Filomat 38:1 (2024), 343–356
https://doi.org/10.2298/FIL2401343S

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Enriched asymptotically nonexpansive mappings with center zero

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Abstract. In this article, a special mapping, the so-called enriched nonexpansive mapping with center zero and its asymptotic version are introduced and corresponding fixed point properties are investigated in the setting of complete normed spaces. Further, using approximate fixed point sequences, the fixed points of such mappings are analysed where Banach spaces have Kadec-Klee Property. Finally, a convexically enriched nonexpansive mapping is launched as a special case of the studied mappings.

1. Introduction

Consider nonempty subsets X, Y, that intersects each other by at least one point, a point $p \in X$ is said to be a fixed point of a mapping $G : X \to Y$ if

$$p = Gp.$$

Fixed point theory (PPT for short) provides substantial approaches of solving many real-life problems from science and engineering through fixed point of certain nonlinear mapping. Recently, PPT has been applied directly to solve convex optimization problems including image restoration, signal processing, radiation therapy, robotic motion control and many further physical phenomena. To the best of our knowledge, the most celebrated fixed point result is the Banach Contraction Mapping Theorem which was initially established in linear spaces (see [1]) and later extended to metric spaces. This result guaranteed the existence and uniqueness of a fixed point for a contraction mapping defined from a complete metric space into itself.

The immediate extension (or the limit case) of the contraction mapping is nonexpansive mapping which may not have a fixed point or even have many fixed points. For examples on the set of real numbers, consider the identity mapping and its positive and negative translation. Using geometric properties of Banach spaces, several results were established and analysed in regards to fixed points of nonexpansive mappings ranging from 1965 to date (see, for example, the standard text [2–7] and the references therein).

2020 *Mathematics Subject Classification*. 47H05, 47H09, 47H10
Keywords. Enriched asymptotically nonexpansive mapping with center zero, fixed point, strictly convex Banach space, weakly continuous mapping
Received: 28 February 2023; Accepted: 19 July 2023
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It is worth mentioning from these results that a nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space possesses a fixed point. Applications of fixed point of nonexpansive mappings can be found in many science phenomena. For instance, these mappings appear as the transition operators for initial value inclusions of the form

$$0 \in \frac{du}{dt} + G(t)u,$$

(see, for example, Bruck [8]). For example where nonexpansive mappings are used in robotic motion control, see [9].

In [10], the class of nonexpansive mappings having a fixed center is studied and analysed. The author, deduced that this class of mappings properly contains the class of quasi-nonexpansive mappings by the fact that the center of the mapping need not be a fixed point. Dehici and Atailia [11] studied the class of asymptotically nonexpansive mappings with each having zero as a center. They established existence of fixed points of the mapping provided it is defined on a bounded convex closed subset of reflexive strictly convex Banach spaces.

On the other hands, Berinde introduced and studied enriched contraction mapping in [12] as an extension of the Banach contraction mapping and introduced enriched nonexpansive in [13]. It is deduced from the definitions that the class of enriched nonexpansive mappings contains nonexpansive mappings, enriched contraction mappings, some strictly pseudocontractive mappings and many other mappings. Furthermore, this class of mappings are widely studied recently up to set-valued mappings and significant applications of such mappings are discussed (see, for example, [14–17]).

The purpose of this paper is to study enriched asymptotically nonexpansive mapping (EANM for short) that has zero as a center. This class of mappings contains several other special classes of mappings with center zero such as nonexpansive mapping, enriched nonexpansive mapping and asymptotically nonexpansive mapping. We establish theorems on the existence of fixed point of the mappings in the setting where Kadec-kalee property holds and discuss substantial special classes of the mappings.

2. Preliminaries and Basic Facts

Throughout this paper, we denote a nonempty subset of a normed linear space $(\mathcal{Y}, \|\cdot\|)$ by $\mathcal{E}$ and the β-relaxation of a mapping G is define by

$$G_\beta u = \frac{\beta}{\beta+1}u + \frac{1}{\beta+1}Gu, \tag{1}$$

for every u in the domain of G.

We now recall the following definitions

Definition 2.1. *A mapping $G : \mathcal{E} \to \mathcal{E}$ is said to be*

(i) a nonexpansive if

$$\|Gu - Gw\| \leq \|u - w\| \quad \textit{for all} \quad u, w \in \mathcal{E}. \tag{2}$$

(ii) an asymptotically nonexpansive (ANM for short) if

$$\limsup_{n\to\infty} \|G^n u - G^n w\| \leq \|u - w\| \quad \textit{for all} \quad u, w \in \mathcal{E}. \tag{3}$$

It is important to note that, according to Goebel and Kirk [18], a mapping $G : \mathcal{E} \to \mathcal{E}$ is asymptotically nonexpansive if there exists a sequence $\{\kappa_n\}$ that converges to 1 and the following inequality holds:

$$\|G^n u - G^n w\| \leq \kappa_n \|u - w\| \quad \text{for all} \quad u, w \in \mathcal{E}. \tag{4}$$

Remark 2.2. *It is known that the notion of ANM in* (3) *is more general than that of* (4). *In fact,* (2) *implies* (4) *and* (4) *implies* (3). *However, the converses are not true. For counterexamples, see [18, pp. 174] and [19, pp. 1208].*

Nonexpansive mappings and their generalizations such as asymptotically nonexpansive mappings are widely studied by many researchers. Most of the researchers investigate the fixed point properties of the mappings. A natural geometrically motivated extensions of the mappings are subsequently define using zero as a center.

