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On the eccentricity-based invariants of uniform hypergraphs

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Abstract. Let $G = (V, E)$ be a simple connected hypergraph with V the vertex set and E the edge set, respectively. The eccentricity of vertex v refers to the farthest distance of vertex v from other vertices of G, denoted by $\varepsilon_G(v)$. The eccentric adjacency index (EAI) of G is described as $\xi^{ad}(G) = \sum_{u\in V(G)} \frac{S_G(u)}{\varepsilon_G(u)}$, where $S_G(u) = \sum_{v\in N_G(u)} d_G(v)$. In this work, we consider the gerneralation of the EAI for hypergraphs to draw several conclusions related to extremal problems to EAI. We first propose several bounds on the EAI of k-uniform hypertrees with fixed maximum degree, diameter and edges, respectively, and characterize the corresponding extremal k-uniform hypertrees. Then we investigate the relationsip between EAI and the adjacent eccentric distance sum. Finally, we present the upper bounds for the difference between the eccentricity distance sum and eccentric connectivity index in the k-uniform hypergraph with diameter 2. It generalizes the previous results of the current authors from the simple graphs to hypergraphs for graph parameters based on eccentricity.

1. Introduction

The vertex set and edge set of the hypergraph G are writed as $V(G)$ and $E(G)$, separately, of which $V(G)$ is not empty as well as every edge in G is noempty subset of vertices. If every edge in the hypergraph has exactly the same number of vertices k, where $k \geq 2$, we call it a k-uniform hypergraph, A 2-uniform hypergraph is usually known as an ordinary simple graph. The degree of a vertex v, refer to the number of edges in G containing v, denoted by $d_G(v)$. We use Δ to represents the maximum degree. Hypergraph theory had a application background in chemistry [8, 13, 14]. The authors of [13] confirmed that hypergraphs are more accurate when describing molecular structure. A walk $H = (V, E)$ is a sequence of vertices and edges of hypergraph, denoted by $W_p = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$, where $v_i \in V$ and $e_i \in E$ such that $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, p$. We refer to all walks with distinct vertices and edges as paths. A hypergraph G is said to be *connected* if any pair of vertices in G connected by path. The length of the shortest path connecting u and v in G, called the distance between them, expressed by $d_G(u, v)$. The *diameter* $d(G)$ refers to the maximum eccentricity in the hypergraph. A connected acyclic hypergraph is called a hypertree. Let the number of vertices and edges of the hypertree be n and m, respectively, where $m, n \geq 1$. Then n and m meet the condition $n = 1 + m(k-1)$. For conceptes and symbols not defined here, we refer to [2].

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Molecular topological index is a kind of invariant of graph, which is an important structural parameter in the study of QSPR/QSAR. Among them, wiener index (WI) is the earliest topological index based on distance, represents the distance sum of any pair of vertices in the graph G [23], namely

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u\in V(G)} D_G(u).$$

Where $D_G(u) = \sum_{v\in V(G)} d_G(u,v)$.

Recent studies on WI of graphs can be founded in [7, 9]. Lately, index based on eccentricity has become a research hotspot. Next, we introduce four such topological indices that are relevant to our conclusion. In 2002, Sardana et al.[20] put forward a new index related to eccentricity named adjacent eccentric distance sum (AEDS), which is described as follows:

$$\xi^{sv}(G) = \sum_{v\in V(G)} \frac{\varepsilon_G(v)D_G(v)}{d_G(v)}.$$

Sardana and Madan [21] later explored the connection between wiener index and adjacent eccentric distance sum. Recent findings on AEDS can be found in references [3, 15, 16, 19, 25] and the literatures they cited.

In 2001, another invariant of graph based on eccentric is proposed by Madan et al. [17], named the eccentric-adjacency index (EAI), in a connected graph G, it is defined as

$$\xi^{ad}(G) = \sum_{u\in V(G)} \frac{S_G(u)}{\varepsilon_G(u)}.$$

Where $S_G(u) = \sum_{v\in N_G(u)} d_G(v)$.

The eccentric-adjacency index has important application in QSPR/QSAR theories. Lately, Malik [15] discussed graph parameters EAI and AEDS in two classes of combinatorial graphs for join and corona products of graphs and obtained the corresponding formulas. The exact upper bound of EAI is given and the relationship between the above two types of graph parameters is explored by Hua et al. [12].

Sharma et al.[22] put forward eccentric connectivity index (ECI) of graph G, labeled as $\xi^c(G)$, which is defined as

$$\xi^c(G) = \sum_{v\in V(G)} \varepsilon_G(v)d_G(v) = \sum_{uv\in E(G)} W(uv). \tag{1}$$

Where $W(uv) = \varepsilon_G(v) + \varepsilon_G(u)$ can be viewed as a edge weight of the edge uv.

The eccentric connectivity index showed strong predictive power in drug properties; see [4, 20]. In addition, the above index has many use in neuroscience and entropy; see [24, 26].

In 2002, Gupta et al. [10] proposed a new graph parameter based on eccentric distance named the eccentricity distance sum (EDS) of graph G, denoted by $\xi^d(G)$, which is expressed as

$$\xi^d(G) = \sum_{\{u,v\}\subseteq V(G)} (\varepsilon_G(v) + \varepsilon_G(u))d_G(u,v) = \sum_{v\in V(G)} \varepsilon_G(v)D_G(v). \tag{2}$$

The eccentricity distance sum is closely related to biological activity and physical properties. As for their mathematical properties, the eccentricity distance sum and eccentric connectivity index were studied extensively, for example, the extremal problems of EDS and ECI are discussed in graphs with given parameters by zhang et al. [27]. In particular, they studied the difference between EDS and ECI, and obtain the strict upper and lower bound for their difference. More results on ECI and EDS are available in [1, 5, 6, 11, 18, 25, 28].

Given all the issues stated above, researchers have been exploring the extremal problems and realations between various graph invariants. We address the problems above and extend these to hypergraphs. The rest of the paper is arranged as follows. In Section 2, Some graph transformations are presented. In Section 3, with the application of transformatios. we prove that in all k-uniform hypertrees given edges, the k-uniform loose path is the only graph with minimum EAI and k-uniform hyperstar is the unique one with maximum EAI. In Section 4, Several bounds on the EAI of k-uniform hypertrees given diameter and maximum degree are proposed, furthermore, the corresponding extremal graph are characterized. In the last part, the tight upper bound on the difference between eccentriccity distance sum and eccentric connectvity index among linear k-uniform hypergraphs of diameter 2 is determined. Furthermore, the relationship between EAI and AEDS is investigated.

Before going any further, we introduce other terms and symbols. Given a hypergraph G and $X \subseteq V(G)$, we use $G\backslash X$ to denote the subgraph obtained by removing the set of vertices in X from G, and $G[X]$ the subgraph derived by X. For $v \in V(G)$, we simply use $G\backslash v$ instead of $G\backslash\{v\}$. A vertex-edge sequence $(v_0, e_1, v_1, \cdots, v_{s-1}, e_s, v_s)$ in a hypergraph G is known as a pendant path on v_0, if $d_G(v_0) \geq 2$, $d_G(v_i) = 2$ for $1 \leq i \leq s-1$, $d_G(v) = 1$ for $v \in e_i\backslash\{v_{i-1}, v_i\}$ with $1 \leq i \leq s$ and $d_G(v_s) = 1$. If an edge E_i is a pendant edge in G, satisfy the condition $|E_i| \geq 2$, and E_i share only one vertex with other edges. A pendent vertex refers to the vertex of degree one. A vertex of hypergraph G is a cut vertex if it is a coalescence vertex of two nontrivial connected sub-hypergraphs. A loose path of length m is a hypergraph with m edges $e_1, \cdots, e_m$ such that $|e_i \cap e_{i+1}| = 1$ for $1 \leq i \leq m-1$ and $|e_i \cap e_j| = 0$ otherwise, which is denoted by P_m^k. For a k-uniform hypertree T given m hyperedges. If there exists a vertex $v \in V(T)$, satisfy for every edge $e \in E(T)$, there is $v \in e$, then T is called hyperstar with v as the center, write it as S_m^k. Specially, S_1^k is a hypergraph with only one edge.

2. Hypergraph transformation changing eccentric-adjacency index

In this section, in order to characterize the extremal structure of hypertree on EAI, we study some transformations which can change the value of EAI.

