Filomat 38:1 (2024), 315–323
https://doi.org/10.2298/FIL2401315E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Meta-metallic Riemannian manifolds

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Abstract. In this study, motivated by the Meta-Golden-Chi ratio, we develop essentially Meta-Metallic manifolds by using Meta-Metallic-Chi ratio and Metallic manifolds, provide an example and explore certain features of its Meta-Metallic structure. We give the conditions for integrability of the almost Meta-Metallic structure and examine its relation to the curvature tensor field. We further demonstrate that the Meta-Metallic Riemannian manifold is flat if and only if its curvature is constant. As a result, we show that a different notion of sectional curvature is needed in Meta-Metallic Riemannian manifolds.

1. Introduction

A close connection has been demonstrated between the Metallic ratio and the transition from Newtonian physics to relativistic mechanics. For example the Golden ratio, which is a special case of the Metallic ratio, has been used to get special theory of relativity, the Golden rectangle, the Lorentz contraction of lengths and the expansion of time intervals. At the same time, the Metallic (resp., Golden) ratio produces interesting and important results in Kantor spacetime, conformal field theory, topology of 4-manifolds, mathematical probability theory, Kantor fractal theory and El Naschie's field theory [1], [2]. This case shows that researchers have been looking for many things that meet the needs of the Golden ratio all over the world. The one was the idea that the Golden ratio can be found in a logarithmic spiral. But Barlett [3], has recently shown that this is not true by showing that the Meta-Golden-Chi ratio works well with an important class of logarithmic spirals. With the same sense he built the Meta-Golden-Chi ratio

$$\chi = \frac{1+\sqrt{4\phi+5}}{2\phi},$$

where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Şahin [4], introduced a new manifold and named this manifold as Meta-Golden Riemannian manifold. This manifold was composed by means of the Meta-Golden-Chi ratio and Golden manifolds.

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 57R15
Keywords. Chi ratio, Golden structure, Meta-Golden structure, Metallic structure, Meta-Metallic-Chi ratio.
Received: 09 May 2023; Revised: 02 July 2023; Accepted: 10 July 2023
Communicated by Mića S. Stanković
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Different polynomial and geometric structures give rise to important consequences while investigating differential and geometric properties of submanifolds in Riemannian (as well as, semi-Riemannian) manifolds. Manifolds with such differential-geometric structures have been studied by several authors and for some we may refer to [5–10]. Riemannian manifolds alone have important applications. One of these applications in recent years is related to the modeling of the COVID pandemic (see, [11]) by developing an approach, namely SBDIEM, which models COVID and similar epidemics. In addition Riemann manifolds, on which polynomial structures are defined, allow very different applications and interdisciplinary aspects.

Golden Riemannian manifolds [6] and Metallic structures on Riemannian manifolds [5] were presented by Crasmareanu and Hretcanu. After these pioneering articles, many new studies were added to the literature for such manifolds, structures defined on them and their submanifolds.

A Metallic manifold is essentially a differentiable manifold and has an extra (1,1) type tensor field which is called a Metallic structure (see [5]) that satisfies certain conditions. A differentiable manifold with a Riemannian metric compatible with Metallic structure is known as a Metallic Riemannian manifold. Metallic Riemannian manifolds are a broad class that considers Golden Riemannian manifolds a special class and includes many other different types of manifolds such as Silver, Nickel etc. Riemannian manifolds. In light of what has been stated, one can see that Metallic manifolds provide a robust geometric theory.

In this study, we introduce a new type manifold, called by us Meta-Metallic manifolds, inspired by the concept of Meta-Metallic-Chi ratio and Metallic manifolds. This type manifolds form a wide class that includes the Meta-Golden manifolds defined in [4].

This article is divided into three sections. In the second section, the fundamental terms and ideas that will be used throughout the text are presented. In section 3, the Meta-Metallic structure is defined, its presence on Metallic manifolds is examined and an example is provided. Moreover, the relationships between the almost Meta-Metallic structure and the product structure is investigated. Finally, integrability conditions of almost Meta-Metallic structure, certain curvature identities of a Meta-Metallic manifold and a different notion of sectional curvature are found.

