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A study of the q-analogue of the paranormed Cesàro sequence spaces

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Abstract. In this article, we introduce and investigate the q-Cesàro matrix $C(q) = (c^q_{uv})$ with $q \in (0,1)$ for which we have

$$c^q_{uv} = \begin{cases} \dfrac{q^v}{[u+1]_q} & (0 \leqq v \leqq u) \\ \\ 0 & (v > u), \end{cases}$$

where the q-number $[\kappa]_q$ is given, as usual in the q-theory, by

$$[\kappa]_q := \begin{cases} \dfrac{1-q^\kappa}{1-q} & (\kappa \in \mathbb{C}) \\ \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\kappa = n \in \mathbb{N}), \end{cases}$$

$\mathbb{C}$ and $\mathbb{N}$ being the sets of complex numbers and positive integers, respectively. The q-Cesàro matrix $C(q)$ is a q-analogue of the Cesàro matrix C_1. We study the sequence spaces $X^q(p)$, $X^q_0(p)$, $X^q_c(p)$ and $X^q_\infty(p)$, which are obtained by the domain of the matrix $C(q)$ in the Maddox spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively. We derive the Schauder basis and the alpha-, beta- and gamma-duals of these newly-defined spaces. Moreover, we state and prove several theorems characterizing matrix transformation from the spaces $X^q(p)$, $X^q_0(p)$, $X^q_c(p)$ and $X^q_\infty(p)$ to anyone of the spaces c_0, c or ℓ_∞.

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1. Introduction, Definitions and Preliminaries

We begin this introductory section by giving a brief overview concerning Cesàro sequence spaces. For this purpose, we define the Cesàro matrix $C_1 = (c_{uv})$ of order 1 by

$$c_{uv} = \begin{cases} \dfrac{1}{u+1} & (0 \leqq v \leqq u) \\ 0 & (v > u). \end{cases}$$

Recently, Ng and Lee [17] studied the sequence spaces X_p and X_∞ defined by

$$X_p := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \sum_{u=0}^{\infty} \left| \frac{1}{u+1} \sum_{v=0}^{u} f_v \right|^p < \infty \quad (1 \leqq p < \infty) \right\}$$

and

$$X_\infty := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \sup_{u \in \mathbb{N}_0} \left| \frac{1}{u+1} \sum_{v=0}^{u} f_v \right| < \infty \right\},$$

where, and in what follows, we have

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad (\mathbb{N} := \{1, 2, 3, \cdots\}).$$

We thus observe that

$$X_p = (\ell_p)_{C_1} \qquad \text{and} \qquad X_\infty = (\ell_\infty)_{C_1}.$$

More recently, Şengönül and Başar [19] studied the Cesàro sequence spaces given by

$$X_c = c_{C_1} \qquad \text{and} \qquad X_0 = (c_0)_{C_1},$$

derived their alpha-, beta- and gamma-duals and characterize matrix transformations related to these spaces. Several results on matrix transformations on Cesàro sequence spaces are investigated by (for example) Ng [16], Başar [3] and Şavas [18] (see also some related recent developments [7], [11] and [23], which are based upon Cesàro sequence spaces).

The studies on the generalization of Cesàro sequence spaces can also be found in [1] and [2], wherein they investigated the following sequence spaces:

$$r^n(p) = \big(\ell(p)\big)_{R^n}, \quad r_0^t(p) = \big(c_0(p)\big)_{R^n}, \quad r_c^t(p) = \big(c(p)\big)_{R^n} \quad \text{and} \quad r_\infty^t(p) = \big(\ell_\infty(p)\big)_{R^n},$$

where $R^n = (r_{uv}^n)$ is Riesz matrix defined by

$$r_{uv}^n = \begin{cases} \dfrac{n_v}{N_u} & (0 \leqq v \leqq u) \\ 0 & (v > u), \end{cases}$$

$n = (n_v)$ being a sequence of positive integers with

$$N_u = \sum_{v=0}^{u} n_v.$$

We remark that, when $n = p = e$, the unit sequence, then

$$\lambda(p) \qquad \big(\lambda \in \{r^n, r_0^n, r_c^n, r_\infty^n\}\big),$$

reduces to the sequence spaces as defined in [17] and [19].

We turn now to the q-calculus which has emerged as an interesting field of study in many fields such as science, architecture and engineering. In the field of mathematics, several researchers are using the q-theory to establish fascinating results in algebra, combinatorics, special function, approximation theory, fractals, and so on.

For $q \in (0,1)$, the q-number $[\kappa]_q$ is defined by (see, for example, [10]; see also the recent work [22])

$$[\kappa]_q := \begin{cases} \dfrac{1-q^\kappa}{1-q} & (\kappa \in \mathbb{C}) \\ \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\kappa = n \in \mathbb{N}), \end{cases}$$

where $\mathbb{C}$ and $\mathbb{N}$ are the sets of complex numbers and positive integers, respectively.

One may notice that, in the limit when $q \to 1-$, the q-number $[\kappa]_q$ reduces to $\kappa \in \mathbb{C}$, that is,

$$\lim_{q \to 1-} \left\{ [\kappa]_q \right\} = \kappa \qquad (\kappa \in \mathbb{C}).$$

We refer to [10] and the recently-published survey-cum-expository review article [20] for detailed studies in q-number and q-theory. It is regretful to see that a large number of seemingly amateurish-type researchers on these and other related topics continue to produce and publish obvious and inconsequential variations and straightforward translations of the known q-results in terms of the so-called (p,q)-calculus by unnecessarily forcing-in an obviously superfluous (or redundant) parameter p into the classical q-calculus and thereby falsely claiming "generalization" (see [20, p. 340] and [21, Section 5, pp. 1511–1512]). Such tendencies to produce and flood the literature with trivialities should be discouraged by all means (see also a recently-published survey-cum-expository review article by Srivastava [22]).

In recent years, we can find several studies which are based upon q-analogues of well-known sequence spaces. For example, in terms of the q-Cesàro matrix $C(q) = (c^q_{uv})$, where

$$c^q_{uv} = \begin{cases} \dfrac{q^v}{[u+1]_q} & (0 \leqq v \leqq u) \\ \\ 0 & (v > u), \end{cases}$$

Demiriz and Şahin [8] introduced the following sequence spaces:

$$X^q_c = c_{C(q)} \qquad \text{and} \qquad X^q_0 = (c_0)_{C(q)}$$

and studied their duals and matrix transformations. Yaying et al. [28] introduced the sequence spaces given by

$$X^q_p = (\ell_p)_{C(q)} \qquad \text{and} \qquad X^q_\infty = (\ell_\infty)_{C(q)}$$

and studied their associated operator ideals. Yaying et al. (see [27], [29] and [30]) further studied the q-analogues of the Catalan, Euler and Pascal sequence spaces. In particular, in the paper [27], some known q-results based upon the q-Euler matrix were trivially translated by involving an obviously redundant (or superfluous) parameter p.

