_3} be three distinct minimal layers of $\mathcal{A}$, i.e., $L_{q_1} \cap L_{q_2} = \emptyset$, $L_{q_1} \cap L_{q_3} = \emptyset$ and $L_{q_2} \cap L_{q_3} = \emptyset$. Then $A_1 = (L_{q_1}, F, \delta|_{L_{q_1} \times (L_{q_1})^n \times F_n})$, $A_2 = (L_{q_2}, F, \delta|_{L_{q_2} \times (L_{q_2})^n \times F_n})$ and $A_3 = (L_{q_3}, F, \delta|_{L_{q_3} \times (L_{q_3})^n \times F_n})$ are distinct subsystems of $\mathcal{A}$. Also, $A_{12} = ((L_{q_1} \cup L_{q_2}), F, \delta|_{(L_{q_1} \cup L_{q_2}) \times (L_{q_1} \cup L_{q_2})^n \times F_n})$ and $A_{13} = ((L_{q_1} \cup L_{q_3}), F, \delta|_{(L_{q_1} \cup L_{q_3}) \times (L_{q_1} \cup L_{q_3})^n \times F_n})$ are distinct subsystems of $\mathcal{A}$ as L_{q_1}, L_{q_2} and L_{q_3} are disjoint. Thus, $\mathcal{A}_1 \sqsubseteq \mathcal{A}_{12}$ and $\mathcal{A}_1 \sqsubseteq \mathcal{A}_{13}$, which contradict the fact that ℓ is a tree. □

A number of compositions of L-valued tree transition systems such as product, cascade product, wreath product of L-valued tree transition systems have been proposed and investigated in [19]. Now in this study, another composition, namely $\oplus$-composition of L-valued tree transition systems, is introduced.

Definition 4.4. Let $\mathcal{A}_1 = (Q_1, F, \delta_1)$ and $\mathcal{A}_2 = (Q_2, F, \delta_2)$ be two L-valued tree transition systems such that $Q_1 \cap Q_2 = \emptyset$. Also, Let $\mathcal{S}$ be the set of all minimal layers of $\mathcal{A}_1$ and let $\mathcal{T}$ be the set of all minimal layers of A_2 such that for all $R \in \mathcal{S}$ there exists a maximal layer $\mathcal{S}_R$ in $\mathcal{T}$ with $\{\mathcal{S}_R | R \in \mathcal{S}\} = \mathcal{T}$. Then a $\oplus$-composition of $\mathcal{A}_1$ and $\mathcal{A}_2$ is an L-valued tree transition system $\mathcal{A}_1 \oplus \mathcal{A}_2 = (Q_1 \cup Q_2, F, \delta)$, where $\delta : (Q_1 \cup Q_2) \times (Q_1 \cup Q_2)^n \times F_n \to [0,1]$ is a map such that $\forall p \in Q_1 \cup Q_2, (q_1, \ldots, q_n) \in (Q_1 \cup Q_2)^n$ and $\forall \sigma \in F_n$

$$\delta(p,(q_1,\ldots,q_n),\sigma) = \begin{cases} \delta_1(p,(q_1,\ldots,q_n),\sigma), & if p \in Q_1, (q_1 \ldots, q_n) \in Q_1^n \\ & \text{such that } p \text{ and } q_i \text{ are not} \\ & \text{in a minimal layer of } \mathcal{A}_1 \\ \delta_2(p,(q_1,\ldots,q_n),\sigma), & if p \in Q_2, (q_1 \ldots, q_n) \in Q_2^n \\ 1, & \text{if } p \text{ is a minimal layer } R \text{ of } \mathcal{A}_1 \text{ and} \\ & q_i = q_x \text{ for unique } q_x \in \mathcal{S}_R \\ 0, & otherwise. \end{cases}$$

Proposition 4.5. *Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong (\mathcal{T}(\mathcal{A}_1), \sqsubseteq) \oplus (\mathcal{T}(\mathcal{A}_2), \sqsubseteq)$.*