2. Preliminaries

The Meta-Metallic-Chi ratio which will be used throughout the article is structured as follows:

With a similar approach as in Figure 1 in [4], we write $\dot{\chi} = \frac{\mathfrak{q}}{\not{c}} + \frac{\mathfrak{p}}{\dot{\chi}}$, which suggests that $\dot{\chi}^2 - \frac{\mathfrak{q}}{\not{c}}\dot{\chi} - \mathfrak{p} = 0$, where $\not{c}$ is the well-known Metallic ratio. Thus, the roots are found as

$$\frac{\frac{\mathfrak{q}}{\not{c}} \mp \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{\not{c}^2}}}{2}.$$

The correlation between continued fractions and the Meta-Golden-Chi ratio was found in [7]. We define the positive root by

$$\dot{\chi} = \frac{\frac{\mathfrak{q}}{\not{c}} + \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{\not{c}^2}}}{2},$$

which is said to be the Silver ratio of inverse of Metallic ratio. We also obtain the negative roots as

$$\ddot{\chi} = \frac{\frac{\mathfrak{q}}{\not{c}} - \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{\not{c}^2}}}{2}.$$

Also, by a direct computation, it is seen that

$$\ddot{\chi} = \frac{\mathfrak{q}}{\not{c}} - \dot{\chi}, \tag{1}$$

$$\dot{\chi}^2 = \mathfrak{p} + \frac{\mathfrak{q}}{\cancel{k}}\dot{\chi}, \tag{2}$$

and

$$\ddot{\chi}^2 = \mathfrak{p} + \frac{\mathfrak{q}}{\cancel{k}}\ddot{\chi}. \tag{3}$$

Hretcanu and Crasmareanu [5] stated that an endomorphism $\tilde{\mathfrak{I}}$ is a Metallic structure on a differentiable manifold $\mathcal{M}^*$, if

$$\tilde{\mathfrak{I}}^2\mathbb{X}_1 = \mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_1 + \mathfrak{q}\mathbb{X}_1, \tag{4}$$

is satisfied for $\mathbb{X}_1 \in \mathfrak{X}(\mathcal{M}^*)$, where $\mathfrak{p}, \mathfrak{q}$ are positive integers. Additionally, if there exists a Riemannian metric on $\mathcal{M}^*$ satisfying

$$\tilde{\mathfrak{g}}(\tilde{\mathfrak{I}}\mathbb{X}_1, \mathbb{Y}_1) = \tilde{\mathfrak{g}}(\mathbb{X}_1, \tilde{\mathfrak{I}}\mathbb{Y}_1), \tag{5}$$

for any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$, then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{I}})$ is called a Metallic Riemannian structure on $\mathcal{M}^*$ and the triple $(\mathcal{M}^*, \tilde{\mathfrak{g}}, \tilde{\mathfrak{I}})$ is said to be a Metallic Riemannian manifold. From (5), we see that

$$\tilde{\mathfrak{g}}(\tilde{\mathfrak{I}}\mathbb{X}_1, \tilde{\mathfrak{I}}\mathbb{Y}_1) = \mathfrak{p}\tilde{\mathfrak{g}}(\mathbb{X}_1, \tilde{\mathfrak{I}}\mathbb{Y}_1) + \mathfrak{q}\tilde{\mathfrak{g}}(\mathbb{X}_1, \mathbb{Y}_1). \tag{6}$$

3. Meta-Metallic Manifolds

Here, we introduce a new type of manifold, as a generalization of Meta-Golden manifolds, which we define as follow by using the Meta-Metallic-Chi ratio.

Definition 3.1. *An endomorphism $\mathfrak{Y}$ on an almost Metallic manifold $(\mathcal{M}^*, \tilde{\mathfrak{I}})$ satisfying*

$$\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{X}_1 = \mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_1 + \mathfrak{q}\mathfrak{Y}\mathbb{X}_1, \tag{7}$$

for every $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$, is named an almost Meta-Metallic structure and $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y})$ is said to be an almost Meta-Metallic manifold.

Theorem 3.2. *Let $\mathfrak{Y}$ be an endomorphism on $(\mathcal{M}^*, \tilde{\mathfrak{I}})$. At that case, $\mathfrak{Y}$ is an almost Meta-Metallic structure iff*

$$\mathfrak{Y}^2 = \tilde{\mathfrak{I}}\mathfrak{Y} - \mathfrak{p}\mathfrak{Y} + \mathfrak{p}I \tag{8}$$

where I is the identity map .