The present study is a natural continuation of the studies which were reported in [8] and [28]. In Section 2, we introduce the paranormed sequence spaces $X^q(p)$, $X^q_0(p)$, $X^q_c(p)$ and $X^q_\infty(p)$, which are obtained in the domain of the q-Casàro matrix $C(q)$ in the familiar Maddox spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively, and obtain the Schauder basis of these newly-defined spaces. In Section 3, we determine the alpha-, beta- and gamma-duals of the spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$. In Section 4, we state and prove several theorems characterizing matrix transformations from the spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ to anyone of the spaces c_0, c or ℓ_∞.

Some further definitions and notations are being presented next. Let ω denote the set of all real- or complex-valued sequences. Then any linear $\lambda \subset \omega$ is called a sequence space. The sets ℓ_p, c_0, c and ℓ_∞ are standard notations for sequence spaces containing absolutely p-summable, null, convergent and bounded sequences, respectively. The notations bs and cs stand for the spaces of all bounded and convergent series, respectively.

Definition 1. Let $\lambda \subset \omega$ and $\rho : \lambda \to \mathbb{C}$. Then (λ, ρ) is said to be a paranormed space if the following conditions are satisfied for all $f, g \in \lambda$ and $\alpha \in \mathbb{C}$:

(C1) $\rho(f) \geqq 0$ and $f = \theta$ implies that $\rho(f) = 0$, where θ is zero of λ;

(C2) $\rho(-f) = \rho(f)$;

(C3) $\rho(f + g) \leqq \rho(f) + \rho(g)$;

(C4) If $(\alpha_v) \in \omega$ and $(f_v) \in \lambda$ with $\alpha_v - \alpha \to 0$ and $\rho(f_v - f) \to 0$ as $v \to \infty$, then $\rho(\alpha_v f_v - \alpha f) \to 0$ as $v \to \infty$.

In this case, ρ is said to be a paranorm for λ.

Each of the following sequence spaces is well known (see [14] and [13]):

$$\ell(p) := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \sum_{v=0}^{\infty} |f_v|^{p_v} < \infty \qquad (0 < p_v \leqq P < \infty) \right\},$$

$$c(p) := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \lim_{v \to \infty} |f_v - l|^{p_v} = 0 \qquad (l \in \mathbb{C}) \right\},$$

$$c_0(p) := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \lim_{v \to \infty} |f_v|^{p_v} = 0 \right\}$$

and

$$\ell_\infty(p) := \left\{ f : f = (f_k) \in \omega \quad \text{and} \quad \sup_{v \in \mathbb{N}_0} |f_v|^{p_v} < \infty \right\}.$$

Here, as we have already stated above,

$$\mathbb{N}_0 := \{0, 1, 2, \cdots\} = \mathbb{N}_0 \cup \{0\}$$

and $p = (p_v)$ is a bounded sequence of positive real numbers with

$$P = \sup_{v \in \mathbb{N}_0} p_v \qquad \text{and} \qquad Q = \max\{1, P\}.$$

The spaces $\ell(p)$ and $\lambda = \{c_0(p), c(p), \ell_\infty(p)\}$ are complete spaces which are paranormed by

$$\rho_1(f) = \left(\sum_{u=0}^{\infty} |f_u|^{p_u}\right)^{1/Q} \qquad \text{and} \qquad \rho_2(f) = \sup_{u\in\mathbb{N}_0} |f_u|^{p_u/Q},$$

respectively. We also refer to the review article by Başar and Yeşilkayagil [4] for studies concerning paranormed spaces which are obtained by infinite matrices.

A sequence $(s_v) \in \lambda$, which is paranormed by ρ, is said to be a Schauder basis for the space λ if there exists a unique sequence (a_v) of scalars such that, for every $f \in \lambda$, we have

$$\lim_{u\to\infty} \rho\left(f - \sum_{v=0}^{u} a_v s_v\right) = 0.$$

Definition 2. For $\lambda, \mu \subset \omega$, we define the set $M(\lambda, \mu)$ by

$$M(\lambda, \mu) := \{t = (t_k) \in \omega : tf = (t_k f_k) \in \mu \quad (\forall\ f = (f_k) \in \lambda)\}.$$

By using this notation, the alpha-, beta- and gamma-duals of the space λ is defined by

$$\lambda^\alpha = M(\lambda, \ell_1),\ \lambda^\beta = M(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = M(\lambda, bs),$$

respectively.

Definition 3. Let $D = (d_{uv})_{u,v=0}^{\infty}$ be an infinite matrix over $\mathbb{C}$. Suppose also that

$$D_u = \left(d_{uv}\right)_{v=0}^{\infty} \qquad \text{and} \qquad Df = (Df)_u,$$

where

$$(Df)_u = \sum_{v=0}^{\infty} d_{uv} f_v$$

for any $f \in \omega$, provided that the infinite sum exists. The sequence Df is known as D-transform of the sequence f. If we let $\lambda, \mu \subset \omega$, then (λ, μ) denotes the family of all matrices that map λ into μ, that is, $D \in (\lambda, \mu)$ if and only if $Df \in \mu$ for all $f \in \lambda$. The set λ_D given by

$$\lambda_D = \left\{f : f \in \omega \quad \text{and} \quad Df \in \lambda\right\}$$

is called the domain of D in λ.