Definition 2.3. *A mapping $G : \mathcal{E} \to \mathcal{E}$ is called*

(i) a nonexpansive mapping with center zero if

$$\|Gu\| \leq \|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{5}$$

(ii) an asymptotically nonexpansive mapping with center zero if

$$\limsup_{n\to\infty} \|G^n u\| \leq \|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{6}$$

Remark 2.4. *Clearly every nonexpansive mapping with zero as a fixed point is nonexpansive mapping with center zero. However, the converse is not always true. For a counterexample, take $\mathcal{E} = [0, 1] \subset \mathbb{R}$ and G defined by $Gu = u^2$. Similar ideas can be taken to the case of asymptotically nonexpansive mappings.*

Following Berinde [13], a mapping $G : \mathcal{E} \to \mathcal{E}$ is said to be *β-enriched nonexpansive* if there exists $\beta \geq 0$ such that

$$\|\beta(u - w) + Gu - Gw\| \leq (\beta + 1)\|u - w\|, \ \forall u, w \in \mathcal{E}. \tag{7}$$

Remark 2.5. *It is easy to see that every nonexpansive mapping is β-enriched nonexpansive mapping with $\beta = 0$. However, the converse is not true as in the subsequent example.*

Example 2.6. *Let $G : \mathbb{R} \to \mathbb{R}$ be defined by $Gu = -3u$ for all $u \in \mathbb{R}$. It is obvious that G is not a nonexpansive mapping. However G is* 1*-enriched nonexpansive mapping since for all $u, w \in \mathbb{R}$,*

$$|1(u - w) + Gu - Gw| = |2w - 2u| \leq (1 + 1)|u - w|.$$

Remark 2.7. *Observe that an enriched nonexpansive mapping must be Lipchitzs continuous. Indeed if G is a β-enriched nonexpansive mapping, then for all $u, w \in Dom(G)$, we have*

$$\begin{aligned}\|Gu - Gw\| &= \|\beta(u - w) + Gu - Gw - \beta(u - w)\| \\ &\leq \|\beta(u - w) + Gu - Gw\| + \beta\|u - w\| \\ &\leq (2\beta + 1)\|u - w\|.\end{aligned}$$

3. Enriched Asymptotically Nonexpansive

In this paper we shall focus on mappings that need not be continuous starting with the following definitions to be used for the main results.

Definition 3.1. *A mapping $G : \mathcal{E} \to \mathcal{E}$ is called an enriched nonexpansive mapping with center zero if there exists $\beta \geq 0$ such that*

$$\|\beta u + Gu\| \leq (\beta + 1)\|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{8}$$

Remark 3.2. *Clearly, every β-enriched nonexpansive mapping with zero as a fixed point is an enriched nonexpansive mapping with center zero. The converse may not be true as shown in the following example.*

Example 3.3. *Let $\sigma < -1$ and consider a mapping $G : \mathbb{R} \to \mathbb{R}$ such that*

$$u \mapsto \begin{cases} 0 & \text{if } u = \sigma; \\ \sigma u & \text{if } u \neq \sigma. \end{cases}$$

Observe that 0 is a fixed point of G. Also by Remark 2.7, we have that G is not a β-enriched nonexpansive mapping since G is not continuous. However, for any $\beta \geq \frac{-\sigma - 1}{2}$, we have $|\beta + \sigma| \leq |\beta + 1|$ which leads to

$$\|\beta u + Gu\| = |\beta u + \sigma u| = |\beta + \sigma||u| \leq (\beta + 1)|u|$$

whenever $u \neq \sigma$. When $u = \sigma$, we have

$$\|\beta u + Gu\| = |\beta u| = \beta|u| \leq (\beta + 1)|u|$$

for any $\beta \geq 0$. Hence G is a β-enriched nonexpansive mapping with center zero for $\beta \geq \frac{-\sigma - 1}{2}$.

Definition 3.4. *Let $G : \mathcal{E} \to \mathcal{E}$ be a mapping. Then we call G an enriched asymptotically nonexpansive mapping if one of the following conditions is satisfied:*

(C1) there exists $\beta_1 \geq 0$ such that

$$\limsup_{n\to\infty} \|\beta_1(u - w) + G^n u - G^n w\| \leq (\beta_1 + 1)\|u - w\| \quad \text{for all} \quad u, w \in \mathcal{E}, \tag{9}$$

(C2) there exists $\beta_2 \geq 0$ such that the n-th composition of G_{β_2} ($n \in \mathbb{N}$) is well-defined and

$$\limsup_{n\to\infty} \|G^n_{\beta_2} u - G^n_{\beta_2} w\| \leq \|u - w\| \quad \text{for all} \quad u, w \in \mathcal{E}. \tag{10}$$

Remark 3.5. *An enriched asymptotically nonexpansive mapping differs from an enriched nonexpansive mapping with center zero. This can be seen from the following example.*

Example 3.6. *For $\mu > 1$, consider a mapping $G : \mathbb{R} \to \mathbb{R}$ such that*

$$w \mapsto \begin{cases} \mu & \text{if } w = 0; \\ \mu|w|^{1/2} + w - \mu w & \text{otherwise.} \end{cases}$$

It is worth noting that

1. *1 is a fixed point of G.*
2. *G is not a β-enriched nonexpansive mapping since G is not continuous.*
3. *G is not an enriched nonexpansive mapping with center zero since for $w = 0$,*

$$|\beta w + Gw| = \left|\mu\right| > 0 = (\beta + 1)|w| \quad \text{for all } \beta \geq 0.$$