Proof. Let $\mathfrak{Y}$ be an almost Meta-Metallic structure. Then for any $\mathbb{X}_1 \in \mathfrak{X}(\mathcal{M}^*)$, by applying $\tilde{\mathfrak{I}}$ to both sides of equation (7) and considering equation (4), we get

$$\mathfrak{p}\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{X}_1 + \mathfrak{q}\mathfrak{Y}^2\mathbb{X}_1 = \mathfrak{p}^2\tilde{\mathfrak{I}}\mathbb{X}_1 + \mathfrak{p}\mathfrak{q}\mathbb{X}_1 + \mathfrak{p}\tilde{\mathfrak{I}}\mathfrak{Y}\mathbb{X}_1.$$

In this equation, if we consider equation (7) again, we obtain

$$\mathfrak{Y}^2\mathbb{X}_1 = \tilde{\mathfrak{I}}\mathfrak{Y}\mathbb{X}_1 - \mathfrak{p}\mathfrak{Y}\mathbb{X}_1 + \mathfrak{p}\mathbb{X}_1.$$

Conversely, we assume that the equation (8) is satisfied for any $\mathbb{X}_1 \in \mathfrak{X}(\mathcal{M}^*)$. If $\tilde{\mathfrak{I}}$ is applied to both sides of equation (8), we find

$$\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{X}_1 = \tilde{\mathfrak{I}}^2\mathfrak{Y}\mathbb{X}_1 - \mathfrak{p}\tilde{\mathfrak{I}}\mathfrak{Y}\mathbb{X}_1 + \mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_1,$$

which implies

$$\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{X}_1 = \mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_1 + \mathfrak{q}\mathfrak{Y}\mathbb{X}_1, \tag{9}$$

via (4). Thus we ends the proof. □

Definition 3.3. *Let $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y})$ be an almost Meta-Metallic manifold and $\tilde{\mathfrak{g}}$ be a Riemannian metric on $\mathcal{M}^*$. If $\tilde{\mathfrak{g}}$ is compatible with $\mathfrak{Y}$ on $\mathcal{M}^*$, namely*

$$\tilde{\mathfrak{g}}(\mathfrak{Y}\mathbb{X}_1, \Upsilon_1) = \tilde{\mathfrak{g}}(\mathbb{X}_1, \mathfrak{Y}\Upsilon_1), \tag{10}$$

or equivalently

$$\tilde{\mathfrak{g}}(\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\Upsilon_1) = \tilde{\mathfrak{g}}(\tilde{\mathfrak{I}}\mathbb{X}_1, \mathfrak{Y}\Upsilon_1) - \mathfrak{p}\tilde{\mathfrak{g}}(\mathbb{X}_1, \mathfrak{Y}\Upsilon_1) + \mathfrak{p}\tilde{\mathfrak{g}}(\mathbb{X}_1, \Upsilon_1), \tag{11}$$

for any $\mathbb{X}_1, \Upsilon_1 \in \mathfrak{X}(\mathcal{M}^*)$, then $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ is called an almost Meta-Metallic Riemannian manifold.

Proposition 3.4. *An almost Meta-Metallic structure $\mathfrak{Y}$ is an isomorphism on $T_p\mathcal{M}^*$, for each point p in $\mathcal{M}^*$.*

Proof. Let $Ker\mathfrak{Y}$ denotes the kernel space of almost Meta-Metallic structure $\mathfrak{Y}$. Therefore, from

$$Ker\mathfrak{Y} = \left\{\mathbb{X}_1 \in \Gamma(T_p\mathcal{M}^*) \mid \mathfrak{Y}(\mathbb{X}_1) = 0, \forall\, p \in \mathcal{M}^*\right\}$$

and since $\mathfrak{Y}$ is non-singular, we find $\mathfrak{Y}(\mathbb{X}_1) = 0$, namely $\mathbb{X}_1 = 0$. Then $Ker\mathfrak{Y} = 0$. □

Proposition 3.5. *Let $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ be an almost Meta-Metallic Riemannian manifold. In that case*

1. If $\not{c}$ is the *eigenvalue* of the almost Metallic structure $\tilde{\mathfrak{I}}$, then $\dot{\chi}$ and $\ddot{\chi}$ are the *eigenvalues* of the almost Meta-Metallic structure.

2. If $\mathfrak{p} - \not{c}$ is the *eigenvalue* of the almost Metallic structure $\tilde{\mathfrak{I}}$, then

$$G_m = \frac{\frac{\mathfrak{q}}{\mathfrak{p}-\not{c}} + \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{(\mathfrak{p}-\not{c})^2}}}{2}, \bar{G}_{\overline{m}} = \frac{\frac{\mathfrak{q}}{\mathfrak{p}-\not{c}} - \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{(\mathfrak{p}-\not{c})^2}}}{2}$$

are the *eigenvalues* of the almost Meta-Metallic structure.