We introduce some key lemmas as follows.

Lemma 2.1. *Let G be a k-uniform connect hypergraph with $|E(G)| \geq 1$ and $w \in V(G)$. P is a k-uniform loose path with $p+q$ hyperedges, denoted by $P = (u_0, e_1, u_1, \cdots, u_{p+q-1}, e_{p+q}, u_{p+q})$, Let H (H',respectively) be the hypergraph constructed by identifying w of G and u_p of P (u_{p+1} of P ,respectively), where $p \geq q \geq 1$, then*

$$\xi^{ad}(H) > \xi^{ad}(H').$$

Proof. Note that vertex w in G is coincides with u_p of H and u_{p+1} of H', respectively. If $z \in V(G)\backslash w$, apparently, $S_H(z) = S_{H'}(z)$ and $\varepsilon_H(z) \leq \varepsilon_{H'}(z)$. As $p \geq q$, then

$$\sum_{z \in V(G)\backslash w} \left(\frac{S_H(z)}{\varepsilon_H(z)} - \frac{S_{H'}(z)}{\varepsilon_{H'}(z)}\right) \geq 0.$$

Let $\varepsilon_G(w) = a$, We discuss this in two cases.

Case 1. $a \leq p$.

If $q = 1$, for $z \in V(P)\backslash\{u_{p-1}, u_p, u_{p+1}\}$, we have $S_H(z) \geq S_{H'}(z)$ and $\varepsilon_H(z) \leq \varepsilon_{H'}(z)$, then

$$\sum_{z \in V(P)\backslash\{u_{p-1}, u_p, u_{p+1}\}} \left(\frac{S_H(z)}{\varepsilon_H(z)} - \frac{S_{H'}(z)}{\varepsilon_{H'}(z)}\right) \geq 0.$$

Note that

$$
\begin{aligned}
&S_H(w) = \sum_{u \in N_G(w)} d_G(u) + 2(k-2) + 3 = \sum_{u \in N_G(w)} d_G(u) + 2k - 1, \ \ \varepsilon_H(w) = p \\
&S_H(u_{P+1}) = (k-2) + d_G(w) + 2 = d_G(w) + k, \ \ \varepsilon_H(u_{P+1}) = p + 1 \\
&S_H(u_{P-1}) = 2(k-2) + d_G(w) + 4 = d_G(w) + 2k, \ \ \varepsilon_H(u_{P-1}) = max\{p-1, a+1\} \\
&S_{H'}(w) = \sum_{u \in N_G(w)} d_G(u) + (k-2) + 2 = \sum_{u \in N_G(w)} d_G(u) + k, \ \ \varepsilon_{H'}(w) = p + 1 \\
&S_{H'}(u_P) = 2(k-2) + d_G(w) + 1 + 2 = d_G(w) + 2k - 1, \ \ \varepsilon_{H'}(u_P) = max\{p, a+1\} \\
&S_{H'}(u_{P-1}) = 2(k-2) + 4 = 2k, \ \ \varepsilon_H(u_{P-1}) = max\{p-1, a+2\}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \left(\frac{S_H(w)}{\varepsilon_H(w)} + \frac{S_H(u_{P+1})}{\varepsilon_H(u_{P+1})} + \frac{S_H(u_{P-1})}{\varepsilon_H(u_{P-1})}\right) \\
&- \left(\frac{S_{H'}(w)}{\varepsilon_{H'}(w)} + \frac{S_{H'}(u_P)}{\varepsilon_{H'}(u_P)} + \frac{S_{H'}(u_{P-1})}{\varepsilon_{H'}(u_{P-1})}\right) \\
&= \left(\frac{\sum_{u \in N_G(w)} d_G(u) + 2k - 1}{p} + \frac{d_G(w) + k}{p+1} + \frac{d_G(w) + 2k}{max\{p-1, a+1\}}\right) \\
&- \left(\frac{\sum_{u \in N_G(w)} d_G(u) + k}{p+1} + \frac{d_G(w) + 2k - 1}{max\{p, a+1\}} + \frac{2k}{max\{p-1, a+2\}}\right) \\
&> \left(\frac{2k-1}{p} - \frac{k}{p+1}\right) + (d_G(w) + k)\left(\frac{1}{p+1} - \frac{1}{max\{p, a+1\}}\right) \\
&+ \frac{d_G(w)}{max\{p-1, a+2\}} - \frac{k-1}{max\{p, a+1\}} \\
&= \left(\frac{2k-1}{p} - \frac{2k-1}{max\{p, a+1\}}\right) + \left(\frac{d_G(w)}{max\{p-1, a+2\}} - \frac{d_G(w)}{max\{p, a+1\}}\right) \\
&+ \frac{d_G(w)}{p+1} \\
&> 0.
\end{aligned}
$$

Thus

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

If $q \geq 2$, for $z \in V(P) \backslash \{u_{p-1}, u_p, u_{p+1}, u_{p+2}\}$, we have $S_H(z) \geq S_{H'}(z)$ and $\varepsilon_H(z) \leq \varepsilon_{H'}(z)$, then

$$\sum_{z \in V(P) \backslash \{u_{p-1}, u_p, u_{p+1}, u_{p+2}\}} \left(\frac{S_H(z)}{\varepsilon_H(z)} - \frac{S_{H'}(z)}{\varepsilon_{H'}(z)}\right) \geq 0.$$

note that

$$
\begin{aligned}
&S_H(u_{P-1}) = S_H(u_{P+1}) = d_G(w) + 2k = S_{H'}(u_{P+2}) = S_{H'}(u_P) \\
&S_H(w) = S_{H'}(w) = \sum_{u \in N_G(w)} d_G(u) + 2k \ \ \ \varepsilon_H(w) \leq \varepsilon_{H'}(w) \\
&S_H(u_{P+2}) = 2k = S_{H'}(u_{P-1}) \\
&\varepsilon_H(u_{P+i}) \leq \varepsilon_{H'}(u_{P+i}) \ \ i = -1, 0, 2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\xi^{ad}(H)-\xi^{ad}(H^{'}) &\geq (\frac{S_H(w)}{\varepsilon_H(w)}+\frac{S_H(u_{P+1})}{\varepsilon_H(u_{P+1})}+\frac{S_H(u_{P-1})}{\varepsilon_H(u_{P-1})}+\frac{S_H(u_{P+2})}{\varepsilon_H(u_{P+2})}) \\
&- (\frac{S_{H'}(w)}{\varepsilon_{H'}(w)}+\frac{S_{H'}(u_{P-1})}{\varepsilon_{H'}(u_{P-1})}+\frac{S_{H'}(u_P)}{\varepsilon_{H'}(u_P)}+\frac{S_{H'}(u_{P+2})}{\varepsilon_{H'}(u_{P+2})}) \\
&\geq \frac{S_H(u_{p-1})-S_{H'}(u_{p-1})}{\varepsilon_{H'}(u_{P-1})}+S_H(w)(\frac{1}{\varepsilon_{H(w)}}-\frac{1}{\varepsilon_{H'(w)}}) \\
&+ S_H(u_{p+1})(\frac{1}{\varepsilon_H(u_{P+1})}-\frac{1}{\varepsilon_{H'}(u_P)}) \\
&+ \frac{S_H(u_{p+2})-S_{H'}(u_{p+2})}{\varepsilon_{H'}(u_{p+2})} \\
&\geq \frac{d_G(w)}{\varepsilon_{H'}(u_{P-1})}-\frac{d_G(w)}{\varepsilon_{H'}(u_{P+2})}>0.
\end{aligned}
$$