4. *G is an enriched asymptotically nonexpansive mapping. To see this, take $\beta = \mu - 1$ and recall that $G_\beta : \mathbb{R} \to \mathbb{R}$ is such that $w \mapsto \frac{\beta}{\beta + 1} w + \frac{1}{\beta + 1} Gw$. So, G_β reduces to the following*

$$G_\beta w = \begin{cases} 1 & \text{if } w = 0; \\ |w|^{1/2} & \text{otherwise.} \end{cases}$$

Consequently, we have

$$G_\beta^2 w = \begin{cases} 1 & \text{if } w = 0; \\ |w|^{\frac{1}{2^2}} & \text{otherwise,} \end{cases} \quad \cdots \quad G_\beta^n w = \begin{cases} 1 & \text{if } w = 0; \\ |w|^{\frac{1}{2^n}} & \text{otherwise.} \end{cases}$$

Therefore, we get

$$\limsup_{n\to\infty} |G_\beta^n u - G_\beta^n w| = 0 \le |u - w| \quad \text{for all} \quad u, w \in \mathbb{R}. \tag{11}$$

Definition 3.7. *Let $G : \mathcal{E} \to \mathcal{E}$ be a mapping. Then we call G an enriched asymptotically nonexpansive mapping with center zero if one of the following conditions is satisfied:*

(C1) there exists $\beta_1 \ge 0$ such that

$$\limsup_{n\to\infty} \|\beta_1 u + G^n u\| \le (\beta_1 + 1)\|u\| \quad \text{for all} \quad u \in \mathcal{E}, \tag{12}$$

(C2) there exists $\beta_2 \ge 0$ such that the n-th composition of G_{β_2} ($n \in \mathbb{N}$) is well-defined and

$$\limsup_{n\to\infty} \|G_{\beta_2}^n u\| \le \|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{13}$$

Remark 3.8. *It is worth noting that every enriched asymptotically nonexpansive mapping with zero as a fixed point is an enriched asymptotically nonexpansive mapping with center zero. The immediate example furnishes a counterexample for the converse.*

Example 3.9. *Let $G : [0, 1] \to [0, 1]$ be defined by $Gw = w^2$ for all $w \in [0, 1]$. Then it is clear that*

1. *0 and 1 are fixed points of G.*
2. *G is not an enriched asymptotically nonexpansive mapping. To show this, observe that*

$$G^n u = u^{2^n} \to \begin{cases} 0 & \text{if } u \ne 1; \\ 1 & \text{if } u = 1. \end{cases}$$

Thus for $u = 1$ and $w = \frac{1}{2}$ we can see that

$$\limsup_{n\to\infty} |\beta_1(u - w) + G^n u - G^n w| = \frac{1}{2}\beta_1 + 1 > \frac{1}{2}(\beta_1 + 1) = (\beta_1 + 1)|u - w|.$$

Hence (9) is not possible. Moreover, for any $\beta_2 \ge 0$, we have $G_{\beta_2}^n(1) = 1 \to 1$ and $G_{\beta_2}^n(1/2) \to 0$ as $n \to \infty$. Thus

$$\limsup_{n\to\infty} \left| G_{\beta_2}^n(1) - G_{\beta_2}^n(1/2) \right| = 1 > \frac{1}{2} = \left| 1 - \frac{1}{2} \right|.$$

Hence (10) is impossible. Therefore G cannot be an enriched asymptotically nonexpansive mapping.

3. *G is an asymptotically enriched nonexpansive mapping with center zero. Indeed for any $\beta \ge 0$,*

$$\limsup_{n\to\infty} |\beta u + G^n u| = \beta u \le (\beta + 1)|u| \quad \text{for all} \quad u \ne 1$$

and

$$\limsup_{n\to\infty} |\beta \cdot 1 + G^n(1)| = (\beta + 1) = (\beta + 1) \cdot |1|.$$

For $r > 0$, let $\bar{B}(0, r)$ denote the closed ball center at 0 with radius r. We recall some beautiful normed spaces.

Definition 3.10. *A Banach space $(\mathcal{Y}, \|\cdot\|)$ is called uniform convex if for $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for all $u, w \in \bar{B}(0, 1)$,*

$$\|u - w\| > \epsilon \quad \Longrightarrow \quad \left\| \frac{u + w}{2} \right\| \leq \delta.$$

Definition 3.11. *A Banach space $(\mathcal{Y}, \|\cdot\|)$ is called strictly convex if for $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for all $u, w \in \mathcal{Y}$, $u \neq w$, we have*

$$\|u\| = 1 = \|w\| \quad \Longrightarrow \quad \left\| \frac{u + w}{2} \right\| \leq 1.$$

Remark 3.12. *Every uniformly convex Banach space is strictly convex. However, the converse is not true in general. To furnish a counterexample, we state the following example (see [20] for details).*

Example 3.13. *Let $\mathcal{Y} = C([a, b])$ be the space of scalar continuous functions defined on $[a, b] \subset \mathbb{R}$. Consider $\|\cdot\|$ defined by*

$$\|g\| = \sup_{t \in [a,b]} |g(t)| + \|g\|_{L^2([a,b])}.$$

Then $(\mathcal{Y}, \|\cdot\|)$ is not uniformly convex but a strictly convex Banach space.

4. Enriched Asymptotically Nonexpansive and Fixed Point

In the what follows, $u_n \rightharpoonup u$ means $\{u_n\}$ converges weakly to u and $u_n \to u$ means $\{u_n\}$ converges strongly to u and β-EANM stands for β-enriched asymptotically nonexpansive mapping.