Proof. Let λ be the *eigenvalue* of the almost Meta-Metallic structure $\mathfrak{Y}$ on $T_p\mathcal{M}^*$. In this case, for any $\mathbb{X}_1 \in T_p\mathcal{M}^*$, we have

$$\mathfrak{Y}\mathbb{X}_1 = \lambda\mathbb{X}_1. \tag{12}$$

If $\mathfrak{Y}$ and $\tilde{\mathfrak{I}}$ is applied to both sides of the equation (12) respectively, we find

$$\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{X}_1 = \lambda^2\tilde{\mathfrak{I}}\mathbb{X}_1.$$

Then from equation (7), we have

$$\mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_1 + \mathfrak{q}\mathfrak{Y}\mathbb{X}_1 = \lambda^2\tilde{\mathfrak{I}}\mathbb{X}_1.$$

If ϖ is the *eigenvalue* of the almost Metallic structure, we obtain

$$(\varpi\lambda^2 - \mathfrak{q}\lambda - \mathfrak{p}\varpi)\mathbb{X}_1 = 0.$$

The *eigenvalues* of the almost Metallic structure $\tilde{\mathfrak{I}}$ are $\not{c}$ and $\mathfrak{p} - \not{c}$, and if we consider the *eigenvalue* $\not{c}$, we find

$$\dot{\chi} = \frac{\frac{\mathfrak{q}}{\not{c}} + \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{\not{c}^2}}}{2}, \quad \ddot{\chi} = \frac{\frac{\mathfrak{q}}{\not{c}} - \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{\not{c}^2}}}{2},$$

and if we consider the *eigenvalue* $\mathfrak{p} - \not{c}$, we obtain

$$G_m = \frac{\frac{\mathfrak{q}}{\mathfrak{p}-\not{c}} + \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{(\mathfrak{p}-\not{c})^2}}}{2}, \quad \bar{G}_{\overline{m}} = \frac{\frac{\mathfrak{q}}{\mathfrak{p}-\not{c}} - \sqrt{4\mathfrak{p} + \frac{\mathfrak{q}^2}{(\mathfrak{p}-\not{c})^2}}}{2}.$$

□

Proposition 3.6. *Let $(\mathcal{M}^*, \tilde{\mathfrak{g}})$ be an m-dimensional Riemannian manifold and $\mathcal{F}$ be an almost product structure on $\mathcal{M}^*$. Then $\mathcal{F}$ reduces two almost Meta-Metallic structures on $(\mathcal{M}^*, \tilde{\mathfrak{g}})$ given as*

$$\mathfrak{Y} = \mathfrak{p}\mathcal{A}_*\mathcal{F} + \mathfrak{p}\mathcal{B}_*\mathcal{I}, \tag{13}$$

where $\not{c}(\mathfrak{p}\mathcal{A}_* + \mathfrak{q}\mathcal{B}_*)^2 = \mathfrak{p}\not{c} + \mathfrak{q}(\mathcal{A}_* + \mathcal{B}_*)$.

Proof. We know that an almost product structure $\mathcal{F}$ induces an almost Metallic structure on $\mathcal{M}^*$ given by $\tilde{\mathfrak{I}} = \frac{1}{2}(\mathfrak{p}I + (2\not{c} - \mathfrak{p})\mathcal{F})$, where I is the identity map. Then, $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \tilde{\mathfrak{g}})$ becomes an almost Metallic Riemannian manifold. The remainder is derived via direct computation. □

Example 3.7. *Let E^5 be the 5-dimensional Euclidean space. Then, E^5 is an almost Metallic manifold with an almost Metallic structure $\tilde{\mathfrak{I}}$ defined as*

$$\begin{aligned} \tilde{\mathfrak{I}} : E^5 &\to E^5 \\ (\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) &\to (\not{c}\mathbb{X}_1, \not{c}\mathbb{X}_2, \not{c}\mathbb{X}_3, (\mathfrak{p} - \not{c})\Upsilon_1, (\mathfrak{p} - \not{c})\Upsilon_2). \end{aligned}$$

Now, we define an endomorphism $\mathfrak{Y}$ on $(E^5, \tilde{\mathfrak{I}})$ by

$$\mathfrak{Y}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) = (\dot{\chi}\mathbb{X}_1, \dot{\chi}\mathbb{X}_2, \dot{\chi}\mathbb{X}_3, -\tilde{\chi}\Upsilon_1, -\tilde{\chi}\Upsilon_2) \tag{14}$$

where $\tilde{\chi} = \frac{\not{c} + \sqrt{\not{c}^2 + 4\mathfrak{p}}}{2}$*, which is called "the Silver mean of Metallic mean", satisfies*

$$\begin{aligned} \tilde{\chi}^2 &= \frac{2\not{c}^2 + 2\not{c}\sqrt{\not{c}^2 + 4\mathfrak{p}}}{4} + \mathfrak{p} \\ &= \not{c}\tilde{\chi} + \mathfrak{p}I \end{aligned}$$