Proof. Let $\mathcal{A} = (Q, F, \delta)$, $\mathcal{A}_1 = (Q_1, F, \delta_1)$ and $\mathcal{A}_2 = (Q_2, F, \delta_2)$ be L-valued tree transition systems such that $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Also, from Proposition 4.1, it can be assumed that the layers of $\mathcal{A}_1$ and $\mathcal{A}_2$ consist of a singleton. Now, for $q \in Q_1$, let $\{q\}$ be a minimal layer of $\mathcal{A}_1$ and let $\mathcal{S}_q = \cup\{p \in Q | \exists R \in \mathcal{L}, (q,p) \in R\}$. Then $\mathcal{A}_q = (\mathcal{S}_q, F, \delta|_{\mathcal{S}_q \times (\mathcal{S}_q)^n \times F_n}) \in \mathcal{T}(\mathcal{A})$ but $\mathcal{A}_q \notin \mathcal{T}(\mathcal{A}_2)$ as $q \notin Q_2$. Again, let $\mathcal{B} = (\mathcal{T}, F, \delta|_{\mathcal{T} \times (\mathcal{T})^n \times F_n}) \in \mathcal{T}(\mathcal{A}_2)$. Then $\mathcal{T} \subset \mathcal{S}_q$ for some q, where $\{q\}$ is a minimal layer of $\mathcal{A}_1$. Further, define a map $f : \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A}_1) \oplus \mathcal{T}(\mathcal{A}_2)$ such that for all $\mathcal{B} \in \mathcal{T}(\mathcal{A})$,

$$f(\mathcal{B}) = \begin{cases} \mathcal{B}, & if \;\; \mathcal{B} \in \mathcal{T}(\mathcal{A}_q), \text{ where } q \text{ is a minimal layer of } \mathcal{A}_1 \\ \mathcal{B}', & if \;\; \mathcal{B} \in \mathcal{T}(\mathcal{A}) \backslash \mathcal{T}(\mathcal{A}_2), \text{ where } \mathcal{B}' = (\mathcal{T} \cap Q_2, F, \delta|_{(\mathcal{T} \cap Q_2) \times (\mathcal{T} \cap Q_2)^n \times F_n}). \end{cases}$$

Obviously, f is an isomorphism and hence $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong (\mathcal{T}(\mathcal{A}_1), \sqsubseteq) \oplus (\mathcal{T}(\mathcal{A}_2), \sqsubseteq)$. □

Proposition 4.6. *Let $\mathcal{A}$ be an L-valued tree transition system. Then there exist positive integers $n_1, n_2, \ldots, n_k$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell(n_1) \oplus \ldots \oplus \ell(n_k)$.*

Proof. Let $\mathcal{A} = (Q, F, \delta)$ be an L-valued tree transition system and let $\{L_{q_1}, L_{q_2}, \ldots, L_{q_{n_1}}\}$ be the set of all minimal layers of $\mathcal{A}$. Also, let $\mathcal{A}_{n_1} = (L_{q_1} \cup L_{q_2} \cup \ldots \cup L_{q_{n_1}}, F, \delta \mid_{(L_{q_1} \cup L_{q_2} \cup \ldots \cup L_{q_{n_1}}) \times (L_{q_1} \cup L_{q_2} \cup \ldots \cup L_{q_{n_1}})^n \times F_n})$. Then from Definition 2.6, $\mathcal{T}(\mathcal{A}_{n_1})$ is a finite upper semilattice, i.e., $(\mathcal{T}(\mathcal{A}_{n_1}), \sqsubseteq) \cong \ell(n_1)$. =Now, consider an L-valued tree transition system $\mathcal{A}' = (Q \backslash L_{q_1} \cup \ldots \cup L_{q_{n_1}}, F, \lambda)$, where $\lambda(p,(p_1,\ldots,p_n),\sigma) = \delta(p,(p_1,\ldots,p_n),\sigma)$, $\forall p \in Q \backslash (L_{q_1} \cup L_{q_2} \cup \ldots \cup L_{q_{n_1}}), (p_1, \ldots, p_n) \in (Q \backslash (L_{q_1} \cup L_{q_2} \cup \ldots \cup L_{q_{n_1}}))^n, \forall \sigma \in F_n$. Then $\mathcal{A} = \mathcal{A}' \oplus \mathcal{A}_{n_1}$, whereby from Proposition 4.5, $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong (\mathcal{T}(\mathcal{A}_{n_1}), \sqsubseteq) \oplus (\mathcal{T}(\mathcal{A}'), \sqsubseteq) \cong \ell(n_1) \oplus (\mathcal{T}(\mathcal{A}'), \sqsubseteq)$. Similar procedure for $\mathcal{A}'$ leads us to $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell(n_1) \oplus \ell(n_2) \oplus (\mathcal{T}(\mathcal{A}''), \sqsubseteq)$, for some L-valued tree transition system $\mathcal{A}''$. Hence, by continuing the same, we get $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_k)$. □