Where $\varepsilon_{H'}(u_{p-1})<\varepsilon_{H'}(u_{p+2})$. Then

$$\xi^{ad}(H)>\xi^{ad}(H^{'}).$$

Case 2. $a \geq p+1$.
If $q=1$

$$
\begin{aligned}
\xi^{ad}(H)-\xi^{ad}(H^{'}) &\geq \sum_{v\in V(P)}(\frac{S_H(v)}{\varepsilon_H(v)}-\frac{S_{H'}(v)}{\varepsilon_{H'}(v)}) \\
&\geq (\frac{S_H(w)}{\varepsilon_H(w)}+(k-2)\sum_{i=1}^{p-1}\frac{k+1}{i+a}+(k-2)\frac{d_G(w)+k+1}{a+1} \\
&+ \frac{k(k-1)}{p+a}+\frac{(k-1)(d_G(w)+k)}{1+a}+\sum_{i=2}^{p-1}\frac{2k}{i+a}+\frac{d_G(w)+2k}{a+1}) \\
&- (\frac{S_{H'}(w)}{\varepsilon_{H'}(w)}+(k-2)\sum_{i=1}^{p}\frac{k+1}{i+a}+(k-2)\frac{d_G(w)+k}{a+1} \\
&+ \frac{k(k-1)}{p+a+1}+\frac{d_G(w)+2k-1}{1+a}+\sum_{i=2}^{p}\frac{2k}{i+a}) \\
&= \frac{1}{a}(S_H(w)-S_{H'}(w))-\frac{k^2-k-2}{p+a}-\frac{2k}{p+a}+\frac{k^2-k}{p+a} \\
&- \frac{k^2-k}{p+a+1}+\frac{k-1}{a+1}+\frac{(k-1)d_G(w)}{a+1}+\frac{k^2-k}{a+1} \\
&= \frac{k-1}{a}-\frac{2(k-1)}{p+a}+\frac{k-1}{a+1}+\frac{k^2-k}{a+1}-\frac{k^2-k}{p+a+1}+\frac{d_G(w)(k-1)}{a+1} \\
&> \frac{k-1}{a}-\frac{k-1}{p+a}+\frac{k-1}{a+1}-\frac{k-1}{p+a}+\frac{d_G(w)(k-1)}{a+1} \\
&> 0,
\end{aligned}
$$

where

$$S_H(w)=\sum_{u\in N_G(w)}d_G(u)+2k-1,$$

$$S_{H'}(w)=\sum_{u\in N_G(w)}d_G(u)+k.$$

Then

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

If $q \geq 2$, by direct calculation, we have

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \sum_{v \in V(P)} (\frac{S_H(v)}{\varepsilon_H(v)} - \frac{S_{H'}(v)}{\varepsilon_{H'}(v)}) \\
&= (\frac{S_H(w)}{\varepsilon_H(w)} + (k-2)\sum_{i=1}^{p-1}\frac{k+1}{i+a} + 2(k-2)\frac{d_G(w)+k+1}{a+1} \\
&+ (k^2-k)(\frac{1}{p+a} + \frac{1}{q+a}) + \sum_{i=2}^{p-1}\frac{2k}{i+a} + \sum_{i=2}^{q-1}\frac{2k}{i+a} \\
&+ (k-2)\sum_{i=1}^{q-1}\frac{k+1}{i+a}) \\
&- (\frac{S_{H'}(w)}{\varepsilon_{H'}(w)} + (k-2)\sum_{i=1}^{p}\frac{k+1}{i+a} + 2(k-2)\frac{d_G(w)+k+1}{a+1} \\
&+ (k^2-k)(\frac{1}{p+1+a} + \frac{1}{q-1+a}) + \sum_{i=2}^{p}\frac{2k}{i+a} + \sum_{i=2}^{q-2}\frac{2k}{i+a} \\
&+ (k-2)\sum_{i=1}^{q-2}\frac{k+1}{i+a}) \\
&= (k^2-k-2)(\frac{1}{q-1+a} - \frac{1}{p+a}) - (k^2-k)(\frac{1}{q-1+a} - \frac{1}{p+a}) \\
&+ 2k(\frac{1}{q-1+a} - \frac{1}{p+a}) + (k^2-k)(\frac{1}{p+a} - \frac{1}{p+1+a}) \\
&= (2k-2)(\frac{1}{q-1+a} - \frac{1}{p+a}) + (k^2-k)(\frac{1}{p+a} - \frac{1}{p+1+a}) \\
&> 0.
\end{aligned}$$

Where $q-1 < p$, $k \geq 2$ and $p < p+1$. Then

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

To sum up the evidence of cases 1 and 2, we have

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

This completes the proof. □

Lemma 2.2. *Let G be a k-uniform hypergraph with $|E(G)| \geq 2$ and $k \geq 3$. Let u, v be the two vertices of an edge e in G, where $d_G(u) = d_G(v) = 1$. Let $P = (u_0, e_1, u_1, \cdots, u_{p-1}, e_p, u_p)$ and $Q = (v_0, e_1^{'}, v_1, \cdots, v_{q-1}, e_q^{'}, v_q)$ be the pendent path with lengths p on u_0 and q on v_0 respectively, where $p \geq q \geq 1$. Let H be the k-uniform hypergraph constructed by identifying u_0 and u and by identifying v_0 of Q and v respectively, Let $H^{'}$ be the hypergraph constructed from H by moving edge $e_q^{'}$ from u_{q-1} to u_p. Then*

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

Proof. If $z \in V(G)\backslash e$, apparently, $S_H(z) = S_{H'}(z)$ and $\varepsilon_H(z) \leq \varepsilon_{H'}(z)$, as $p \geq q$. Let $\varepsilon_G(u) = a$, then $\varepsilon_G(v) = a$. Clearly, $a \geq 2$, since G is a k-uniform hypergraph with $|E(G)| \geq 2$ and $d_G(u) = d_G(v) = 1$, we discuss it in the following two cases.

Case 1. $p+1 \leq a$.

For $x \in G\backslash\{u,v\}$, we have $S_H(x) = S_{H'}(x)$ and $\varepsilon_H(x) \leq \varepsilon_{H'}(x)$. If $q=1$, note that

$$\begin{aligned}
&S_H(u) = 2(k-2)+4 = 2k \quad \varepsilon_H(u) = a\\
&S_H(v) = 2(k-2)+3 = 2k-1 \quad \varepsilon_H(v) = a\\
&S_{H'}(u) = 2(k-2)+1+2 = 2k-1, \quad \varepsilon_{H'}(u) = a\\
&S_{H'}(v) = k-3+2 = k-1, \quad \varepsilon_{H'}(v) = a.
\end{aligned}$$

We have

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H') &\geq \sum_{x \in V(P)\cup V(Q)} \left(\frac{S_H(x)}{\varepsilon_H(x)} - \frac{S_{H'}(x)}{\varepsilon_{H'}(x)}\right)\\
&= \left(\frac{2k}{a} + (k-2)\sum_{i=1}^{p-1}\frac{k+1}{i+a} + \frac{k(k-1)}{a+p} + \sum_{i=1}^{p-2}\frac{2k}{i+a} + \frac{2k-1}{p-1+a}\right.\\
&+ \left.\frac{2k-1}{a} + \frac{k(k-1)}{a+1} + \frac{(k-2)(k+1)}{a}\right)\\
&- \left(\frac{2k-1}{a} + (k-2)\sum_{i=1}^{p}\frac{k+1}{i+a} + \frac{k(k-1)}{a+p+1} + \sum_{i=1}^{p-1}\frac{2k}{i+a} + \frac{2k-1}{p+a}\right.
\end{aligned}$$

$$\begin{aligned}
&+ \left.\frac{k-1}{a} + \frac{(k-2)k}{a}\right)\\
&= \frac{k+1}{a} - \frac{(k-2)(k+1)}{a+p} + (k^2-k)\left(\frac{1}{p+a} - \frac{k^2-k}{p+a+1}\right) - \frac{2k}{p+a-1}\\
&+ (2k-1)\left(\frac{1}{p-1+a} - \frac{1}{p+a}\right) + \frac{k^2-k}{1+a} + \frac{k-2}{a}\\
&= \frac{k+1}{a} + \frac{2}{p+a} - \frac{k^2-k}{p+1+a} - \frac{1}{p-1+a} - \frac{2k-1}{p+a} + \frac{k^2-k}{1+a} + \frac{k-2}{a}\\
&= \left(\frac{k+1}{a} - \frac{k-1}{p+a}\right) - \frac{k-2}{p+a} + \left(\frac{k^2-k}{1+a} - \frac{k^2-k}{p+1+a}\right) + \frac{k-2}{a} - \frac{1}{p+a-1}\\
&= \left(\frac{k}{a} - \frac{k-1}{p+a}\right) + \left(\frac{k-2}{a} - \frac{k-2}{p+a}\right) + \left(\frac{k^2-k}{1+a} - \frac{k^2-k}{p+1+a}\right) + \left(\frac{1}{a} - \frac{1}{p-1+a}\right)\\
&> 0
\end{aligned}$$