Theorem 4.1. *Let $\mathcal{E}$ be a nonempty subset of a complete normed space and $G : \mathcal{E} \to \mathcal{E}$ be a β-EANM with center zero. If $0 \in \mathcal{E}$ and G is continuous then $0 \in Fix(G)$.*

Proof. Since G is β-EANM with center zero, we consider two cases.
Case 1: $(C1)$ holds. Then β_1 is such that

$$\limsup_{n \to \infty} \|\beta_1 u + G^n u\| \leq (\beta_1 + 1)\|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{14}$$

This and the hypothesis that $0 \in \mathcal{E}$ imply that

$$\limsup_{n \to \infty} \|G^n 0\| \leq 0.$$

This implies

$$\lim_{n \to \infty} \|G^n 0\| \leq 0$$

and consequently, we have

$$\lim_{n \to \infty} G^n 0 = 0.$$

Since G is continuous, we have

$$\lim_{n \to \infty} G^{n+1} 0 = G0.$$

Uniqueness of limit guaranteed that $G0 = 0$.
Case 2: $(C2)$ holds. Then β_2 is such that

$$\limsup_{n \to \infty} \|G^n_{\beta_2} u\| \leq \|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{15}$$

Following similar lines as in Case 1 with G_{β_2} in place of G and $\beta_1 = 0$, we have that 0 is a fixed point of G_{β_2}. Consequently, we have from (1) that 0 is a fixed point of G. $\square$

The following corollary follows from the fact that every asyptotically nonexpansive mapping is 0-EANM.

Corollary 4.2. *Let $\mathcal{E}$ be a nonempty subset of a complete normed space and $G : \mathcal{E} \to \mathcal{E}$ be an asymptotically nonexpansive mapping with center zero. If $0 \in \mathcal{E}$ and G is continuous then $0 \in Fix(G)$.*

In the sequel, we say that a subset $\mathcal{E}$ of a normed space satisfies *Property* (ℓ) if there exists a unique $w \in \mathcal{E}$ such that

$$\|w\| = \inf\left\{\|z\| : z \in \mathcal{E}\right\}.$$

Remark 4.3. *It is a known fact (see, for example, [11, Theorem 3.1]) that every nonempty closed convex subset of a reflexive strictly convex Banach space satisfies Property (ℓ).*

Theorem 4.4. *Let $\mathcal{E}$ be a closed convex subset of a complete normed space that satisfies satisfies Property (ℓ) and $G : \mathcal{E} \to \mathcal{E}$ be a β-EANM with center zero. If $0 \notin \mathcal{E}$ and G is weakly continuous then $Fix(G) \neq \emptyset$.*

Proof. Following Property (ℓ), we have the existence of a unique point $u_o \in \mathcal{E}$ with

$$\|u_o\| = \inf\left\{\|u\| : u \in \mathcal{E}\right\}. \tag{16}$$

Since G is β-EANM, then, we consider two cases.
Case 1: ($C1$) holds. Then β_1 is such that

$$\limsup_{n\to\infty} \|\beta_1 u_o + G^n u_o\| \leq (\beta_1 + 1)\|u_o\|. \tag{17}$$

Now, let z be in the weak closure of the sequence $\left\{G^n u_o\right\}$. Then there exists a subsequence $\left\{G^{n_k} u_o\right\}$ that converges weakly to z. By lower semi-continuity of the norm, we have

$$\begin{aligned} \|\beta_1 u_o + z\| &\leq \liminf_{k\to\infty} \|\beta_1 u_o + G^{n_k} u_o\| \\ &\leq \liminf_{n\to\infty} \|\beta_1 u_o + G^n u_o\| \\ &\leq \limsup_{n\to\infty} \|\beta_1 u_o + G^n u_o\| \\ &\leq (\beta_1 + 1)\|u_o\|. \end{aligned}$$

This implies that

$$\left\|\frac{\beta_1}{\beta_1 + 1} u_o + \frac{1}{\beta_1 + 1} z\right\| \leq \left\|u_o\right\|.$$

Since $\mathcal{E}$ is closed, $z \in \mathcal{E}$. The last inequality and the convexity of $\mathcal{E}$ imply that

$$z \in \left\{u \in \mathcal{E} : \|u\| = \|u_o\|\right\}.$$

Thus, Property (ℓ) yields that $\dfrac{\beta_1}{\beta_1 + 1} u_o + \dfrac{1}{\beta_1 + 1} z = u_o$. Therefore, $z = u_o$. Hence, every weakly convergence subsequence of $\left\{G^n u_o\right\}$ converges to u_o. Therefore, the sequence $\left\{G^n u_o\right\}$ converges weakly to u_o. So is the $\left\{G^{n+1} u_o\right\}$. Consequently, since $G^{n+1} u_o = G(G^n u_o)$ for all $n \geq 1$, we can pass the weak limit to obtain the desired result.
Case 2: ($C2$) holds. Then β_2 is such that

$$\limsup_{n\to\infty} \|G^n_{\beta_2} u\| \leq \|u\| \quad \text{for all} \quad u \in \mathcal{E}. \tag{18}$$

Following similar lines as in Case 1 with G_{β_2} in place of G and $\beta_1 = 0$, we have that u_o is a fixed point of G_{β_2}. Consequently, we have from (1) that u_o is a fixed point of G. □

Following Theorem 4.1, Remark 4.3 and Theorem 4.4, we have the following corollaries.