2. Paranormed q-Cesàro Sequence Spaces

In this section, we first introduce a sequence $g = (g_v)$ whose v^{th} term is given by $g_v = \left(C^q f\right)_v$. This means that the sequence g is the C^q-transform of the sequence f. Equivalently, for all $u \in \mathbb{N}_0$, we have

$$g_u = \sum_{v=0}^{u} \frac{q^v}{[u+1]_q} f_v. \tag{1}$$

Let $p = (p_v)$ be a bounded sequence of positive real numbers. Then the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ are given by

$$X^q(p) := \left\{f : f = (f_v) \in \omega \quad \text{and} \quad g = C^q f \in \ell(p)\right\},$$

$$X_0^q(p) := \Big\{f : f = (f_v) \in \omega \quad \text{and} \quad g = C^q f \in c_0(p)\Big\},$$

$$X_c^q(p) := \Big\{f : f = (f_v) \in \omega \quad \text{and} \quad g = C^q f \in c(p)\Big\}$$

and

$$X_\infty^q(p) := \Big\{f : f = (f_v) \in \omega \quad \text{and} \quad g = C^q f \in \ell_\infty(p)\Big\},$$

respectively. It is easily seen that the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ can be expressed as follows:

$$X^q(p) = (\ell_p)_{C(q)},\ X_0^q(p) = (c_0)_{C(q)},\ X_c^q(p) = c_{C(q)} \quad \text{and} \quad X_\infty^q(p) = (\ell_\infty)_{C(q)}.$$

We emphasize that the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ yield the following class of sequence spaces in the special case when $p = (p_k)$ and for q:

(i) If $q \to 1-$, then the sequences spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ reduce to the spaces $X(p) = \big(\ell(p)\big)_{C_1}$, $X_0(p) = \big(c_0(p)\big)_{C_1}$, $X_c(p) = \big(c(p)\big)_{C_1}$ and $X_\infty(p) = \big(\ell_\infty(p)\big)_{C_1}$, respectively;

(ii) If $p_v = p$ for all $v \in \mathbb{N}_0$, then the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ yield the spaces X_p^q, X_0^q, X_c^q and X_∞^q, respectively, as studied by Yaying et al. [28] and Demiriz and Şahin [8];

(iii) If $q \to 1-$ and $p_v = p$ for all $v \in \mathbb{N}_0$, then the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ reduce to the spaces X_p, X_0, X_c and X_∞, respectively, as studied by Ng and Lee [17], and Şengönül and Başar [19].

The equality (1) can also be rewritten in terms of the sequence $g = (g_v)$ as follows:

$$f_u = \sum_{v=u-1}^{u} (-1)^{u-v} \frac{[v+1]_q}{q^u}\, g_v \qquad \Big(u \in \mathbb{N}_0;\ f_0 = g_0\Big). \tag{2}$$

We now state and prove our first main result as Theorem 1 below.

Theorem 1. *The sequence spaces $X^q(p)$ and $X_0^q(p)$ are complete linear metric spaces paranormed by*

$$\rho(f) = \left(\sum_{u=0}^{\infty} \left|\frac{1}{[u+1]_q} \sum_{v=0}^{u} q^v f_v\right|^{p_u}\right)^{1/Q} \quad \text{and} \quad \rho_\infty(f) = \sup_{u\in\mathbb{N}_0} \left|\frac{1}{[u+1]_q} \sum_{v=0}^{u} q^v f_v\right|^{p_u/Q},$$

respectively, where $0 \leqq p_v \leqq P < \infty$. The spaces $X_c^q(p)$ and $X_\infty^q(p)$ are paranormed by ρ_∞ only in the trivial case $\inf p_v > 0$, when $X_\infty^q(p) = X_\infty^q$ and $X_c^q(p) = X_c^q$.

Proof. We give the proof for the space $X_0^q(p)$. One can observe that the axioms (C1) and (C2) of Definition 1 suffice for ρ and for all $f \in X_0^q(p)$. Let $f_1, f_2 \in X_0^q(p)$ and $z_1, z_2 \in \mathbb{C}$. Then, by using the following known inequality [15, p. 30]):

$$|a_v + b_v|^{p_v/Q} \leqq |a_v|^{p_v/Q} + |b_v|^{p_v/Q},$$

and, in view of the linearity of $C(q)$, we get

$$\begin{aligned}\rho(z_1 f_1 + z_2 f_2) &= \sup_{u\in\mathbb{N}_0} \left|(C(q)(z_1 f_1 + z_2 f_2))_u\right|^{p_u/Q} \\ &\leqq \max\{1,|z_1|\} \sup_{u\in\mathbb{N}_0} \left|(C(q)f_1)_u\right|^{p_u/Q} + \max\{1,|z_2|\} \sup_{u\in\mathbb{N}_0} \left|(C(q)f_2)_u\right|^{p_u/Q} \\ &= \max\{1,|z_1|\}\rho_\infty(f_1) + \max\{1,|z_2|\}\rho_\infty(f_2).\end{aligned}$$

Thus the axiom (C3) of Definition 1 holds true.

We now assume that $\{f^{(u)}\}$ is any sequence of points in $X_0^q(p)$ satisfying $\rho_\infty(f^{(u)} - f) \to 0$ as $u \to \infty$ and (z_v) is any sequence of scalars such that $z_v \to z$ as $v \to \infty$. Then, by using the subadditivity of ρ_∞, we find that

$$\begin{aligned}\rho_\infty(z_u f^{(u)} - z) &= \sup_{l\in\mathbb{N}_0} \left|\left(C(q)(z_u f^{(u)} - z f)\right)_l\right|^{p_l/Q} \\ &= \sup_{l\in\mathbb{N}_0} \left|\sum_{v=0}^{l} \frac{q^v (z_u f_v^{(u)} - z f_v)}{[l+1]_q}\right|^{p_l/Q} \\ &\leqq |z_n - z|^{p_v/Q} \rho_\infty(f^{(u)}) + |z|^{p_v/Q} \rho_\infty(f^{(u)} - f) \\ &\to 0 \quad \text{as} \quad u \to \infty.\end{aligned}$$

Thus, clearly, the axiom (C4) of Definition 1 also holds true. This concludes that ρ_∞ is a paranorm on the space $X_0^q(p)$.

Next, we establish the completeness of the space $X_0^q(p)$. Let $f^i = \{f^{(i)_k}\}$ be any Cauchy sequence in $X_0^q(p)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $m(\varepsilon)$ such that

$$\rho_\infty(f^i - f^j) < \varepsilon \tag{3}$$

for all $i, j \geqq m(\varepsilon)$. Therefore, by using (3), we obtain

$$\left|\left(C(q)f^i\right)_v - \left(C(q)f^j\right)_v\right| \leqq \sup_{v\in\mathbb{N}_0} \left|\left(C(q)f^i\right)_v - \left(C(q)f^j\right)_v\right|^{p_v/Q} < \varepsilon \tag{4}$$

for all $i, j \geqq m(\varepsilon)$. This yields the fact that

$$\left\{\left(C(q)f^0\right)_v, \left(C(q)f^1\right)_v, \left(C(q)f^2\right)_v, \cdots\right\}$$

is a Cauchy sequence in $\mathbb{C}$ for each $v \in \mathbb{N}_0$. Furthermore, since $\mathbb{C}$ is complete, the sequence $\left\{\left(C(q)f^i\right)_v\right\}$ converges to, say, $(C(q)f)_v$ for each v as $i \to \infty$. Now, upon proceeding to the limits as $j \to \infty$ in (4), we find, for each $v \in \mathbb{N}_0$, that

$$\left|\left(C(q)f^i\right)_v - (C(q)f)_v\right| < \varepsilon \tag{5}$$

for all $i \geqq m(\varepsilon)$. Again, since $f^i \in X_0^q(p)$,

$$\left|\left(C(q)f^i\right)_v\right|^{p_v/Q} < \varepsilon. \tag{6}$$

Thus the equations (5) and (6) together imply that

$$\left|(C(q)f)_v\right|^{p_v/Q} < \varepsilon.$$

Consequently, $C(q)f \in X_0^q(p)$. Hence $X_0^q(p)$ is a complete space. □

Theorem 2. *The sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ are linearly isomorphic to $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively.*