Then

$$\xi^{ad}(H) > \xi^{ad}(H').$$

If $q \geq 2$, then

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \sum_{x \in V(P) \cup V(Q)} (\frac{S_H(x)}{\varepsilon_H(x)} - \frac{S_{H'}(x)}{\varepsilon_{H'}(x)}) \\
&= (\frac{2k}{a} + (k-2)\sum_{i=1}^{p-1} \frac{k+1}{i+a} + \frac{k(k-1)}{a+p} + \sum_{i=1}^{p-2} \frac{2k}{i+a} + \frac{2k-1}{p-1+a}) \\
&+ \frac{2k}{a} + (k-2)\sum_{i=1}^{q-1} \frac{k+1}{i+a} + \frac{k^2-k}{q+a} + \sum_{i=1}^{q-2} \frac{2k}{i+a} + \frac{2k-1}{q-1+a}) \\
&- (\frac{2k}{a} + (k-2)\sum_{i=1}^{p} \frac{k+1}{i+a} + \frac{k(k-1)}{a+p+1} + \sum_{i=1}^{p-1} \frac{2k}{i+a} + \frac{2k-1}{p+a} \\
&+ \frac{2k}{a} + (k-2)\sum_{i=1}^{q-2} \frac{k+1}{i+a} + \frac{k(k-1)}{q-1+a} + \sum_{i=1}^{q-3} \frac{2k}{i+a} + \frac{2k-1}{q-2+a}) \\
&= -(k-2)\frac{k+1}{p+a} + (k^2-k)(\frac{1}{p+a} - \frac{1}{p+a+1}) - \frac{2k}{p-1+a} \\
&+ (2k-1)(\frac{1}{p-1+a} - \frac{1}{p+a}) + (k-2)\frac{k+1}{q-1+a} + \frac{2k}{q-2+a} \\
&+ (k^2-k)(\frac{1}{q+a} - \frac{1}{q+a-1}) + (2k-1)(\frac{1}{q-1+a} - \frac{1}{q-2+a}) \\
&= \frac{2}{p+a} - \frac{k^2-k}{a+p+1} - \frac{1}{p-1+a} - (2k-1)\frac{1}{p+a} - \frac{2}{q-1+a}
\end{aligned}$$

$$\begin{aligned}
&+ (k^2-k)\frac{1}{q+a} + \frac{1}{q-2+a} + \frac{2k-1}{q-1+a} \\
&= (\frac{2}{p+a} - \frac{2}{q-1+a}) + (k^2-k)(\frac{1}{q+a} - \frac{1}{p+a+1}) \\
&+ (\frac{1}{q-2+a} - \frac{1}{p-1+a}) + (2k-1)(\frac{1}{q-1+a} - \frac{1}{p+a}) \\
&= (k^2-k)(\frac{1}{q+a} - \frac{1}{p+a+1}) + (\frac{1}{q-2+a} - \frac{1}{p-1+a}) + (2k-3)(\frac{1}{q-1+a} - \frac{1}{p+a}) \\
&> 0.
\end{aligned}$$

Then

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

Case 2. $p \geq a$.

Let $u = u_0$, $v = v_0$. Let $e_i = \{u_{i,1}, \cdots, u_{i,k}\}$. Where $u_{i,1} = u_{i-1}$ and $u_{i,k} = u_i$. Note that $u_{i,k} = u_{i+1,1}$ for $i = 1, \cdots, p$. Let $e_{p+1} = \{u_{p+1,1}, \cdots, u_{p+1,k}\}$ be the pendent edge at u_p in $H^{'}$, where $u_{p+1,1} = u_p$ and $u_{p+1,k} = u_{p+1}$. For $z \in e \backslash \{u, v\}$, $\varepsilon_H(z) = p + 1 \leq \varepsilon_{H'}(z) = p + 2$. To prove our conclusion, we discuss the following two subcases.

Subcase 2.1. $q = 1$.

For $i = 1, 2, \cdots, p$ and $t = 2, \cdots, k-1$, we have $S_H(u_{i,t}) = S_{H'}(u_{i+1,t})$.

$$\varepsilon_H(u_{i,t}) - \varepsilon_{H'}(u_{i+1,t}) = max\{i + a, p + 1 - i\} - max\{i + 1 + a, p + 1 - i\} \leq 0.$$

If $t = k \neq p$, then $S_H(u_{i,t}) \geq S_{H'}(u_{i+1,t})$

$$\varepsilon_H(u_{i,t}) - \varepsilon_{H'}(u_{i+1,t}) = max\{i + a, p - i\} - max\{i + 1 + a, p - i\} \leq 0.$$

Hence the contribution of vertices in $\cup_{i=1}^{p-1} e_i \backslash \{u_0\}$ to $\xi^{ad}(H)$ is not as less as the contribution of vertices in $\cup_{i=1}^{p} e_i \backslash \{u_1\}$ to $\xi^{ad}(H^{'})$. Therefore, we have

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \sum_{x \in V(e \cup e_1^{'} \cup e_p)} (\frac{S_H(x)}{\varepsilon_H(x)} - \frac{S_{H'}(x)}{\varepsilon_{H'}(x)}) \\
&= (\sum_{z \in e \backslash \{u,v\}} \frac{S_H(z)}{p+1} + \frac{2k}{p} + \frac{2k-1}{p+1} + \frac{k^2-k}{p+2} + \frac{2k-1}{p+a-1} + \frac{k^2-k}{p+a}) \\
&- (\sum_{z \in e \backslash \{u,v\}} \frac{S_H(z)}{p+2} + \frac{2k-1}{p+1} + \frac{k}{p+2} + \frac{2k}{p+a-1} + \frac{k^2-3k+2}{p+a} \\
&+ \frac{2k-1}{p+a} + \frac{k^2-k}{p+a+1})
\end{aligned}$$

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq (\frac{k^2-k-2}{p+1} + \frac{2k}{p} + \frac{k^2-k}{p+2} + \frac{2k-1}{p+a-1} + \frac{k^2-k}{p+a}) \\
&- (\frac{k^2-k}{p+2} + \frac{2k}{p+a-1} + \frac{k^2-k+1}{p+a} + \frac{k^2-k}{p+a+1}) \\
&> \frac{k^2-k-2}{p+1} + \frac{2k}{p} - \frac{k^2-k+1}{p+a-1} - \frac{1}{p+a} \\
&= (k^2-k+1)(\frac{1}{p+1} - \frac{1}{p+a-1}) + (\frac{k}{p} - \frac{3}{p+1}) + (\frac{k}{p} - \frac{1}{p+a}) \\
&> 0.
\end{aligned}$$

Then

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

Subcase 2.2. $q \geq 2$.

For $i = 1, 2, \cdots, q$. Let $e_i^{'} = \{v_{i,1}, \cdots, v_{i,k}\}$, where $v_{i,1} = v_{i-1}$ and $v_{i,k} = v_i$. Note that $v_{i,k} = v_{i+1,1}$ for $i = 1, 2, \cdots, q-1$. For $i = 1, 2, \cdots, p-1$ and $t = 2, \cdots, k-1$. we have $S_H(u_{i,t}) = S_{H'}(u_{i+1,t})$. If $t \geq k$ and $a \leq q$, we have

$$\varepsilon_H(u_{i,t}) - \varepsilon_{H'}(u_{i+1,t}) = max\{i+q+1, p+1-i\} - max\{i+1+q, p+1-i\} \leq 0.$$

If $t \geq k$ and $a \geq q+1$, we have

$$\varepsilon_H(u_{i,t}) - \varepsilon_{H'}(u_{i+1,t}) = max\{i+a, p-i+1\} - max\{i+1+a, p-i+1\} \leq 0.$$

Thus the contribution of vertices in $\cup_{i=1}^{p-1} e_i \backslash \{u_{p-1}\}$ to $\xi^{ad}(H)$ is not as less as the contribution of vertices in $\cup_{i=2}^{p} e_i \backslash \{u_p\}$, and the contribution of vertices in $\cup_{i=2}^{q-1} e_i^{'} \backslash \{v_{q-1}\}$ to $\xi^{ad}(H)$ is not as less as the contribution of vertices in $\cup_{i=1}^{q-2} e_i^{'} \backslash \{v_{q-2}\}$ to $\xi^{ad}(H^{'})$. Does not lose its generality, we suppose that $a \geq q+1$, we get