Corollary 4.5. *Let $\mathcal{E}$ be a nonempty closed convex subset of a Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be a β-EANM with center zero.*

1. *If $0 \in \mathcal{E}$ and G is continuous then $0 \in Fix(G)$.*
2. *If $\mathcal{Y}$ is reflexive and strictly convex, $0 \notin \mathcal{E}$ and G is weakly continuous, then G has a fixed point.*

Proof.
1. Theorem 4.1 yields the result by the fact that every Banach space is a complete normed space.
2. Following Remark 4.3, we have that $\mathcal{Y}$ satisfies Property (ℓ). Consequently Theorem 4.4 completes the proof.
□

The following theorem is the main result of Dehici and Atailia [11]. See Theorem 3.2 and Remark 3.2 of their article.

Corollary 4.6. *Let $\mathcal{E}$ be a nonempty closed convex subset of a Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be an asymptotically nonexpansive mapping with center zero.*

1. *If $0 \in \mathcal{E}$ and G is continuous then $0 \in Fix(G)$.*
2. *If $\mathcal{Y}$ is reflexive and strictly convex, $0 \notin \mathcal{E}$ and G is weakly continuous, then G has a fixed point.*

Proof. The proof follows from Corollary 4.5 and the fact that every asymptotically nonexpansive mapping with center zero is 0-enriched asymptotically nonexpansive mapping with center zero. □

We now state another important results that can be deduced from our main results.

Corollary 4.7. *Let $\mathcal{E}$ be a nonempty closed convex subset of a complete normed space $(\mathcal{Y}, \|\cdot\|)$ that satisfies Property (ℓ) and $G : \mathcal{E} \to \mathcal{E}$ be a nonexpansive mapping with center zero.*

1. *If $0 \in \mathcal{E}$ then $0 \in Fix(G)$.*
2. *If $0 \notin \mathcal{E}$ and $\mathcal{Y}$ is reflexive strictly convex Banach space, then G has a fixed point.*

Proof. Using the fact that every nonexpansive mapping with center zero is continuous and is 0-enriched asymptotically nonexpansive mapping with center zero, the proof follows from Theorem 4.1 and Theorem 4.4. □

We state below another result.

Corollary 4.8. *Let $\mathcal{E}$ be a nonempty closed convex subset of a reflexive strictly convex Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be a β-enriched nonexpansive mapping with center zero. Then G has a fixed point.*

Proof. The proof follows from the fact that every β-enriched nonexpansive mapping with center zero is β-enriched asymptotically nonexpansive mapping with center zero. □

A substantial fixed point results for generalized nonexpansive mapping (see, for example, [21, Theorem 3] and [22, Theorem 2.1]) asserted that a nonexpansive self-mapping with center zero defined on a weakly compact convex subset of a strictly convex Banach space always possesses a fixed point. The assumption that the domain of the mapping need to be weakly compact is too strong since such a set has to be bounded.

Following the aforementioned theorems, we can have such a result with the domain not necessary weakly compact.

Corollary 4.9. *Let $\mathcal{E}$ be a nonempty closed convex subset of a reflexive strictly convex Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be a nonexpansive mapping with center zero. Then G has a fixed point.*

Proof. The proof follows directly from Corollary 4.7. □

Several existing and further results can be deduced from our results even when dealing with composition or sum of finite mappings. For instance, consider the following corollaries.

Corollary 4.10. *Let $\mathcal{E}$ be a nonempty closed convex subset of a reflexive strictly convex Banach space $(\mathcal{Y}, \|\cdot\|)$. Suppose that $G : \mathcal{E} \to \mathcal{E}$ is a mapping such that for some $\beta \geq 0$,*

$$\|\beta(u+w) + Gu + Gw\| \leq (\beta+1)\|u+w\|, \quad \text{for all} \quad u \in \mathcal{E}. \tag{19}$$

Then G has a fixed point.

Proof. Setting $w = u$ in (19), we get that G is $\beta-$enriched nonexpansive mapping with center zero. Thus Corollary 4.8 yields the desired result. □

Corollary 4.11. *Let $\mathcal{E}$ be a nonempty closed convex subset of a reflexive strictly convex Banach space $(\mathcal{Y}, \|\cdot\|)$ Suppose that $G : \mathcal{E} \to \mathcal{E}$ and $F : \mathcal{E} \to \mathcal{E}$ are two mappings, F is into and there exists $\beta \geq 0$ such that*

$$\left\|\beta Fu + GFu\right\| \leq (\beta+1)\|Fu\| \quad \forall u \in \mathcal{E}.$$

Then G has a fixed point.

Proof. The condition that $\left\|\beta Fu + GFu\right\| \leq (\beta+1)\|Fu\|$ for all $u \in \mathcal{E}$ is equivalent to G is $\beta-$enriched nonexpansive mapping. Thus Corollary 4.8 completes the proof. □

5. The Case of Banach spaces with Kadec-Klee Property

In the sequel, a Banach space $(\mathcal{Y}, \|\cdot\|)$ is said to have Kadec-Klee Property (KKP) if

$$\left.\begin{array}{r} u_n \rightharpoonup u \\ \|u_n\| \to \|u\| \end{array}\right\} \Longrightarrow \quad u_n \to u.$$

Example of spaces with KKP include Hilbert spaces and the $L_p(t)$ spaces for $1 < p < \infty$.
The following definitions are crucial in our subsequent results.