Proof. The proof is similar for each case. Hence to avoid unnecessary repetition of the statements, we present the proof only for the space $X^q(p)$.

Upon defining the mapping $\mathcal{T} : X^q(p) \to \ell(p)$ by $f \mapsto g = \mathcal{T}f = C(q)f$, it is fairly straightforward to see that $\mathcal{T}$ is linear and injective. Let $g = (g_v) \in \ell(p)$. Then, by using (2), we deduce the fact that

$$\begin{aligned}
\rho(f) &= \left(\sum_{u=0}^{\infty}\left|\frac{1}{[u+1]_q}\sum_{v=0}^{u} q^v f_v\right|^{p_u}\right)^{1/Q}\\
&= \left(\sum_{u=0}^{\infty}\left|\frac{1}{[u+1]_q}\sum_{v=0}^{u} q^v\left(\sum_{l=v-1}^{v}(-1)^{v-l}\frac{[l+1]_q}{q^v}g_l\right)\right|^{p_u}\right)^{1/Q}\\
&= \left(\sum_{u=0}^{\infty}|g_u|^{p_u}\right)^{1/Q} = \rho_1(g) < \infty.
\end{aligned}$$

Thus $f \in X^q(p)$, that is, $\mathcal{T}$ is onto and paranorm-preserving. Consequently, $X^q(p) \cong \ell(p)$. □

We conclude this section by constructing sequences in the spaces $X^q(p)$, $X_0^q(p)$ and $X_c^q(p)$ that will act as the Schauder basis for the respective spaces.

Theorem 3. *Assume that $0 < p_k \leqq P < \infty$ and $g = C(q)f$. Define the sequence $s^v(q) = \left(s_u^{(v)}(q)\right)_{u\in\mathbb{N}_0}$ by*

$$s_u^{(v)} = \begin{cases} (-1)^{u-v}\,\dfrac{[v+1]_q}{q^u} & (v \leqq u \leqq v+1)\\[2ex] 0 & (0 \leqq u < v;\ u > v+1)\end{cases}$$

for each fixed $v \in \mathbb{N}_0$. Then each of the following assertions holds true:

1. *The sequence $s^v(q)$ is a Schauder basis for the spaces $X^q(p)$ and $X_0^q(p)$ and for every $f \in X^q(p)$ or $X_0^q(p)$ is uniquely expressed in the following form:*
$$f = \sum_{v=0}^{\infty} g_k s^k(q);$$

2. *The set $\{e, s^k(q)\}$ is a Schauder basis for the space $X_c^q(p)$ and every $f \in X_c^q(p)$ is uniquely expressed in the following form:*
$$f = ke + \sum_{v=0}(g_v - k)s^k(q),$$
where $k = \lim_{v\to\infty} g_v$.

3. *The sequence space $X_\infty^q(p)$ has no Schauder basis.*

Proof. We consider each of the following cases:

(1) For each $v \in \mathbb{N}_0$, we have

$$C(q)s^k(q) = e^{(v)} \in \ell(p), \tag{7}$$

which implies that $\{s^k(q)\} \subset X^q(p)$. Let us take any sequence $f \in X^q(p)$ and set, for each $m \in \mathbb{N}_0$,

$$f^{[r]} = \sum_{v=0}^{r} g_v s^v(q). \tag{8}$$

Then, by using (8) together with (7), we obtain

$$C(q)f^{[r]} = \sum_{v=0}^{r} g_v C(q)s^v(q) = \sum_{v=0}^{r} g_v e^v$$

and

$$\Big(C(q)(f - f^{[r]})\Big)_v = \begin{cases} 0 & (0 \leqq v \leqq r) \\ \Big(C(q)f\Big)_v & (v > r). \end{cases}$$

Thus, for any given $\varepsilon > 0$, there exists an integer r_0 such that

$$\left(\sum_{l=r}^{\infty} \left|\Big(C(q)\Big)_l\right|^{p_l}\right)^{1/Q} < \frac{\varepsilon}{2} \qquad (\forall\ r \geqq r_0).$$

Hence we deduce that

$$\rho(f - f^{[r]}) = \left(\sum_{l=r}^{\infty} \left|\Big(C(q)f\Big)_l\right|^{p_l}\right)^{1/Q} \leqq \left(\sum_{l=r_0}^{\infty} \left|\Big(C(q)f\Big)_l\right|^{p_l}\right)^{1/Q} < \frac{\varepsilon}{2} < \varepsilon$$

for all $r \geqq r_0$. Thus we are led to the following representation:

$$f = \sum_{v=0}^{\infty} g_v s^v(q).$$

Next, in order to prove the uniqueness of this representation, we assume that there exists another representation, say,

$$f = \sum_{v=0}^{\infty} g'_v s^v(q).$$

Then we have

$$\Big(C(q)f\Big)_u = \sum_{v=0}^{\infty} g'_v \Big(C(q)s^k(q)\Big)_u = \sum_{v=0}^{\infty} g'_v e_u^{(v)} = g'_u,$$

which contradicts the fact that $(C(q)f)_u = g_u$. Thus the representation is unique.

By applying similar techniques, one can prove that $s^v(q)$ is a Schauder basis for the space $X_0^q(p)$ by replacing the space $X^q(p)$ by $X_0^q(p)$, and the paranorm ρ by ρ_∞ in the above proof. Hence we exclude the details.

(2) This is similar to the proof of Theorem 2 and hence we omit the details involved.

(3) This result is immediate from the fact that the space $\ell_\infty(p)$ has no Schauder basis. □

3. Alpha-, Beta- and Gamma-Duals

In this section, we compute the α-, β- and γ-duals of the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$. We provide proof for the space $X^q(p)$ only. One can obtain the proofs for the other spaces by following similar techniques.