and

$$\begin{aligned} (\mathfrak{p} - \not{c})\tilde{\chi}^2\Upsilon_j &= (\mathfrak{p} - \not{c})\not{c}\tilde{\chi}\Upsilon_j + \mathfrak{p}(\mathfrak{p} - \not{c})\Upsilon_j \\ &= -\mathfrak{q}\tilde{\chi}\Upsilon_j + \mathfrak{p}(\mathfrak{p} - \not{c})\Upsilon_j \\ &= \mathfrak{q}\mathfrak{Y}\Upsilon_j + \mathfrak{p}\tilde{\mathfrak{I}}\Upsilon_j \end{aligned}$$

where I is the identity map and $j = 1, 2$. Also $\dot{\chi}$ is the Meta-Metallic-Chi ratio and we have

$$\begin{aligned} \not{c}\dot{\chi}^2 &= \mathfrak{q}\dot{\chi} + \mathfrak{p}\not{c}, \\ \not{c}\dot{\chi}^2\mathbb{X}_i &= \mathfrak{q}\dot{\chi}\mathbb{X}_i + \mathfrak{p}\not{c}\mathbb{X}_i \\ &= \mathfrak{p}\tilde{\mathfrak{I}}\mathbb{X}_i + \mathfrak{q}\mathfrak{Y}\mathbb{X}_i, \end{aligned}$$

for $i = 1, 2, 3$. If we apply $\mathfrak{Y}$ to both sides of equation (14) and consider equation (2), we find

$$\begin{aligned} \tilde{\mathfrak{I}}\mathfrak{Y}^2(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) &= (\not{c}\dot{\chi}^2\mathbb{X}_1, \not{c}\dot{\chi}^2\mathbb{X}_2, \not{c}\dot{\chi}^2\mathbb{X}_3, (\mathfrak{p} - \not{c})\tilde{\chi}^2\Upsilon_1, (\mathfrak{p} - \not{c})\tilde{\chi}^2\Upsilon_2) \\ &= \mathfrak{p}\tilde{\mathfrak{I}}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2) + \mathfrak{q}\mathfrak{Y}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \Upsilon_1, \Upsilon_2). \end{aligned}$$

Therefore $\mathfrak{Y}$ is an almost Meta-Metallic structure. Thus $(E^5, \tilde{\mathfrak{I}}, \mathfrak{Y})$ is an almost Meta-Metallic manifold.

3.1. Integrability of almost Meta-Metallic structures

The Nijenhuis tensor field of $\mathfrak{Y}$ is defined by

$$N_{\mathfrak{Y}}(\mathbb{X}_1, \Upsilon_1) = \mathfrak{Y}^2[\mathbb{X}_1, \Upsilon_1] + [\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\Upsilon_1] - \mathfrak{Y}[\mathbb{X}_1, \mathfrak{Y}\Upsilon_1] - \mathfrak{Y}[\mathfrak{Y}\mathbb{X}_1, \Upsilon_1], \quad \mathbb{X}_1, \Upsilon_1 \in \mathfrak{X}(\mathcal{M}^*).$$

Initially, we state

Lemma 3.8. *Let* $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ *be an almost Meta-Metallic Riemannian manifold. Then we have*

$$N_{\mathfrak{Y}}(\mathbb{X}_1, \mathbb{Y}_1) = (\nabla_{\mathfrak{Y}\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 - (\nabla_{\mathfrak{Y}\mathbb{Y}_1}\mathfrak{Y})\mathbb{X}_1 - \mathfrak{Y}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 + \mathfrak{Y}(\nabla_{\mathbb{Y}_1}\mathfrak{Y})\mathbb{X}_1, \tag{15}$$

where for any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$ *and* ∇ *is the Levi-Civita conection on* $\mathcal{M}^*$ *.*

Proof. If we use (8) in the Nijenhuis tensor field of $\mathfrak{Y}$, for any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$, we obtain

$$\begin{aligned} N_{\mathfrak{Y}}(\mathbb{X}_1, \mathbb{Y}_1) = {} & \tilde{\mathfrak{I}}\mathfrak{Y}\,[\mathbb{X}_1, \mathbb{Y}_1] - \mathfrak{p}\mathfrak{Y}\,[\mathbb{X}_1, \mathbb{Y}_1] + \mathfrak{p}\,[\mathbb{X}_1, \mathbb{Y}_1] \\ & + [\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1] - \mathfrak{Y}\,[\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1] - \mathfrak{Y}\,[\mathfrak{Y}\mathbb{X}_1, \mathbb{Y}_1]\,. \end{aligned}$$