Definition 5.1. Let $\mathcal{E}$ be a nonempty subset of a Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be a mapping. Then a sequence $\{u_n\}$ is said to be *an approximate fixed point sequence* (AFPS) of G provided that

$$\|u_n - Gu_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Remark 5.2. *It is worth noting that if $\mathcal{E}$ is convex, then a sequence $\{u_n\}$ is an AFPS of $G : \mathcal{E} \to \mathcal{E}$ if and only if it is an AFPS of G_β for any $\beta \geq 0$. Indeed, this is justify by the fact that*

$$\left\|u - G_\beta u\right\| = \frac{1}{\beta+1}\|u - Gu\| \quad \forall u \in \mathcal{E}.$$

Definition 5.3. Let $\mathcal{E}$ be a nonempty subset of a Banach space $(\mathcal{Y}, \|\cdot\|)$. A mapping $G : \mathcal{E} \to \mathcal{E}$ is said to satisfy Condition (L) if the following two conditions hold:

(CL1) If a subset $\mathcal{D} \subseteq \mathcal{E}$ is nonempty closed convex and G-invariant, then there exists $\{u_n\} \subset \mathcal{D}$ an AFPS of G.

(CL2) For any AFPS of G (say $\{u_n\}$) and all $u \in \mathcal{E}$,

$$\limsup_{n\to\infty} \|u_n - Gu\| \leq \limsup_{n\to\infty} \|u_n - u\|.$$

Remark 5.4. *It is known (see, for example [23]) that several substantial class of mappings satisfy Condition* (L)*. This mappings include but not limited to*

(RM1) Suziki's mapping *which maps* $\mathcal{E}$ *into itself in such away that for any* $u, w \in \mathcal{E}$,

$$\frac{1}{2}\|u - Tu\| \leq \|u - y\| \quad \Longrightarrow \quad \|Tu - Tw\| \leq \|u - w\|.$$

(RM2) generalized nonexpansive mapping *which maps* $\mathcal{E}$ *into itself in such away that for any* $u, w \in \mathcal{E}$,

$$\|Tu - Tw\| \leq \alpha_1\|u - w\| + \alpha_2\Big(\|u - Tu\| + \|w - Tw\|\Big) + \alpha_3\Big(\|u - Tw\| + \|w - Tu\|\Big),$$

where $\alpha_1, \alpha_2, \alpha_3$ *are nonnegative numbers satisfying* $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 1$.

Following $(RM1)$ of Remark 5.4, we get the following.

Remark 5.5. *Every nonexpansive mapping satisfies Condition* (L)*. However* $G : \left[0, \frac{2}{3}\right] \to \left[0, \frac{2}{3}\right]$ *defined by* $Gu = u^2$ *furnishes a counterexample for the converse. Additionally, G is* 0*-enriched nonexpansive mapping since* $|Gu| = u^2 \leq u = |u|$*, for all* $u \in [0, 1]$.

Theorem 5.6. *Let* $\mathcal{E}$ *be a weakly compact convex subset of a Banach space* $(\mathcal{Y}, \|\cdot\|)$ *and let* $G : \mathcal{E} \to \mathcal{E}$ *be a* β*-enriched nonexpansive mapping with center zero.*

(P1) If $0 \in \mathcal{E}$*, then G has a fixed point.*

(P2) If $0 \notin \mathcal{E}$ *and G is a mapping which satisfies Condition* (L)*, then G has a fixed point.*

Proof. $(P1)$ follows trivially from the definition of β-enriched nonexpansive mapping with center zero. To establish $(P2)$, let $\mathcal{D}^* := \{w \in \mathcal{E} \,:\, \|w\| = \gamma\}$ where $\gamma = \inf\{\|w\| \,:\, w \in \mathcal{E}\}$. It is clear that $\gamma > 0$ and $\mathcal{D}^*$ is a nonempty weakly compact convex subset of $\mathcal{Y}$. Since G is β-enriched nonexpansive mapping with center zero, then

$$\|\beta u + Gu\| \leq (\beta + 1)\|u\|.$$

This implies that

$$\left\|G_\beta u\right\| \leq \|u\|.$$

Thus $G_\beta\left(\mathcal{D}^*\right) \subseteq \mathcal{D}^*$. Since G satisfies Condition (L), there exists $\{u_n\} \subset \mathcal{D}^*$ an AFPS of G. By Remark 5.2, $\{u_n\}$ is an AFPS of G_β. Since $\mathcal{D}^*$ is weakly compact, we have $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ that converges weakly to $u^* \in \mathcal{D}^*$. In addition, definition of $\mathcal{D}^*$ yields that $\left\|u_{n_k}\right\| = \|u^*\| = \gamma$. Furthermore, since $\mathcal{Y}$ have KKP, $\{u_{n_k}\}$ converges strongly to u^*. Using Condition (CL2) of Definition 5.3 and the fact that $\{u_{n_k}\}$ is an AFPS of G, we get that

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\left\|u_{n_k} - G_\beta u^*\right\| \\
&\leq \limsup_{k\to\infty}\left\|u_{n_k} - G_\beta u^*\right\| \\
&= \limsup_{k\to\infty}\left\|\frac{\beta}{\beta+1}\left(u_{n_k} - u^*\right) + \frac{1}{\beta+1}\left(u_{n_k} - Gu^*\right)\right\| \\
&\leq \frac{\beta}{\beta+1}\limsup_{k\to\infty}\left\|u_{n_k} - u^*\right\| + \frac{1}{\beta+1}\limsup_{k\to\infty}\left\|u_{n_k} - Gu^*\right\| \\
&\leq \limsup_{k\to\infty}\left\|u_{n_k} - u^*\right\| = 0.
\end{aligned}$$

Consequently, we have

$$\left\|u^* - G_\beta u^*\right\| = \lim_{k\to\infty} \left\|u_{n_k} - G_\beta u^*\right\| = 0.$$

Thus, $u^* = G_\beta u^*$ which implies that $u^* = Gu^*$. Therefore, u^* is a fixed point of G. □

Theorem 5.7. *Let $\mathcal{E}$ be a weakly compact convex subset of a Banach space $(\mathcal{Y}, \|\cdot\|)$ and let $G : \mathcal{E} \to \mathcal{E}$ be a mapping.*

(P1) If $0 \in \mathcal{E}$ and G is a β-EANM with center zero, then 0 is a fixed point of G.