The following lemmas are needed for our investigation. Throughout the paper, $\mathcal{N}$ denotes the family of all finite subsets of $\mathbb{N}_0$. We also assume that $D = (d_{uv})$ is an infinite matrix over the complex field.

Lemma 1. (see [9, Theorem 5.1.0]) *Each of the following statements holds true*:

(i) Assume that $1 < p_v \leqq P < \infty$ *for each* $v \in \mathbb{N}_0$. *Then*

$$D = (d_{uv}) \in \Big(\ell(p), \ell_1\Big)$$

if and only if there exists an integer $I > 1$ *such that*

$$\sup_{U \in \mathcal{N}} \sum_{v=0}^{\infty} \left| \sum_{u \in U} d_{uv} I^{-1} \right|^{p_v'} < \infty;$$

(ii) Assume that $0 < p_v \leqq 1$ *for each* $v \in \mathbb{N}_0$. *Then*

$$D = (d_{uv}) \in \Big(\ell(p), \ell_1\Big)$$

if and only if

$$\sup_{U \in \mathcal{N}} \sup_{v \in \mathbb{N}_0} \left| \sum_{u \in U} d_{uv} \right|^{p_v} < \infty.$$

Lemma 2. (see [12, Theorem 1]) *Each of the following statements holds true*:

(i) Assume that $1 < p_v \leqq P < \infty$ *for each* $v \in \mathbb{N}_0$. *Then*

$$D = (d_{uv}) \in \Big(\ell(p), \ell_\infty\Big)$$

if and only if there exists an integer $I > 1$ *such that*

$$\sup_{u \in \mathbb{N}_0} \sum_{v=0}^{\infty} \left| d_{uv} I^{-1} \right|^{p_v'} < \infty; \tag{9}$$

$$\tag{10}$$

(ii) Assume that $0 < p_v \leqq 1$ *for each* $v \in \mathbb{N}_0$. *Then*

$$D = (d_{uv}) \in \Big(\ell(p), \ell_\infty\Big)$$

if and only if

$$\sup_{u,v \in \mathbb{N}_0} |d_{uv}|^{p_v} < \infty. \tag{11}$$

Lemma 3. (see [12, Theorem 1]) *Assume that* $0 < p_v \leqq P < \infty$ *for each* $v \in \mathbb{N}_0$. *Then*

$$D = (d_{uv}) \in \big(\ell(p), c\big)$$

if and only if the conditions (9) *and* (11), *together with the following limit condition*:

$$\lim_{u\to\infty} d_{uv} = \alpha_v \qquad (\forall\, v \in \mathbb{N}_0),$$

hold true.

Theorem 4. *Let the sets* δ_i $(1 \leqq i \leqq 5)$ *be defined by*

$$\delta_1 := \bigcup_{I>1} \left\{ t : t = (t_k) \in \omega \quad \text{and} \quad \sup_{U\in\mathcal{N}} \sum_{v=0}^{\infty} \left| \sum_{u\in U} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, t_u I^{-1} \right|^{p'_v} < \infty \right\},$$

$$\delta_2 := \left\{ t : t = (t_k) \in \omega \quad \text{and} \quad \sup_{U\in\mathcal{N}} \sup_{v\in\mathbb{N}_0} \left| \sum_{u\in U} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, t_u I^{-1} \right|^{p_v} < \infty \right\},$$

$$\delta_3 := \bigcup_{I>1} \left\{ t : t = (t_k) \in \omega \quad \text{and} \quad \sup_{V\in\mathcal{N}} \sum_{u=0}^{\infty} \left| \sum_{v\in V} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, t_u I^{-1/p_v} \right| < \infty \right\},$$

$$\delta_4 := \bigcup_{I>1} \left\{ t : t = (t_k) \in \omega \quad \text{and} \quad \sum_{u=0}^{\infty} \left| \sum_{v=0}^{\infty} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, t_u \right| < \infty \right\}$$

and

$$\delta_5 := \bigcup_{I>1} \left\{ t : t = (t_k) \in \omega \quad \text{and} \quad \sup_{V\in\mathcal{N}} \sum_{u=0}^{\infty} \left| \sum_{v\in V} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, t_u I^{1/p_v} \right| < \infty \right\}.$$

Then each of the following assertions holds true:

(i) $$[X^q(p)]^\alpha = \begin{cases} \delta_1 & (1 < p_v \leqq P < \infty) \\ \delta_2 & (0 < p_v \leqq 1); \end{cases}$$

(ii) $\left[X_0^q(p)\right]^\alpha = \delta_3$, $\left[X_c^q(p)\right]^\alpha = \delta_3 \cap \delta_4$ *and* $\left[X_\infty^q(p)\right]^\alpha = \delta_5$.

Proof. In the light of (2), we observe that the following equality:

$$t_u f_u = \sum_{v=u-1}^{u} (-1)^{u-v} \, \frac{[v+1]_q}{q^u} \, g_v t_u = (A(q)g)_u$$

holds true for $t = (t_u) \in \omega$, where $A(q) = (a_{uv}^q)$ is a triangle defined by

$$a_{uv}^q = \begin{cases} (-1)^{u-v} \, \dfrac{[v+1]_q}{q^u} \, t_u & (u-1 \leqq v \leqq u) \\ 0 & (\text{otherwise}). \end{cases} \tag{12}$$

Thus we have $tf = (t_u f_u) \in \ell_1$ whenever $f \in X^q(p)$ if and only if $A(q)g \in \ell_1$ whenever $g \in \ell(p)$. This implies that $t = (t_n) \in [X^q(p)]^\alpha$ if and only if $A(q) \in (\ell(p), \ell_1)$. So, by using Lemma 1, we find that

$$\exists I > 1 \ni \sup_{U \in \mathcal{N}} \sum_{v=0}^{\infty} \left| \sum_{u \in U} (-1)^{u-v} \frac{[v+1]_q}{q^u} t_u I^{-1} \right|^{p'_v} < \infty \qquad (1 < p_v \leqq P < \infty)$$

and

$$\sup_{U \in \mathcal{N}} \sup_{v \in \mathbb{N}_0} \left| \sum_{u \in U} (-1)^{u-v} \frac{[v+1]_q}{q^u} t_u I^{-1} \right|^{p_v} < \infty \qquad (0 < p_v \leqq 1).$$