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \sum_{x\in V(e_p \cup e_q^{'} \cup e_{q-1}^{'})} \left(\frac{S_H(x)}{\varepsilon_H(x)} - \frac{S_{H'}(x)}{\varepsilon_{H'}(x)}\right) \\
&= \left(\frac{2k-1}{a+p-1} + \frac{k^2-k}{a+p} + \frac{2k}{p+q-1} + \frac{2k-1}{p+q-1}\right. \\
&+ \left.\frac{(k-2)(k+1)}{p+q-1} + \frac{k^2-k}{q+p}\right) \\
&- \left(\frac{2k}{a+p-1} + \frac{k+1}{a+p} + \frac{2k-1}{a+p} + \frac{k^2-k}{p+q-1}\right. \\
&+ \left.\frac{2k-1}{p+q-1} + \frac{k^2-k}{a+p+1}\right)
\end{aligned}$$

$$\begin{aligned}
\xi^{ad}(H) - \xi^{ad}(H^{'}) &\geq \frac{2k-2}{p+q-1} - \frac{1}{a+p-1} + \frac{k^2-4k}{a+p} + (k^2-k)\left(\frac{1}{p+q} - \frac{1}{a+p+1}\right) \\
&> \frac{2k-3}{a+p-1} + \frac{k^2-4k}{a+p} + (k^2-k)\left(\frac{1}{p+q} - \frac{1}{a+p+1}\right) \\
&> \frac{k^2-2k-3}{a+p} + (k^2-k)\left(\frac{1}{p+q} - \frac{1}{a+p+1}\right) \\
&> \frac{(k-1)^2-4}{a+p} \\
&> 0.
\end{aligned}$$

Where $a \geq q+1$ and $k \geq 3$. Thus

$$\xi^{ad}(H) > \xi^{ad}(H^{'}).$$

Similarly, the same is true for $a < q$. Synthesizing the proof of Case 1 and Case 2, we get $\xi^{ad}(H) > \xi^{ad}(H^{'})$. It yields the result. □

Lemma 2.3. *Let G be k-uniform hypergraph with a cut edge $e = \{v_1, v_{1,1}, \cdots, v_{1,k-2}, v_2\}$ satisfy $G - e$ consists of two componets G_1 and G_2, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let $G^{'}$ be the hypergraph constructed from G by moving G_2 from v_2 to v_1, then $\xi^{ad}(G) < \xi^{ad}(G^{'})$.*

Proof. For convenience, let $\varepsilon_{G_1}(v_1) = a \geq b = \varepsilon_{G_2}(v_2)$. If $v \in V(G_1 \cup G_2)\backslash\{v_1, v_2\}$, apparently, $S_G(v) \leq S_{G'}(v)$ and $\varepsilon_G(v) \geq \varepsilon_{G'}(v)$. We have

$$\sum_{v\in V(G_1\cup G_2)\backslash\{v_1,v_2\}} \left(\frac{S_G(v)}{\varepsilon_G(v)} - \frac{S_{G'}(v)}{\varepsilon_{G'}(v)}\right) \leq 0.$$

If $v \in e\backslash\{v_1, v_2\}$, it is easy to see that $\varepsilon_G(v) = \varepsilon_{G'}(v)$ and $S_G(v) \leq S_{G'}(v)$, we have

$$\sum_{v\in e\backslash\{v_1,v_2\}} \left(\frac{S_G(v)}{\varepsilon_G(v)} - \frac{S_{G'}(v)}{\varepsilon_{G'}(v)}\right) \leq 0.$$

Thus the contribition of vertices in $V(G_1 \cup G_2)\backslash\{v_1, v_2\}$ and $e\backslash\{v_1, v_2\}$ to $\xi^{ad}(G)$ in G is no more than the contribition of these vertices to $\xi^{ad}(G^{'})$ in $G^{'}$. Therefore, we have

$$\begin{aligned}
\xi^{ad}(G)-\xi^{ad}(G^{'}) &= \sum_{v\in V(G)}(\frac{S_H(v)}{\varepsilon_H(v)}-\frac{S_{H'}(v)}{\varepsilon_{H'}(v)})\\
&= (\sum_{v\in V(G_1\cup G_2)\backslash\{v_1,v_2\}}+\sum_{v\in e\backslash\{v_1,v_2\}}+\sum_{v\in\{v_1,v_2\}})(\frac{S_G(v)}{\varepsilon_G(v)}-\frac{S_{G'}(v)}{\varepsilon_{G'}(v)})\\
&\le \sum_{v\in\{v_1,v_2\}}(\frac{S_G(v)}{\varepsilon_G(v)}-\frac{S_{G'}(v)}{\varepsilon_{G'}(v)}).
\end{aligned}$$

$$\begin{aligned}
\xi^{ad}(G)-\xi^{ad}(G^{'}) &\le \frac{S_G(v_1)}{\varepsilon_G(v_1)}-\frac{S_{G'}(v_1)}{\varepsilon_{G'}(v_1)}+\frac{S_G(v_2)}{\varepsilon_G(v_2)}-\frac{S_{G'}(v_2)}{\varepsilon_{G'}(v_2)}\\
&= \frac{\sum_{u\in N_{G_1}(v_1)}d_{G_1}(u)+(k-2)+d_{G_2}(v_2)}{max\{a,b+1\}}\\
&+ \frac{\sum_{w\in N_{G_2}(v_2)}d_{G_2}(w)+(k-2)+d_{G_1}(v_1)}{a+1}\\
&- \frac{\sum_{u\in N_{G_1}(v_1)}d_{G_1}(u)+(k-1)+\sum_{w\in N_{G_2}(v_2)}d_{G_2}(w)}{a}\\
&- \frac{(k-2)+d_{G_1}(v_1)+d_{G_2}(v_2)}{a+1}\\
&\le \frac{-1+d_{G_2}(v_2)-\sum_{w\in N_{G_2}(v_2)}d_{G_2}(w)}{a}\\
&+ \frac{\sum_{w\in N_{G_2}(v_2)}d_{G_2}(w)-d_{G_2}(v_2)}{a+1}\\
&\le -\frac{1}{a}\\
&< 0.
\end{aligned}$$

Thus

$$\xi^{ad}(G)<\xi^{ad}(G^{'}).$$

This completes the proof. □

Lemma 2.4. *Let H be a k-uniform connected hypergraph and u be a vertex of H. For $t\ge 1$ is a positive integer, Let G be a k-uniform hypergraph that adds a pendent edge to the vertex u and then adds one pendent edge to each of some t vertices of the pendent edge at u, where these t vertices different from u. Let $G^{'}$ be the k-uniform hypergraph obtained from H with new $t+1$ pendent edges attaching to u. Then $\xi^{ad}(G)<\xi^{ad}(G^{'})$.*

Proof. It is evident that $S_{G'}(x)\ge S_G(x)$ and $\varepsilon_{G'}(x)\le\varepsilon_G(x)$ for $x\in V(H)$. Let $\varepsilon_H(u)=b$, obviously,

$$\varepsilon_G(u)\ge\varepsilon_{G'}(u)=b>1.$$

$$S_{G'}(u)=\sum_{w\in N_H(u)}d_H(w)+(t+1)(k-1).$$

Let

$$P_1=\{v\mid \text{ v is the vertex of t pendent edges in G}\}$$

and

$$P_2=\{v\mid \text{ v is the vertex of the pendent edge e at u in G}\}$$

respectively. Note that if $x\in P_1\cup P_2$, $S_{G'}(x)=k-2+d_H(u)+t$ and $\varepsilon_{G'}(x)=b+1$, $S_G(u)=\sum_{w\in N_H(u)}d_H(w)+k+t-1$.