(P2) Suppose that $\mathcal{Y}$ has KKP, $0 \notin \mathcal{E}$ and

- *(A1) If G satisfies Condition (L), then G has a fixed point;*
- *(A2) $\{G^n u_1\}$ is an AFPS of G;*
- *(A3) For $\gamma = \inf\{\|w\| : w \in \mathcal{E}\}$, the set $\mathcal{E}_\gamma := \mathcal{E} \bigcap \{w \in \mathcal{Y} : \|w\| = \gamma\}$ is G-invariant.*

Then for any $\beta \geq 0$, G_β has a fixed point in $\mathcal{E}$.

Proof. The assertion in (P1) follows trivially from Theorem 4.1. Following the assumption that $\mathcal{E}$ is weakly compact convex subset of $\mathcal{Y}$, we have that $\gamma > 0$ and $\mathcal{E}_\gamma \neq \emptyset$. By $(A3)$, we have that $\mathcal{E}_\gamma$ is a G-invariant weakly compact convex subset of $\mathcal{Y}$. Now, let $u_1 \in \mathcal{E}_\gamma$ and let $\{G^{n_k} u_1\}$ be a subsequence of $\{G^n u_1\}$ that converges weakly to $u^* \in \mathcal{E}_\gamma$. Since $\mathcal{Y}$ has KKP, we get that $\{G^{n_k} u_1\}$ converges strongly to u^*. Since $u_{n_k} = G^{n_k} u_1$ is an AFPS and G satisfies Condition (L), it follows that

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty} \left\|u_{n_k} - G_\beta u^*\right\| \\
&\leq \limsup_{k\to\infty} \left\|u_{n_k} - G_\beta u^*\right\| \\
&= \limsup_{k\to\infty} \left\|\frac{\beta}{\beta+1}\left(u_{n_k} - u^*\right) + \frac{1}{\beta+1}\left(u_{n_k} - Gu^*\right)\right\| \\
&\leq \frac{\beta}{\beta+1} \limsup_{k\to\infty} \left\|u_{n_k} - u^*\right\| + \frac{1}{\beta+1} \limsup_{k\to\infty} \left\|u_{n_k} - Gu^*\right\| \\
&\leq \limsup_{k\to\infty} \left\|u_{n_k} - u^*\right\| = 0.
\end{aligned}$$

Consequently, we have

$$\left\|u^* - G_\beta u^*\right\| = \lim_{k\to\infty} \left\|u_{n_k} - G_\beta u^*\right\| = 0.$$

Thus, $u^* = G_\beta u^*$. Moreover, $u^* = Gu^*$. □

6. Alternate Convexically Enriched Nonexpansive

In [22], a substantial class of mappings was studied and later on analysed by many scholars. These mappings are usually refers to as *alternate convexically nonexpansive mappings*. In the literature, fixed point of such mappings are proved to be useful and can be obtained using AFPS and minimal set properties.

Consider a normed space $(\mathcal{Y}, \|\cdot\|)$ with a nonempty subset $\mathcal{E}$. A mapping $G : \mathcal{E} \to \mathcal{E}$ is called *alternate convexically nonexpansive* provided

$$\left\|\frac{1}{m}\sum_{j=1}^{m}(-1)^{j+1}Tu_j - Tw\right\| \leq \left\|\frac{1}{m}\sum_{j=1}^{m}(-1)^{j+1}u_j - w\right\| \tag{20}$$

for all $u_j, w \in \mathcal{E}$ and $m \in \mathbb{N}$. It can be observed that alternate convexically nonexpansive mappings are special cases of nonexpansive mappings with center zero. This can easily be established. In fact, it suffices to take $m = 2$ and choose $u_1 = u_2 = w \in \mathcal{E}$. Consequently, all the results about fixed points as prior obtained herein directly hold for the class of convexically nonexpansive mappings.

An immediate superclass of the class of convexically nonexpansive mappings was analysed in different articles as in the following sense. For a normed space $(\mathcal{Y}, \|\cdot\|)$ with a nonempty subset $\mathcal{E}$, a mapping $G : \mathcal{E} \to \mathcal{E}$ is called *κ-alternate convexically nonexpansive* provided

$$\left\| \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} T u_j - T w \right\| \leq \left\| \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} u_j - w \right\| \tag{21}$$

for all $u_j, w \in \mathcal{E}$ and $1 \leq m \leq \kappa$.

Fallowing [12], we can enrich the alternate convexically mapping as in the following definition.

Definition 6.1. *Let $\mathcal{Y}$ be a normed space with a nonempty subset $\mathcal{E}$. A mapping $G : \mathcal{E} \to \mathcal{E}$ is called* alternate convexically β-enriched nonexpansive *if there exists $\beta \geq 0$ such that*

$$\left\| \beta u + T u + \frac{\beta}{m} \sum_{j=1}^{m} (-1)^{j} w_j - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} T w_j \right\| \leq (\beta + 1) \left\| u - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} w_j \right\| \tag{22}$$

for all $u, w_j \in \mathcal{E}$ and $m \in \mathbb{N}$.