These conclude that

$$[X^q(p)]^\alpha = \begin{cases} \delta_1 & (1 < p_v \leqq P < \infty) \\ \delta_2 & (0 < p_v \leqq 1). \end{cases}$$

□

Theorem 5. *Suppose that the sets δ_i $(6 \leqq i \leqq 9)$ are defined by*

$$\delta_6 := \bigcup_{I>1} \left\{ t : t = (t_u) \in \omega, \quad \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) I^{-1} \right|^{p'_v} < \infty \right.$$
$$\left. \text{and} \quad \left(\frac{[v+1]_q t_v}{q^v} I^{-1} \right)^{p'_v} \in \ell_\infty \right\},$$

$$\delta_7 := \left\{ t : t = (t_u) \in \omega, \quad \left([v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) \right)^{p_v} \in \ell_\infty \right.$$
$$\left. \text{and} \quad \left(\frac{[v+1]_q t_v}{q^v} \right)^{p_v} \in \ell_\infty \right\},$$

$$\delta_8 := \bigcup_{I>1} \left\{ t : t = (t_u) \in \omega, \quad \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) \right| I^{-1/p_v} < \infty \right.$$
$$\left. \text{and} \quad \left(\frac{[v+1]_q t_v}{q^v} I^{-1/p_v} \right) \in \ell_\infty \right\},$$

$$\delta_9 := \bigcap_{I>1} \left\{ t : t = (t_u) \in \omega : \sum_{v=0} \left| [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) \right|^{1/p_v} < \infty \right.$$
$$\left. \text{and} \quad \left(\frac{[v+1]_q t_v}{q^v} \right)^{1/p_v} \in c_0 \right\}$$

and

$$\delta_{10} := \bigcap_{I>1} \left\{ t : t = (t_u) \in \omega, \quad \sum_{v=0} \left| [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) \right| I^{1/p_v} < \infty \right.$$
$$\left. \text{and} \quad \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) I^{1/p_v} \in \ell_\infty \right\}.$$

Then each of the following assertions holds true:

(i) $[X^q(p)]^\beta = [X^q(p)]^\gamma = \begin{cases} \delta_6 & (1 < p_v \leqq P < \infty) \\ \delta_7 & (0 < p_v \leqq 1); \end{cases}$

(ii) $\left[X_0^q(p)\right]^\beta = \left[X_0^q(p)\right]^\gamma = \delta_8$;

(iii) $\left[X_c^q(p)\right]^\beta = \delta_8 \cap cs$ *and* $\left[X_c^q(p)\right]^\gamma = \delta_8 \cap bs$;

(iv) $\left[X_\infty^q(p)\right]^\beta = \delta_9$ *and* $\left[X_\infty^q(p)\right]^\gamma = \delta_{10}$.

Proof. For $t = (t_v) \in \omega$, the following equality holds true:

$$\begin{aligned} \sum_{v=0}^{u} t_v f_v &= \sum_{v=0}^{u} t_v \left(\sum_{l=v-1}^{v} (-1)^{v-l} \frac{[l+1]_q}{q^v} g_l \right) \\ &= \sum_{v=0}^{u-1} [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) g_v + \frac{[u+1]_q}{q^u} g_u t_u \\ &= (B(q)g)_u, \ u \in \mathbb{N}_0, \end{aligned} \tag{13}$$

where $B(q) = (b_{uv}^q)$ is a triangle defined by

$$b_{uv}^q = \begin{cases} [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) & (0 \leqq v < u) \\ \frac{[u+1]_q}{q^u} t_u & (v = u) \\ 0 & (\text{otherwise}). \end{cases} \tag{14}$$

In view of (13), we observe that $tf = (t_u f_u) \in cs$ whenever $f = (f_v) \in X^q(p)$ if and only if $B(q)g \in c$ whenever $g = (g_v) \in \ell(p)$. This implies that $t = (t_n) \in [X^q(p)]^\beta$ if and only if $A(q) \in (\ell(p), c)$. Thus, by using Lemma 3, we deduce, for $1 < p_v \leqq P < \infty$, that

$$\sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{t_v}{q^v} - \frac{t_{v+1}}{q^{v+1}} \right) I^{-1} \right|^{p_v'} < \infty$$

and

$$\left\{ \left(\frac{[v+1]_q t_v}{q^v} I^{-1} \right)^{p_v'} \right\} \in \ell_\infty.$$

Moreover, for $0 < p_v \leqq 1$, we have

$$\left\{\left([v+1]_q\left(\frac{t_v}{q^v}-\frac{t_{v+1}}{q^{v+1}}\right)\right)^{p_v}\right\} \in \ell_\infty$$

and

$$\left\{\left(\frac{[v+1]_q t_v}{q^v}\right)^{p_v}\right\} \in \ell_\infty.$$

These conclude that

$$[X^q(p)]^\beta = \begin{cases} \delta_6 & (1 < p_v \leqq P < \infty) \\ \delta_7 & (0 < p_v \leqq 1). \end{cases}$$

The gamma-dual of the space $X^q(p)$ can be obtained by applying a similar technique as detailed above and by using Lemma 2 instead of Lemma 3. the details involved are being omitted here. □

4. A Set of Matrix Transformations

In this section, we characterize some classes of matrix transformations from the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ to any one of the spaces ℓ_∞, c or c_0. The following inequality will be necessary in our investigation:

$$|xy| \leqq I\left(|xI^{-1}|^{p'} + |y|^p\right) \qquad \left(x, y \in \mathbb{C};\ I > 0\right), \tag{15}$$

where $p > 1$ such that $p^{-1} + p'^{-1} = 1$. We also introduce $T(I)$ defined by

$$T(I) := \sup_{u \in \mathbb{N}_0} \sum_{v=0}^{\infty} \left|[v+1]_q\left(\frac{t_v}{q^v}-\frac{t_{v+1}}{q^{v+1}}\right)I^{-1}\right|^{p'_v}. \tag{16}$$

Theorem 6. *Each of the following assertions holds true*:

(i) *Let $1 < p_v \leqq P < \infty$ for $k \in \mathbb{N}_0$. Then $D = (d_{uv}) \in \left(X^q(p), \ell_\infty\right)$ if and only if there is an integer $I > 1$ such that*

$$T(I) < \infty \tag{17}$$

and

$$\left\{\left(\frac{[v+1]_q d_{uv}}{q^v} I^{-1}\right)^{p'_v}\right\} \in \ell_\infty \qquad (u \in \mathbb{N}_0). \tag{18}$$

(ii) *Let $0 < p_v \leqq 1$ for $k \in \mathbb{N}_0$. Then $D = (d_{uv}) \in \left(X^q(p), \ell_\infty\right)$ if and only if*

$$\sup_{u,v \in \mathbb{N}_0} \left|[v+1]_q\left(\frac{t_v}{q^v}-\frac{t_{v+1}}{q^{v+1}}\right)\right|^{p_v} < \infty \tag{19}$$

and

$$\left\{\left(\frac{[v+1]_q d_{uv}}{q^v}\right)^{p_v}\right\}_{v \in \mathbb{N}_0} \in \ell_\infty \qquad (u \in \mathbb{N}_0). \tag{20}$$

Proof. Suppose $D \in \left(X^q(p), \ell_\infty\right)$ and $1 < p_v \leqq P < \infty$. Then, for each $f \in X^q(p)$, Df exists and belongs to the space ℓ_∞ which implies that $D_u \in [X^q(p)]^\beta$. This ensures the necessity of the conditions (17) and (18).