Here by using the covariant derivative of $\mathfrak{Y}$ with definition of Lie bracket, we obtain

$$\begin{aligned} N_{\mathfrak{Y}}(\mathbb{X}_1, \mathbb{Y}_1) = {} & \tilde{\mathfrak{I}}\mathfrak{Y}\,[\mathbb{X}_1, \mathbb{Y}_1] + (\nabla_{\mathfrak{Y}\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 - (\nabla_{\mathfrak{Y}\mathbb{Y}_1}\mathfrak{Y})\mathbb{X}_1 - \mathfrak{Y}\nabla_{\mathbb{X}_1}\mathfrak{Y}\mathbb{Y}_1 \\ & + \mathfrak{Y}\nabla_{\mathbb{Y}_1}\mathfrak{Y}\mathbb{X}_1 + \mathfrak{p}\mathfrak{Y}\nabla_{\mathbb{Y}_1}\mathbb{X}_1 - \mathfrak{p}\mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + \mathfrak{p}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 - \mathfrak{p}\nabla_{\mathbb{Y}_1}\mathbb{X}_1. \end{aligned}$$

Hence we complete the proof. □

Namely, if the Nijenhuis tensor field of $\mathfrak{Y}$ vanishes, the almost Meta-Metallic structure is called integrable and $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ is said to be a Meta-Metallic Riemannian manifold. In the continuation of this lemma, we can state the following conclusion.

Corollary 3.9. *Let* $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ *be an almost Meta-Metallic Riemannian manifold . If* $\nabla\mathfrak{Y} = 0$*, then the almost Meta-Metallic structure is integrable, and so* $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ *is a Meta-Metallic Riemannian manifold .*

We can also derive the integrability requirement for the Codazzi-like equation from equation (15), which states that this equation must satisfy certain conditions.

Theorem 3.10. *Let* $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ *be an almost Meta-Metallic Riemannian manifold. Then* $\mathfrak{Y}$ *is integrable if Codazzi-like equation*

$$(\nabla_{\mathfrak{Y}\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 - \mathfrak{Y}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 = 0$$

is ensured for any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$*.*

Theorem 3.11. *Let* $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ *be an almost Meta-Metallic Riemannian manifold. If* $\nabla\mathfrak{Y} = 0$*, then* $\nabla\tilde{\mathfrak{I}} = 0$ *.*

Proof. For any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$, we have

$$(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 = \nabla_{\mathbb{X}_1}\mathfrak{Y}\mathbb{Y}_1 - \mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1,$$

then if we use equation (8), we find

$$\begin{aligned} (\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 = {} & -\frac{1}{\mathfrak{p}}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathfrak{Y}\mathbb{Y}_1 - \frac{1}{\mathfrak{p}}\mathfrak{Y}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 \\ & - \frac{1}{\mathfrak{p}}\mathfrak{Y}^2\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + \frac{1}{\mathfrak{p}}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 \\ & + \frac{1}{\mathfrak{p}}\tilde{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 + \frac{1}{\mathfrak{p}}\tilde{\mathfrak{I}}\mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 \\ & + \nabla_{\mathbb{X}_1}\mathbb{Y}_1 - \mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1. \end{aligned}$$

If equation (8) is used again in this equation, we get

$$(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 = -\frac{1}{\mathfrak{p}}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathfrak{Y}\mathbb{Y}_1 - \frac{1}{\mathfrak{p}}\mathfrak{Y}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1 + \frac{1}{\mathfrak{p}}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 + \frac{1}{\mathfrak{p}}\tilde{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\mathfrak{Y})\mathbb{Y}_1. \tag{16}$$

Therefore if $\nabla\mathfrak{Y} = 0$, then $(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 = 0$, for any $\mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$. Since $(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 = 0$, for $\mathfrak{Y}\mathbb{Y}_1 \in \mathfrak{X}(\mathcal{M}^*)$, this equation is provided. Hence, we have

$$\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}}\mathfrak{Y}^2\mathbb{Y}_1 - \tilde{\mathfrak{I}}\nabla_{\mathbb{X}_1}\mathfrak{Y}^2\mathbb{Y}_1 = 0.$$

Here if we use equations (7) and (8), we obtain

$$\left(\begin{array}{c} \mathfrak{p}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathbb{Y}_1 + \mathfrak{p}\tilde{\mathfrak{I}}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + \mathfrak{q}\mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 - \tilde{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 \\ -\tilde{\mathfrak{I}}^2\nabla_{\mathbb{X}_1}\mathfrak{Y}\mathbb{Y}_1 + \mathfrak{p}\tilde{\mathfrak{I}}\nabla_{\mathbb{X}_1}\mathfrak{Y}\mathbb{Y}_1 - \mathfrak{p}\tilde{\mathfrak{I}}\nabla_{\mathbb{X}_1}\mathbb{Y}_1 \end{array}\right) = 0. \tag{17}$$