If $x_1 \in P_1 \backslash P_2$, $S_G(x_1) = k-2+2t$, $\varepsilon_G(x_1) = b+2$.
If $x_2 \in P_2 \backslash P_1$, $S_G(x_2) = k+t-2+d_H(u)$, $\varepsilon_G(x_2) = b+1$.
If $x \in P_1 \bigcap P_2$, $S_G(x) = 2k+t-4+d_H(u)$, $\varepsilon_G(x) = b+1$. Then, we have

$$
\begin{aligned}
\xi^{ad}(G^{'}) - \xi^{ad}(G) &\geq \frac{\sum_{w \in N_H(u)} d_H(w) + (t+1)(k-1)}{b} + \frac{(k+t-2+d_H(u))(t+1)(k-1)}{b+1} \\
&- \frac{\sum_{w \in N_H(u)} d_H(w) + k + t - 1}{\varepsilon_G(u)} - \frac{(k-1)t(k+2t-2)}{b+2} \\
&- \frac{(k-1-t)(k+t-2+d_H(u))}{b+1} - \frac{t(2k+t-4+d_H(u))}{b+1} \\
&> \frac{t(k-2)}{b} + \frac{(t(k-1)+(k-1))(k+t-2+d_H(u))}{b+1} \\
&- \frac{t(k-1)(k+2t-2)}{b+1} - \frac{(k-1-t)(k+t-2+d_H(u))}{b+1} \\
&- \frac{t(2k+t-4+d_H(u))}{b+1} \\
&= \frac{t(k-2)}{b} + \frac{t(k(d_H(u)-2t) - (d_H(u)-3t))}{b+1} \\
&> 0.
\end{aligned}
$$

Where $k \geq 2$, $d_H(u) - 2t \geq d_H(u) - 3t$. Then $\xi^{ad}(G) < \xi^{ad}(G^{'})$. This completes the proof. □

3. Hypertrees with small eccentric-adjacency index

In this section, for eccentric-adjacency index, we characterize the extremal k-uniform hypertree given maximum degree and edges. Denote $S^k(m_1, \cdots, m_\Delta)$ by a k-uniform hypertree with $m_1 + m_2 + \cdots + m_\Delta$ edges acquired from Δ loose paths $P^k_{m_1}, P^k_{m_2}, \cdots, P^k_{m_\Delta}$ by coinciding one terminal vertex of every loose path, where $\Delta \geq 2$. In pariticular, $S^k_{m_1,m_2} = P^k_{m_1+m_2}$, obviously, $S^k(m_1, m_2, \cdots, m_\Delta)$ is a hypertree with maximum degree Δ. Let $B^k_{m,\Delta} = S(\underbrace{1, 1, \cdots, 1}_{\Delta-1}, m-\Delta+1)$, call it a starlike hypertree with maximum degree Δ and number of edges m. In particular, $B^k_{m,\Delta} \cong P^k_m$ if $\Delta = 1, 2$ and $B^k_{m,m} \cong S^k_m$.

Theorem 3.1. *Let T be a k-uniform hypertree with maximum degree Δ and m edges, where $1 \leq \Delta \leq m$. Then $\xi^{ad}(T) \geq \xi^{ad}(B^k_{m,\Delta})$, the equality is true if and only if $T \cong B^k_{m,\Delta}$.*

Proof. It is trival if $\Delta = 1$. Suppose that $\Delta \geq 2$, Let T be a k-uniform hypertree with maximum degree Δ and minimum eccentric-adjacency index. we intend to prove that $T \cong B^k_{m,\Delta}$ below. Let $d_T(u) = \Delta$. Next, we discuss in two different conditions.

Case 1. $\Delta \geq 3$.

First, We assert that there exist only one vertex u of T that has a degree of at least 3. By contradiction assume that there are two vertices in T with degrees at least 3. Choose a vertex v that is as far away from u as possible and has a degree of at least 3. Assume $T - v$ have $d_T(v)$ branches $T_1, \cdots, T_{d_T(v)}$, Suppose that $u \in V(T_1)$ without generality, among the remaining branches except T_1, there exist branches that are not pendent paths. Let T_i be such a branch and $e = \{w_1, \cdots, w_k\} \in E(T_i)$ such that the distance between w_1 and v is as large as possible, Note that $d_H(w_s) = d_H(w_t) = 1$. According to Lemma 2.2, we get a hypertree $T^{'}$ with maximum degree Δ, obviously, we have $\xi^{ad}(T) \geq \xi^{ad}(T^{'})$, which is contradiction, therefore $T[V(T_i) \cup \{v\}]$ is a pendent path adjacent to v with $2 \leq i \leq d_T(v)$. Set the length of the pendent path $T[V(T_i) \cup \{v\}]$ on v to l_i, where $2 \leq i \leq d_T(v)$ and $l_i \geq 1$. Suppose instead we only talked of $l_2 \geq l_3$. Move a pedent edge from the pedent path in T_3 to the pendent edge in T_2, we obtain a new hypertree $T^{''}$ whose maximum degree is Δ. By Lemma 2.1, we have $\xi^{ad}(T^{'}) \geq \xi^{ad}(T^{''})$, which is contradiction, so u is the only vertex in T that has a degree of at least 3. Similar to the discussion above and according to Lemma 2.2, T contains Δ pendent paths on u.

Second, We assert that at most one pendent path is not less than 2 in length. Assume to the contrary that there exist at least two pendent paths on u with length at least two. Let P and Q be two such paths, having length p and q, respectively, may as well suppose that $p \geq q$. By the transformation in Lemma 2.1, we obtain the graph T''' by removing the pendent edge on Q and adding them to the pengdent edge on P, so that the length of P increases by one and the length of Q decreases by one. By Lemma 2.1, we have $\xi^{ad}(T) \geq \xi^{ad}(T''')$, which is contradiction. Thus there is at most one pendent path of length at least 2. From above arguments, it follows that $T \cong B^k_{m,\Delta}$.

Case 2. $\Delta = 2$.

It is trival if $k = 2$. May as well set $k \geq 3$. Assume to the contrary that $T \ncong B^k_{m,2}$, then there exists an edge in T that contains at least three vertices of degree two, let $e = \{w_1, \cdots, w_k\}$ be such an edge, and the distance between u and the endpoint w_1 of e is as large as possible. There are two pendent paths P and Q at two distinct vertices of e, say w_j and w_l, respectively, where $2 \leq j \leq l \leq k$. Let the lengths of P and Q be p and q, respectively, such that $p \geq q \geq 1$. From the transformation in Lemma 2.2, we get a graph T' whose maximum degree is still 2. By Lemma 2.2, we have $\xi^{ad}(T) \geq \xi^{ad}(T')$, which is contradiction. Thus each edge in T has at most two terminal vertices of degree two, it follows that $T \cong B^k_{m,2}$. In summary, this concludes the proof of the theorem. □

Theorem 3.2. *Let T be a k-uniform hypertree with $m \geq 1$ edges. Then Let*

$$\xi^{ad}(T) \geq \begin{cases} \dfrac{2k^2+2k}{m} - \dfrac{2}{m-1} + \sum_{i=1}^{\frac{m}{2}-1} \dfrac{2k^2+2k-4}{m-i} & \text{if } m \text{ is even,} \\ \dfrac{2k^2+2k-6}{m-1} + \dfrac{2k^2-2k}{m} + \dfrac{10k+4}{m+1} + \sum_{i=2}^{\frac{m-3}{2}} \dfrac{2k^2+2k-4}{m-i} & \text{if } m \text{ is odd,} \end{cases}$$

the equation is true if and only if $T \cong P^k_m$.

Proof. The conclusion is obvious for $m = 1, 2$. Let's say that $m \geq 3$. T is the graph with minimal EAI in the k-uniform hypertree with m edges. Denote Δ by the largest degree of T. Thus it follows from Theorem 3.1, $T \cong B^k_{m,\Delta}$. If $\Delta \geq 3$, then by Lemma 2.1, we can obtain that $\xi^{ad}(B_{m,\Delta}) > \xi^{ad}(B_{m,\Delta-1})$, which is contradiction. Then $\Delta = 2$ and thus $T \cong B^k_{m,2} = P^k_m$. If m is even, then by direct computation, we have

$$\xi^{ad}(P^k_m) = \frac{2k^2+2k}{m} - \frac{2}{m-1} + \sum_{i=1}^{\frac{m}{2}-1} \frac{2k^2+2k-4}{m-i}.$$

And if m is odd, then

$$\xi^{ad}(P^k_m) = \frac{2k^2+2k-6}{m-1} + \frac{2k^2-2k}{m} + \frac{10k+4}{m+1} + \sum_{i=2}^{\frac{m-3}{2}} \frac{2k^2+2k-4}{m-i}.$$

This completes the proof. □

4. Hypertrees with large eccentric-adjacency index

In the coming section, we characterize the extremal hepertrees having the largest EAI among k-uniform hypertrees given diameter and number of edges.