Conversely, we assume that the conditions (17) and (18) hold true and let $f \in X^q(p)$. Then $D_u \in [X^q(p)]^\beta$ for each $u \in \mathbb{N}_0$, which, in turn, implies that Df exists.

We now consider the following equality which is obtained by taking the r^{th} partial sum of the series $\sum_{v=0}^{\infty} d_{uv} f_v$ given by

$$\sum_{v=0}^{r} d_{uv} f_v = \sum_{v=0}^{r-1} [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}} \right) g_v + \frac{[r+1]_q}{q^r} \, h_{ur} g_r \qquad (u \in \mathbb{N}_0). \tag{21}$$

Thus, upon proceeding to the limits as $r \to \infty$ in (21) and keeping the condition (18) in mind, we deduce that

$$\sum_{v=0}^{\infty} d_{uv} f_v = \sum_{v=0}^{\infty} [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}} \right) g_v. \tag{22}$$

Therefore, by using (22) together with (15) and (16), we find that

$$\sup_{u \in \mathbb{N}_0} \left| \sum_{v=0}^{\infty} f_v \right| \leqq \sup_{u \in \mathbb{N}_0} \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}} \right) \right| |g_v| \leqq I\left(T(I) + \rho_1(g)\right) < \infty,$$

which implies that $Df \in \ell_\infty$. Thus we have $D \in \left(X^q(p), \ell_\infty\right)$.

In a similar manner, one can complete the proof of Case (ii). □

Theorem 7. *Each of the following assertions holds true*:

(i) *Let* $1 < p_v \leqq P < \infty$ *for* $k \in \mathbb{N}_0$. *Then* $D = (d_{uv}) \in \left(X^q(p), c\right)$ *if and only if* (17) *and* (18) *hold true and there exists a sequence* (α_v) *of scalars such that*

$$\lim_{u \to \infty} [v+1]_q \left(\frac{d_{uv} - \alpha_v}{q^v} - \frac{d_{u,v+1} - \alpha_{v+1}}{q^{v+1}} \right) = 0, \ v \in \mathbb{N}_0. \tag{23}$$

(ii) *Let* $0 < p_v \leqq 1$ *for* $k \in \mathbb{N}_0$. *Then* $D = (d_{uv}) \in \left(X^q(p), c\right)$ *if and only if* (19), (20) *and* (23) *hold true.*

Proof. We present the proof of Case (i). One can give the proof of Case (ii) by following similar arguments.

Let us assume that $1 < p_v \leqq P < \infty$ and $D \in \left(X^q(p), c\right)$. Since $c \subset \ell_\infty$, the necessity part of the conditions (17) and (18) is straightforward from Case (i) of Theorem 6. Let us now consider the sequence $s^v(q)$ defined in Theorem 3. Since $D \in \left(X^q(p), c\right)$, Df exists for each $f \in X^q(p)$ and belongs to the space c. So it is evident that

$$Ds^k(q) = \left\{ [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}} \right) \right\}_{v \in \mathbb{N}_0} \in c$$

for each $u \in \mathbb{N}_0$. This proves the necessity of (23).

Conversely, we assume that the conditions (17), (18) and (23) hold true. Also let $f \in X^q(p)$. Then $D_u \in [X^q(p)]^\beta$. This implies that Df exists.

We now consider the following equality for $I > 1$:

$$\sum_{v=0}^{r}\left|[v+1]_q\left(\frac{d_{uv}}{q^v}-\frac{d_{u,v+1}}{q^{v+1}}\right)I^{-1}\right|^{p'_v} \leqq \sup_{u\in\mathbb{N}_0}\sum_{v=0}^{\infty}\left|[v+1]_q\left(\frac{d_{uv}}{q^v}-\frac{d_{u,v+1}}{q^{v+1}}\right)I^{-1}\right|^{p'_v}.$$

Thus, upon proceeding to the limits as $r, u \to \infty$ and by using (17) and (23), we get

$$\sum_{v=0}^{\infty}\left|[v+1]_q\left(\frac{d_{uv}}{q^v}-\frac{d_{u,v+1}}{q^{v+1}}\right)I^{-1}\right|^{p'_v} < \infty. \tag{24}$$

Again, if we proceed to the limits as $u \to \infty$ in (18), we see that

$$\left\{\left(\frac{\alpha_v[v+1]_q}{q^v}\right)^{p'_v}\right\} \in \ell_\infty.$$

This fact together with (24) yields $(\alpha_v) \in [X^q(p)]^\beta$. Thus the series $\sum_{v=0}^{\infty} \alpha_v f_v$ converges for each $f \in X^q(p)$.

One can now observe from the equality (22) that the following condition holds true:

$$\sum_{v=0}^{\infty}(d_{uv}-\alpha_v)f_v = \sum_{v=0}^{\infty}[v+1]_q\left(\frac{d_{uv}-\alpha_v}{q^v}-\frac{d_{u,v+1}-\alpha_{v+1}}{q^{v+1}}\right)g_v. \tag{25}$$

By using the conditions (17) and (23), it follows immediately from Lemma 2 that

$$\left([v+1]_q\left(\frac{d_{uv}-\alpha_v}{q^v}-\frac{d_{u,v+1}-\alpha_{v+1}}{q^{v+1}}\right)\right)_{u,v\in\mathbb{N}_0} \in (\ell_p, c_0).$$

This statement together with (25) yields

$$\lim_{u\to\infty}\sum_{v=0}^{\infty}(d_{uv}-\alpha_v)f_v = 0.$$

Since $\sum_{v=0}^{\infty} \alpha_v f_v$ converges, it follows immediately that

$$Df = \sum_{v=0}^{\infty} d_{uv} f_v \in c$$

for each $f \in X^q(p)$. This completes the proof of Theorem 7. □

Upon replacing the space c with c_0, in Theorem 7, we are led to the following result.