Proposition 4.7. *Let ℓ be an upper semilattice such that $\ell \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_r)$, for some positive integers $n_1, n_2, \ldots, n_r$. Then there exists an L-valued tree transition system $\mathcal{A}$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_r)$.*

Proof. Let ℓ be an upper semilattice such that $\ell \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_r)$. Construct an L-valued tree transition system $\mathcal{A} = (Q, F, \delta)$, where $Q = \{1, 2, \ldots, n_r\}$, $|F| = \max\{n_1, n_2, \ldots, n_r\}$, and $\delta : Q \times Q^n \times F_n \to [0,1]$ is a map such that $\forall p \in Q, (q_1, \ldots, q_n) \in Q^n$ and $\forall \sigma \in F_n$,

$$\delta(p,(q_1,\ldots,q_n),\sigma) = \begin{cases} t \in (0,1], & if \; p = q_i, 1 \leq i \leq n \\ 0, & otherwise. \end{cases}$$

Then $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell(n_r)$. Now, let $\mathcal{A}' = (Q', F, \delta')$ be an L-valued tree transition system such that $(\mathcal{T}(\mathcal{A}'), \sqsubseteq) \cong \ell(n_k) \oplus \ell(n_{k+1}) \oplus \ldots \oplus \ell(n_r)$, where k is the minimal positive integer. For $k = 1$, there is nothing left to prove. If $k > 1$, let $\mathcal{A}''$ be an L-valued tree transition system such that $(\mathcal{T}(\mathcal{A}''), \sqsubseteq) \cong \ell(n_{k-1})$. Then $(\mathcal{T}(\mathcal{A}' \oplus \mathcal{A}''), \sqsubseteq) \cong (\mathcal{T}(\mathcal{A}''), \sqsubseteq) \oplus (\mathcal{T}(\mathcal{A}'), \sqsubseteq) \cong \ell(n_{k-1}) \oplus \ell(n_k) \oplus \ldots \oplus \ell(n_r)$. This contradicts the minimality of k. Hence, $k = 1$, and therefore, there exists an L-valued tree transition system $\mathcal{A}'$ such that $(\mathcal{T}(\mathcal{A}'), \sqsubseteq) \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_r)$. □

Finally, Proposition 4.6 and Proposition 4.7 lead us to the following.

Proposition 4.8. *Let ℓ be a finite upper semilattice. Then there exists an L-valued tree transition system $\mathcal{A}$ such that $(\mathcal{T}(\mathcal{A}), \sqsubseteq) \cong \ell$ if and only if $\ell \cong \ell(n_1) \oplus \ell(n_2) \oplus \ldots \oplus \ell(n_r)$ for some positive integers $n_1, n_2, \ldots, n_r$.*

5. Conclusion

In this study, an attempt was made to enhance the algebraic study of L-valued tree transition systems through using the concept of their layers. The most distinguished outcomes of this study were (1) the detailed characterization of existing algebraic concepts of L-valued tree transition systems in terms of their layers, and (2) the establishment of isomorphism between the poset of class of subsystem of an L-valued tree transition system and an upper semilattice. It seems that the topological concepts and fuzzy topological observations may also be applied in such investigations. We will definitely try to make an effort to do such studies in near future. As suggestions for further studies, we can replace the crisp relation R with a fuzzy relation related to fuzzy order (L-valued order), comparing the obtained results with those already presented in the current study.

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