Theorem 4.1. *Let T be a connected k-uniform hypertree with $m \geq 2$ edges. Then*

$$\xi^{ad}(T) \leq \frac{m(k-1)(m+k)}{2},$$

the equation is true if and only if $T \cong S^k_m$.

Proof. Let T be the extremal graph with maximum eccentric-adjacency index in k-uniform hypertree with m edges. We start the proof with a claim.

Claim 1. All cut edges in T are pendent edges.

Proof of Claim 1. In reverse we say that, there exists a non-pendent cut edge, by Lemma 2.3, one can get a hypergraph T', meeting the condition $\xi^{ad}(T') > \xi^{ad}(T)$, a contradiction, and hence completes the proof of claim 1.

There is no cycle in T since it is a hypertree, then all edges are cut edges in T, by Claim 1 and Lemma 2.4, all edges share a common vertex, says it u, i.e. $d_T(u) = m$ and $\varepsilon_T(u) = 1$. The other vertices different from u with degree 1 and eccentricity 2. Thus

$$\xi^{ad}(T) = \frac{m(k-1)}{1} + \frac{m(k-1)(k-2+m)}{2} = \frac{m(k-1)(m+k)}{2}.$$

the equality is true when $T \cong S_m^k$. This completes the proof. □

For $2 \le d \le m$. Let

$$G(m,d,k) = \begin{cases} \xi^{ad}(P_m^k) + \dfrac{2t(k-1)}{d} + \dfrac{2t(k-1)(m+d+k+t)}{d+2} & \text{if } m \text{ is even,} \\ \xi^{ad}(P_m^k) + \dfrac{2t(k-1)}{d+1} + \dfrac{2t(k-1)(m+d+k+t)}{d+3} & \text{if } m \text{ is odd,} \end{cases}$$

Let $F_1(m,d)$ be the k-uniform hypertree obtained by adding $m-d$ pendent edges on $u_{\frac{d}{2}}$ from loose path $P_d^k = (u_0, e_1, u_1, \cdots, u_{d-1}, e_d, u_d)$, if d is even. Let $F_2(m,d)$ be the k-uniform hypertree obtained by adding $m-d$ pendent edges on $e_{\frac{d+1}{2}}$ from $P_d^k = (u_0, e_1, u_1, \cdots, u_{d-1}, e_d, u_d)$, when d is odd.

Theorem 4.2. *Let T be a k-uniform connected hypertree with $m \ge 2$ edges and diameter d, where $2 \le d \le m$. Then $\xi^{ad}(T) \le G(m,d,k)$, the equality holds when $T \cong F_1(m,d)$, if d is even; and the equality is true when $T \cong F_2(m,d)$, if d is odd.*

Proof. For $d = 2$ and $d = m$, it is obviously true. Assume that $3 \le d \le m-1$ and T is a k-uniform hypertree with diameter d satisfying that $\xi^{ad}(T)$ is as large as possible. Let $P = (u_0, e_1, u_1, \cdots, u_{d-1}, e_d, u_d)$ be a diametrical path of T. Denoted by $e_i = \{u_{i,1}, \cdots, u_{i,k}\}$ for $i = 1, \cdots, d$, where $u_{i,1} = u_{i-1}$ and $u_{i,k} = u_i$, By using Lemma 2.3 repeatly, we infer that all edges other than edges in P are pendent edges, which are adjacent to vertices of P except these in e_1 and e_d. We discuss this in two cases.

Case 1. d is even.

Suppose P has at least a vertex u other than $u_{\frac{d}{2}}$ adjacent to no less than one pendent edge. Let $E(u)$ be the branch that contains vertex u and does not contain edges in P. Moving the edges in $E(u)$ from u to $u_{\frac{d}{2}}$, We construct a new hypertree T'. Let $|E(u)| = t$ and $H = T[V(T)\backslash E(u)\backslash\{u\}]$ with $\varepsilon_H(u) = a$. Then $a > \dfrac{d}{2}$ and $\varepsilon_H(u_{\frac{d}{2}}) = \frac{d}{2}$, we have

$$\begin{aligned} \xi^{ad}(T') - \xi^{ad}(T) &= \frac{t(k-1)}{d/2} + \frac{t(k-1)(k+t-2+d_T(u_{\frac{d}{2}}))}{d/2+1} \\ &- \left(\frac{t(k-1)}{a} + \frac{t(k-1)(k+t-2+d_T(u))}{a+1}\right) \\ &> t(k-1)\left(\frac{1}{d/2} - \frac{1}{a}\right) + t(k-1)(k+t-2+d_T(u))\left(\frac{1}{d/2+1} - \frac{1}{a+1}\right) \\ &> 0. \end{aligned}$$

Where $d_T(u_{\frac{d}{2}}) \ge d_T(u)$, then $\xi^{ad}(T') > \xi^{ad}(T)$, arrive at a contradiction. Therefore, all edges not on P are pendent edges attached to $u_{\frac{d}{2}}$. Hence, we have

$$\xi^{ad}(T) = \xi^{ad}(P_m^k) + \frac{2t(k-1)}{d} + \frac{2t(k-1)(m+d+k+t)}{d+2}.$$

Case 2. d is odd.

If P contains some vertex $u_{i,j}$ which are adjacent to at least one pendent edge out of P, and furthermore $u_{i,j} \notin e_{\frac{d+1}{2}}$. If $\frac{d+1}{2} < i \leq d-1, 2 \leq j \leq k$, then $\varepsilon_T(u_{i,j}) = i$. By moving each edge adjacent to $u_{i,j}$ outside of P to $u_{\frac{d}{2}}$, we obtain k-uniform hypertree T', similarly, we can derive that $\xi^{ad}(T') > \xi^{ad}(T)$, draw a contradiction. Then all edges other than P are pendent edges attached to the vertices of $e_{\frac{d+1}{2}}$, we have

$$\xi^{ad}(T) = \xi^{ad}(P_m^k) + \frac{2t(k-1)}{d+1} + \frac{2t(k-1)(m+d+k+t)}{d+3}.$$

End of the proof. □

5. Comparing various eccentricity-based graph invariants for k-uniform hypergraphs

In this part, We first explored the connection between AEDS and EAI for k-uniform hypertree, the upper bound of the difference between EDS and ECI in the hypergraph with diameter 2 is also discussed.

Theorem 5.1. *Let T be a k-uniform hypertree with $m \geq 2$ edges. Then $\xi^{sv}(T) > \xi^{ad}(T)$.*

Proof. According to theorem 4.1, we just need to testify $\xi^{sv}(T) > \dfrac{m(k-1)(m+k)}{2}$. Let DS_{m_1,m_2} be the dumbbell hypertree obtained from vertex-disjiont $S_{m_1}^k$ with center u and $S_{m_2}^k$ with center v by adding a hyperedge e with k vertices, write it as $e = \{u, v, w_1, \cdots, w_{k-2}\}$. Let $t(\geq 0)$ be the number of vertices $v_i \in V(T)$ such that $d_T(v_i) \geq \varepsilon_T(v_i) + 1$, We will get the coming claim.

Claim 1. $t \leq \left\lceil \frac{m(k-1)}{2} \right\rceil$ for $m \geq 2$ and $k \geq 2$.