Theorem 8. *Each of the following assertions holds true*:

(i) *Let* $1 < p_v \leqq P < \infty$ *for* $k \in \mathbb{N}_0$. *Then* $D = (d_{uv}) \in \left(X^q(p), c_0\right)$ *if and only if* (17) *and* (18) *hold true and the condition* (23) *with* $\alpha_v = 0$ *is satisfied for all* $v \in \mathbb{N}_0$.

(ii) *Let* $0 < p_v \leqq 1$ *for* $k \in \mathbb{N}_0$. *Then* $D = (d_{uv}) \in \left(X^q(p), c_0\right)$ *if and only if* (19) *and* (20) *hold true and the condition* (23) *with* $\alpha_v = 0$ *is satisfied for all* $v \in \mathbb{N}_0$.

In a similar way, one can characterize matrix transformations from the spaces $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ to any of the spaces ℓ_∞, c or c_0. The lines of proof are analogous to those of Theorems 6, 7 and 8. Hence, in order to avoid unnecessary repetitions, we state the results without proof. Before proceeding, we list several conditions that will be utilized in proving the following results:

$$\left(\frac{[v+1]_q}{q^v} d_{uv} I^{1/p_v}\right)_{v\in\mathbb{N}_0} \in c_0 \qquad (u \in \mathbb{N}_0), \tag{26}$$

$$\sup_{u\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right)\right| I^{1/p_v} < \infty, \tag{27}$$

$$\lim_{u\to\infty} \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right) - \alpha_v\right| I^{1/p_v} < \infty \qquad (\alpha_v \in \mathbb{C}), \tag{28}$$

$$\sup_{u\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right)\right| I^{-1/p_v} < \infty, \tag{29}$$

$$\sup_{u\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right)\right| < \infty, \tag{30}$$

$$\lim_{u\to\infty} \left| \sum_{v=0}^{\infty} [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right) - \alpha\right| < \infty \qquad (\alpha \in \mathbb{C}), \tag{31}$$

$$\lim_{u\to\infty} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right) - \alpha_v\right| < \infty \qquad (\alpha_v \in \mathbb{C}) \tag{32}$$

and

$$\lim_{u\in\mathbb{N}_0} \left| [v+1]_q \left(\frac{d_{uv}}{q^v} - \frac{d_{u,v+1}}{q^{v+1}}\right) - \alpha_v\right| I^{-1/p_v} < \infty \qquad (\alpha_v \in \mathbb{C}). \tag{33}$$

Theorem 9. *Each of the following assertions holds true*:

(1) $D \in \left(X_\infty^q(p), \ell_\infty\right)$ *if and only if* (26) *and* (27) *hold true;*

(2) $D \in \left(X_\infty^q(p), c\right)$ *if and only if* (26), (27) *and* (28) *hold true;*

(3) $D \in \left(X_\infty^q(p), c_0\right)$ *if and only if* (26) *and* (27) *hold true and the condition* (28) *with* $\alpha_v = 0$ *is also satisfied for all* $v \in \mathbb{N}_0$*;*

(4) $D \in \left(X_c^q(p), \ell_\infty\right)$ *if and only if* (26), (29) *and* (30) *hold true;*

(5) $D \in \left(X_c^q(p), c\right)$ *if and only if* (26), (31), (32) *and* (33) *hold true;*

(6) $D \in \left(X_c^q(p), c_0\right)$ *if and only if* (26) *and* (31) *hold true and the conditions* (32) *and* (33) *with* $\alpha_v = 0$ *are also satisfied for all* $v \in \mathbb{N}_0$*;*

(7) $D \in \left(X_c^q(p), \ell_\infty\right)$ *if and only if* (26) *and* (29) *hold true;*

(8) $D \in \left(X_c^q(p), c\right)$ *if and only if* (26), (29), (32) *and* (33) *hold true;*

(9) $D \in \left(X_c^q(p), c_0\right)$ *if and only if* (26) *holds true and the conditions* (32) *and* (33) *with* $\alpha_v = 0$ *are also satisfied for all* $v \in \mathbb{N}_0$.

5. Concluding Remarks and Observations

In our present investigation, we have introduce and systematically study the q-Cesàro matrix $C(q) = (c_{uv}^q)$ with $q \in (0, 1)$ for which we can write

$$c_{uv}^q = \begin{cases} \dfrac{q^v}{[u+1]_q} & (0 \leqq v \leqq u) \\[2ex] 0 & (v > u), \end{cases}$$

where $[\kappa]_q$ denotes, as usual, the basic (or quantum or q-) number. The q-Cesàro matrix $C(q)$, which we have considered herein, is a q-analogue of the familiar Cesàro matrix C_1. We have presented a general theory of the sequence spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$, which are obtained by the domain of the matrix $C(q)$ in the Maddox spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively. In particular, we have derived the Schauder basis and the alpha-, beta- and gamma-duals of each of these spaces which we have defined in this article. Moreover, we have proved a total of nine theorems characterizing matrix transformation from the spaces $X^q(p)$, $X_0^q(p)$, $X_c^q(p)$ and $X_\infty^q(p)$ to anyone of the spaces c_0, c or ℓ_∞.

For the interest of the reader and for encouraging further researches along the lines presented herein, we have chosen to include the citations of several recent developments on the q-theory and the q-analysis. Indeed it is known that the basic (or q-) series and the basic (or q-) polynomials, especially the basic (or q-) gamma and q-hypergeometric functions and the basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [26, pp. 350–351] and [20, p. 328]; see also the recent developments in [5], [6], [24] and [25] on various diversied applications of the q-theory and q-analysis). Moreover, in the recently-published survey-cum-expository review articles by Srivastava (see [20], [21] and [22]), it was exposed, demonstrated and reiterated that the so-called (p, q)-calculus is, in fact, a rather trivial and inconsequential variation or a trivial and inconsequential translation of the classical q-calculus, simply because the additional forced-in parameter p is obviously redundant or superfluous (see, for details, [20, p. 340] and [21, pp. 1511–1512]; see also [22, Sections 5 and 6]). This observation by Srivastava (see [20], [21] and [22]) will surely apply also to any future attempts to produce the rather straightforward (p, q)-variants of the results which we have presented in this article. Such tendencies on the part of some seemingly amateurish-type researchers ought to be discouraged by all means.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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