Example 6.2. *Let*

$$\mathcal{E} = \left\{ (s^n) \in \ell_1 : \sum_{n=1}^{\infty} s^n = 1 \right\}$$

and consider $G : \mathcal{E} \to \mathcal{E}$ defined by

$$G(u) = \left(-3u^1, 4u^1 - 3u^2, 4u^2 - 3u^3, 4u^3 - 3u^4, 4u^4 - 3u^5, \dots \right)$$

for all $u = \left(u^1, u^2, u^3, u^4, \dots\right) \in \mathcal{E}$. Observe that for every $y = \left(y^1, y^2, y^3, \dots\right) \in \mathcal{E}$,

$$3y + Gy = 4\left(0, y^1, y^2, y^3, \dots\right).$$

Now, let $m \in \mathbb{N}$ and $u, w_j \in \mathcal{E}$. Then

$$\begin{aligned}
&\left\| 3u + Gu + \frac{3}{m} \sum_{j=1}^{m} (-1)^{j} w_j - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} G w_j \right\| \\
&\qquad = \left\| 3u + Gu - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} \left(3w_j + G w_j \right) \right\| \\
&\qquad = 4 \left\| \left(0, u^1, u^2, u^3, \dots\right) - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} \left(0, w_j^1, w_j^2, w_j^3, \dots\right) \right\| \\
&\qquad = (3+1) \left\| u - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} w_j \right\|.
\end{aligned}$$

Therefore, $G : \mathcal{E} \to \mathcal{E}$ is alternate convexically β-enriched nonexpansive mapping with $\beta = 3$. Moreover, for $m = 2$, take $w_1 = w_2 \in \mathcal{E}$ and $u = (1, 0, 0, 0, \cdots)$. Then we have

$$\left\| \frac{1}{2} \sum_{j=1}^{2} (-1)^{j+1} T w_j - Tu \right\| = \|Tu\| = \|(-3, 4, 0, 0, \cdots)\| > 1 = \left\| \frac{1}{2} \sum_{j=1}^{2} (-1)^{j+1} w_j - u \right\|$$

which implies that G is not alternate convexically nonexpansive.

Similarly, we can enrich κ-alternate convexically nonexpansive mapping as in below.

Definition 6.3. *Let $\mathcal{Y}$ be a normed space with a nonempty subset $\mathcal{E}$. A mapping $G : \mathcal{E} \to \mathcal{E}$ is called* κ-alternate convexically β-enriched nonexpansive *if there exists $\beta \geq 0$ such that*

$$\left\| \beta u + Tu + \frac{\beta}{m} \sum_{j=1}^{m} (-1)^j w_j - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} T w_j \right\| \leq (\beta + 1) \left\| u - \frac{1}{m} \sum_{j=1}^{m} (-1)^{j+1} w_j \right\| \tag{23}$$

for all $u_j, w \in \mathcal{E}$ and $1 \leq m \leq \kappa$.

Remark 6.4. *The class of alternate convexically β-enriched nonexpansive is properly contained in the class of κ-alternate convexically β-enriched nonexpansive. Note that [22, Example 4] provides example of 0-enriched nonexpansive that is not 2-alternate convexically 0-enriched nonexpansive.*

Remark 6.5. *It is clear from (22) and (23) that every alternate (resp. κ-alternate) convexically nonexpansive mapping is alternate (resp. κ-alternate) convexically 0-enriched nonexpansive*

Remark 6.6. *Note that every κ-alternate convexically β-enriched nonexpansive for $\kappa \geq 2$ is a β-enriched nonexpansive mapping with center zero. Indeed, it suffices to take $\kappa = 2$ and set $u = w_1 = w_2$. Thus the established results also hold for all κ-alternate convexically β-enriched nonexpansive mappings.*

Corollary 6.7. *Let $\mathcal{E}$ be a nonempty closed convex subset of a reflexive strictly convex Banach space $(\mathcal{Y}, \|\cdot\|)$ and $G : \mathcal{E} \to \mathcal{E}$ be a κ-alternate convexically β-enriched nonexpansive mapping. Then G has a fixed point.*

7. Conclusion Remarks

In this work we studied enriched asymptotically nonexpansive mapping with center zero in the setting of complete norm spaces. We proved existence of fixed points of the studied mappings when certain classical conditions are imposed into the spaces. Following the geometry of enriched nonexpansive mappings, we defined κ-alternate convexically β-enriched nonexpansive mappings and show that its contained in the class of the studied mappings. Our results complements and unified several recent results in the literature including the results of [11], [21], [10] and [22].

For future works, it is natural to ask if such results can be achieved in the framework of geodesically connected spaces such as $\mathrm{CAT}_p(0)$ spaces since the setting share many useful properties of strictly convex spaces and search for possible applications as in [9, 24].

Acknowledgements

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research project was supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2023 under project number FRB660073/0164. The first author was supported by the "Petchra Pra Jom Klao Ph.D. Research Scholarship" from "King Mongkut's University of Technology Thonburi" with the contract number 67/2563. The idea of this paper was initiated while the fourth author (Dhananjay Gopal) visited KMUTT during 14–22 June 2022, under a Visiting Professor grant from KMUTT.

Availability of data and material

Not applicable.

Conflict Interests

The authors declare that they have no conflict of interest.

Authors' Contributions

The authors wrote, read and approved the final manuscript.

Funding

This research was funded by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2023 under project number FRB660073/0164. The first author was supported by the "Petchra Pra Jom Klao Ph.D. Research Scholarship" from "King Mongkut's University of Technology Thonburi" with the contract number 67/2563. The idea of this paper was initiated while the last author (Dhananjay Gopal) visited KMUTT during 14–22 June 2022, under a Visiting Professor grant from KMUTT. He thanks Professor Poom Kumam and the University for their hospitality and support. He also thank to administration of GGV Bilaspur.

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