Proof of Claim 1. Let's say the diameter of T is d. If $d = 2$, we have $T \cong S_m^k$ and hence $t = 1 < \left\lceil \frac{m(k-1)}{2} \right\rceil$. If $d = 3$, we have $T \cong DS_{m_1,m_2}$, let's say $m_1 \geq m_2$, then $m_1 + m_2 + 1 = m$ and hence $2 \leq t \leq \left\lceil \frac{m(k-1)}{2} \right\rceil$ for $m \geq 5$. (For $m = 4$,there is $t = 1 < \left\lceil \frac{m(k-1)}{2} \right\rceil$.) If $d = 4$, let T be the hypertree with vertex v such that

$$T - \{v\} = S_{m_1+1}^k \cup S_{m_3+1}^k \cup tm_2.$$

Where v is the unique vertex in T satisfing $\varepsilon_T(v) = 2$ and $m_1 + m_2 + m_3 + 4 = m$. It was very clear $m(k-1) \geq 2t$, that is $t \leq \left\lceil \frac{m(k-1)}{2} \right\rceil$. Or else $d \geq 5$ and $\varepsilon_T(v_i) \geq 3$ for any vertex $v_i \in V(T)$. On the contrary, we prove this result for $\varepsilon_T(v_i) \geq 3$, $v_i \in V(T)$, we assume that $t > \left\lceil \frac{m(k-1)}{2} \right\rceil$, let t be the number of verticecs such that $d_T(v_i) \geq \varepsilon_T(v_i) + 1 \geq 4$ as $\varepsilon_T(v_i) \geq 3$. Thus, we have

$$km = \sum_{i=1}^{n} d_T(v_i) = \sum_{v_i:\varepsilon_T(v_i)\geq 3} d_T(v_i) \geq 4t > 2m(k-1).$$

Draw a contradiction and complete the proof of the claim. □

If $T \cong S_m^k$, then one can easily see that

$$\xi^{sv}(T) = (k-1) + 2m(k-1)^2(2m-1) > \frac{m(k-1)(m+k)}{2}.$$

Otherwise, if $T \ncong S_m^k$, then $D_T(v_i) \geq m(k-1) + 1$ for any $v_i \in V(T)$. From claim 1, we conclude that

$$\begin{aligned}
\xi^{sv}(T) &= \sum_{v_i \in V(T)} \frac{\varepsilon_T(v_i) D_T(v_i)}{d_T(v_i)} \geq \sum_{v_i:\varepsilon_T(v_i)\geq d_T(v_i)} \frac{\varepsilon_T(v_i) D_T(v_i)}{d_T(v_i)} \\
&\geq \sum_{v_i:\varepsilon_T(v_i)\geq d_T(v_i)} D_T(v_i) \geq (m(k-1)+1-t)(m(k-1)+1) \\
&> \frac{(m(k-1)+1)^2}{2} > \frac{m(k-1)(k+m)}{2}.
\end{aligned}$$

This proof is completed. □

Next step, we discussed the upper bound of the difference between EDS and ECI in the k-uniform hypergraph with diameter 2. First, we present one result useful for our main conclusion. A linear hypergraph is one in which any two edges have at most one common vertex.

Lemma 5.2. *Let G be a connected linear k-uniform hypergraph with n vertices, diameter 2 and maximum degree no more than $|E|-1$, where $|E|$ denotes the number of edges of G. Then $|E| \geq \dfrac{n-2+k}{k-1}$, with the equality holds if and only if the maximum degree is $|E|-1$.*

Proof. Let u_0 be a vertex of G with the maximum degree. Then exists at least one edge e satisfies $u_0 \notin e$. Since $diam(G) = 2$, we have $d_G(w,u) \leq 2$ for all $u \in N_G(u_0)$ and $w \in e$. Clearly, the greater the degree of u_0, the fewer the number of edges. Let $V_1 = \{x | x \in e \cap N_G(u_0)\}$. We have

$$\begin{aligned} k|E| &= \sum_{v\in V(G)} d_G(v) = d_G(u_0) + \sum_{v\in V_1} d_G(v) + \sum_{v\in N_G(u_0)\setminus V_1} d_G(v) \\ &\geq |E| - 1 + 2k + n - 1 - k \end{aligned}$$

$$(k-1)|E| \geq k + n - 2, \qquad |E| \geq \frac{n-2+k}{k-1}.$$

With the equality holds if and only if the maximum degree is $|E|-1$. This completes the proof. □

From the arguments above, it can be clearly seen $k > |E|-1$, otherwise $d > 2$. Thus, in the case under considerarion, $k > \dfrac{n-2+k}{k-1} - 1 = \dfrac{n-1}{k-1}$. Therefore, $k^2 > n-1+k$.

Let $S(n,k)$ be a k-uniform hyperstar with n vertices. We now prove our main result.

Theorem 5.3. *Let G be a connected linear k-uniform hypergraph with n vertices and diameter 2. Then*

$$\xi^d(G) - \xi^c(G) \leq 4n^2 - 8n + 4 - 2k(n-1),$$

equality is true when $G \cong S(n,k)$.

Proof. Let G be a k-uniform hypergraph with diameter 2. For any $v \in V(G)$, we have

$$D_G(v) = (k-1)d_G(v) + 2(n-1-(k-1)d_G(v)) = 2n - (k-1)d_G(v) - 2. \tag{3}$$

Let $|E|$ be the number of edges of G, obviously, the degree of any vertex is at most $|E|$. Let $\Delta(G)$ be the maximum degree of G. First, we consider $\Delta(G) = |E|$. As G is linear and satisfied $|E| \geq 2$, it is impossible for two edges to have two common vertices. If there exists one common vertex say u of every edge, then $d_G(u) = |E|$ and $\varepsilon_G(u) = 1$ and for any vertex $v \neq u$, $\varepsilon_G(v) = 2$. Let $A = \{x | \varepsilon_G(x) = 1\}$. In this case, clearly, $|A| = 1$. Together with (3),(1) and (2), we have

$$\begin{aligned} \xi^d(G) - \xi^c(G) &= \sum_{v\in V(G)} \varepsilon_G(v) D_G(v) - \sum_{v\in V(G)} \varepsilon_G(v) d_G(v) \\ &= 2 \sum_{v\in V(G)\setminus A} (2n - (k-1)d_G(v) - 2 - d_G(v)) \\ &= 4(n-1)^2 - 2k \sum_{v\in V(G)\setminus A} d_G(v) \\ &= 4n^2 - 8n + 4 - 2k^2|E| + 2k|E|. \end{aligned} \tag{4}$$

From equation (4), we find that the difference of $\xi^d(G)$ and $\xi^c(G)$ is decreasing on $|E|$, so when the value of $|E|$ is the smallest, then the difference is the largest. Note that $diam(G) = 2$ and G is linear k-uniform connected hypergraph, one has $|E| \geq \frac{n-1}{k-1}$ in (4). By direct computation, we easily see that

$$\begin{aligned} \xi^d(G) - \xi^c(G) &= 4n^2 - 8n + 4 - 2k^2|E| + 2k|E| \\ &= 4n^2 - 8n + 4 - 2k(k-1)|E| \\ &\leq 4n^2 - 8n + 4 - 2k(k-1)\frac{n-1}{k-1} \\ &= 4n^2 - 8n + 4 - 2k(n-1). \end{aligned} \tag{5}$$

Now consider $\Delta(G) \leq |E| - 1$. In this case, $\varepsilon_G(v) = 2$ for every $v \in V(G)$, Together with (3),(1) and (2), we have

$$\begin{aligned} \xi^d(G) - \xi^c(G) &= \sum_{v \in V(G)} \varepsilon_G(v) D_G(v) - \sum_{v \in V(G)} \varepsilon_G(v) d_G(v) \\ &= 2 \sum_{v \in V(G)} (2n - kd_G(v) - 2) \\ &= 4n^2 - 4n - 2k^2|E|. \end{aligned} \tag{6}$$

By Lemma 5.2, we have $|E| \geq \frac{n-2+k}{k-1}$ in (6). By direct computation, we easily see that

$$\begin{aligned} \xi^d(G) - \xi^c(G) &= 4n^2 - 4n - 2k^2|E| \\ &\leq 4n^2 - 4n - \frac{2k^2(n-2+k)}{k-1}. \end{aligned} \tag{7}$$

Let

$$f_1(n,k) = 4n^2 - 4n - \frac{2k^2(n-2+k)}{k-1},$$

$$f_2(n,k) = 4n^2 - 8n + 4 - 2k(n-1).$$

$$\begin{aligned} f_2(n,k) - f_1(n,k) &= 4n + 4 + \frac{2k^2(n-2+k)}{k-1} - 2k(n-1) \\ &= 4n + 4 + 2k\left(\frac{k(n-2+k)}{k-1} - (n-1)\right) \\ &= 4n + 4 + 2k\left(\frac{k^2+n-k-1}{k-1}\right) \\ &= 4n + 4 + 2k\left(\frac{(k^2-k)+(n-1)}{k-1}\right) \\ &> 0. \end{aligned} \tag{8}$$

Where $k \geq 2$ and $n \geq 2$. On the basis of equation (5), (7) and (8) , one can get

$$\xi^d(G) - \xi^c(G) \leq 4n^2 - 8n + 4 - 2k(n-1),$$

with equality if and only if $G \cong S(n,k)$. This completes the proof. □

Conflict of Interest

All authors in the paper have no conflict of interest.

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