Using $\nabla\mathfrak{Y} = 0$ and equation (9), we write

$$\mathfrak{p}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathbb{Y}_1 - \tilde{\mathfrak{I}}(\nabla_{\mathbb{X}_1}\tilde{\mathfrak{I}})\mathfrak{Y}\mathbb{Y}_1 = 0.$$

From the equation (16), we obtain $\nabla\tilde{\mathfrak{I}} = 0$. □

Let us now give another criterion for the integrability condition of the almost Meta-Metallic structure. Let $\mho$ be a pure tensor of $(0, s)$. An operator $\mathfrak{L}$ according to $\mathfrak{Y}$ is given by

$$\begin{aligned}(\mathfrak{L}_{\mathfrak{Y}}\mho)(\mathbb{X}, \mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}_s) &= (\mathfrak{Y}\mathbb{X}\mho(\mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}_s)) \\ &- \mathbb{X}\mho(\mathfrak{Y}\mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}_s) \\ &+ \sum_{\mu=1}^{s}\mho(\mathbb{Y}_1, ..., (\mathfrak{L}_{\mathbb{Y}_\mu}\mathfrak{Y})\mathbb{X}, ..., \mathbb{Y}_s),\end{aligned}$$

for $\mathbb{X}, \mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}_s \in \mathfrak{X}(\mathcal{M}^*)$, where $\mho$ is a pure $(0, s)$-type tensor and $\mathfrak{L}_{\mathbb{Y}_\mu}\mathfrak{Y}$ is Lie derivative according to $\mathbb{Y}_\mu$. Therefore the integrability condition of the almost Meta-Metallic structure can be given as follow by considering Theorem 2.1 in [8].

Theorem 3.12. *Let $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ be an almost Meta-Metallic Riemannian manifold. The almost Meta-Metallic structure is integrable if $\mathfrak{L}_{\mathfrak{Y}}\tilde{\mathfrak{g}} = 0$.*

3.2. *Curvatures of Meta-Metallic manifolds*

Let $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ be a Meta-Metallic Riemannian manifold with $\nabla\mathfrak{Y} = 0$. Let's define the curvature tensor field with

$$\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{Z}_1 = \nabla_{\mathbb{X}_1}\nabla_{\mathbb{Y}_1}\mathbb{Z}_1 - \nabla_{\mathbb{Y}_1}\nabla_{\mathbb{X}_1}\mathbb{Z}_1 - \nabla_{[\mathbb{X}_1,\mathbb{Y}_1]}\mathbb{Z}_1,$$

for any $\mathbb{X}_1, \mathbb{Y}_1, \mathbb{Z}_1 \in \mathfrak{X}(\mathcal{M}^*)$.

Proposition 3.13. *Let $(\mathcal{M}^*, \tilde{\mathfrak{I}}, \mathfrak{Y}, \tilde{\mathfrak{g}})$ be a Meta-Metallic Riemannian manifold. Then, for all $\mathbb{X}_1, \mathbb{Y}_1, \mathbb{Z}_1 \in \mathfrak{X}(\mathcal{M}^*)$, we have followings:*

1. $\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)\mathfrak{Y} = \mathfrak{Y}\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)$,
2. $\mathfrak{R}(\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1) = \mathfrak{R}(\tilde{\mathfrak{I}}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1) - \mathfrak{p}\mathfrak{R}(\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1) + \mathfrak{p}\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)$,
3. $\mathfrak{R}(\tilde{\mathfrak{I}}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1 = \mathfrak{R}(\mathbb{X}_1, \tilde{\mathfrak{I}}\mathfrak{Y}\mathbb{Y}_1)$.

Proof. (1) Since $\mathfrak{Y}$ is parallel, we have $\nabla_{\mathbb{X}_1}\mathfrak{Y}\mathbb{Y}_1 = \mathfrak{Y}\nabla_{\mathbb{X}_1}\mathbb{Y}_1$, therefore we get $\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)\mathfrak{Y} = \mathfrak{Y}\mathfrak{R}(\mathbb{X}_1, \mathbb{Y}_1)$.

(2) $\tilde{\mathfrak{I}}$ is parallel because of the parallellity of $\mathfrak{Y}$. If we use features of curvature tensor, we have

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1, \mathbb{W}_1) = \tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{Z}_1, \mathbb{W}_1)\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1).$$

By the first property and considering equation (15), we find

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathfrak{Y}\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1, \mathbb{W}_1) = \tilde{\mathfrak{g}}(\mathfrak{Y}\mathfrak{R}(\mathbb{Z}_1, \mathbb{W}_1)\mathbb{X}_1, \mathfrak{Y}\mathbb{Y}_1),$$

and

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathfrak{Y}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)=\tilde{\mathfrak{g}}(\tilde{\mathfrak{I}}\mathfrak{R}(\mathbb{Z}_1,\mathbb{W}_1)\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)-\mathfrak{p}\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{Z}_1,\mathbb{W}_1)\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)$$
$$+\mathfrak{p}\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{Z}_1,\mathbb{W}_1)\mathbb{X}_1,\mathbb{Y}_1),$$

respectively. Here since $\tilde{\mathfrak{I}}$ is parallel and also we use properties of curvature tensor, it follows that

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathfrak{Y}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)=\tilde{\mathfrak{g}}(\mathfrak{R}(\tilde{\mathfrak{I}}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)-\tilde{\mathfrak{g}}(\mathfrak{p}\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)$$
$$+\tilde{\mathfrak{g}}(\mathfrak{p}\mathfrak{R}(\mathbb{X}_1,\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1).$$

Thus, the second property is provided.

(3) First of all, we know that

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\tilde{\mathfrak{I}}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)=\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{Z}_1,\mathbb{W}_1)\tilde{\mathfrak{I}}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1).$$

Considering that $\tilde{\mathfrak{I}}$ is parallel here, we find

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\tilde{\mathfrak{I}}\mathbb{X}_1,\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1)=\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\tilde{\mathfrak{I}}\mathfrak{Y}\mathbb{Y}_1)\mathbb{Z}_1,\mathbb{W}_1).$$

Thus, we complete the proof. □

Theorem 3.14. *Let $(\mathcal{M}^*,\tilde{\mathfrak{I}},\mathfrak{Y},\tilde{\mathfrak{g}})$ be a Meta-Metallic Riemannian manifold with dimension m. If $\mathcal{M}^*$ has constant curvature, at least one of the following is satisfied:*

1. *$\mathcal{M}^*$ is flat,*
2. *While the vector fields $\mathbb{X}_1$ and $\mathbb{Z}_1$ on $\mathcal{M}^*$ are perpendicular, the vector fields $\tilde{\mathfrak{I}}\mathbb{X}_1$ and $\mathbb{Z}_1$ are also perpendicular,*
3. *While the vector fields $\mathbb{X}_1$ and $\mathbb{Z}_1$ on $\mathcal{M}^*$ are perpendicular, the vector fields $\mathfrak{Y}\mathbb{X}_1$ and $\mathfrak{Y}\mathbb{Z}_1$ are also perpendicular.*

Finally, we can prove that the concept of holomorphic-like cross-sectional curvature doesn't exist on Meta-Metallic Riemannian manifolds by using the following lemma.

Lemma 3.15. *Let $(\mathcal{M}^*,\tilde{\mathfrak{I}},\mathfrak{Y},\tilde{\mathfrak{g}})$ be a Meta-Metallic Riemannian manifold. Then, for any vector field $\mathbb{X}_1$ on $\mathcal{M}^*$, we have*

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1)=0.$$

Proof. With the help of property of the curvature tensor given by $\mathfrak{R}(\mathbb{X}_1,\mathbb{Y}_1)\mathfrak{Y}=\mathfrak{Y}\mathfrak{R}(\mathbb{X}_1,\mathbb{Y}_1)$, we have

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1)=\tilde{\mathfrak{g}}(\mathfrak{Y}\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathbb{X}_1,\mathbb{X}_1),$$

and from equation (10), we get

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1)=\tilde{\mathfrak{g}}(\mathfrak{Y}\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathbb{X}_1,\mathbb{X}_1)=\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1).$$

Via property of curvature tensor, we find

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1)=\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)=-\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1).$$

Therefore, we obtain

$$\tilde{\mathfrak{g}}(\mathfrak{R}(\mathbb{X}_1,\mathfrak{Y}\mathbb{X}_1)\mathfrak{Y}\mathbb{X}_1,\mathbb{X}_1)=0.$$

□

4. Conclusion

Metallic Riemannian manifolds were presented by Crasmareanu and Hretcanu [5]. Metallic manifolds, which is one of the most studied manifolds with polynomial structures, allowed us to define a new manifold class, namely Meta-Metallic manifolds, by considering the concept of Meta-Metallic-Chi ratio that we introduced. The geometric properties of this new manifold can be studied from many perspectives. Considering our paper, there will be opportunity to lift many geometric structures to the bundle theory. On the other hand, the submanifolds of this new type manifolds and properties of the induced structures to submanifolds create important study areas. Considering the application areas of Riemannian manifolds alone, it is clear that different application areas will emerge with this new class. In this respect, our paper has potential for further research.

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