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On an iterative scheme for approximating fixed points

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Abstract. In this paper, we introduce a new three-steps iteration scheme for approximation of the fixed points for the nonexpansive mappings. We discuss the stability of the introduced scheme. Also, we obtain some results on weak and strong convergence for generalized nonexpansive mappings of Suzuki.

1. Introduction

In recent years, convergence of iterative processes for fixed points is an important problem in the theory of nonlinear analysis. In the literature, there are many results provide different kinds of iterative processes and its convergence results. Now we recall the iteration processes existing in the literature.

Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in the interval $[0,1]$. Let $\mu_0 \in \mathcal{G}$, where $\mathcal{G}$ subset of Banach space $\mathcal{X}$ and $\mathcal{H}$ be a self mapping on $\mathcal{X}$.

- Mann, 1953, [11]: Then the iteration is defined by

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \mathcal{H}\mu_n. \tag{1.1}$$

- Ishikawa, 1974, [10]: Then the iteration is defined by

$$\begin{cases} \nu_n = (1-\beta_n)\mu_n + \beta_n \mathcal{H}\mu_n, \\ \mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \mathcal{H}\nu_n. \end{cases} \tag{1.2}$$

On the iterative methods (1.1) and (1.2) see for example [4, 6, 9, 14, 16, 18, 21]).

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- Noor, 2000, [12]: Then the iteration is defined by

$$\begin{cases} \omega_n = (1-\gamma_n)\mu_n + \gamma_n \mathcal{H}\mu_n, \\ \nu_n = (1-\beta_n)\mu_n + \beta_n \mathcal{H}\omega_n, \\ \mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \mathcal{H}\nu_n. \end{cases} \tag{1.3}$$

- Agarwal et al., 2007, [2]: Then the iteration is defined by

$$\begin{cases} \nu_n = (1-\beta_n)\mu_n + \beta_n \mathcal{H}\mu_n, \\ \mu_{n+1} = (1-\alpha_n)\mathcal{H}\mu_n + \alpha_n \mathcal{H}\nu_n. \end{cases} \tag{1.4}$$

- Abbas et al., 2014, [1]: Then the iteration is defined by

$$\begin{cases} \omega_n = (1-\gamma_n)\mu_n + \gamma_n \mathcal{H}\mu_n, \\ \nu_n = (1-\beta_n)\mathcal{H}\mu_n + \beta_n \mathcal{H}\omega_n, \\ \mu_{n+1} = (1-\alpha_n)\mathcal{H}\nu_n + \alpha_n \mathcal{H}\omega_n. \end{cases} \tag{1.5}$$

- Thakur et al., 2016, [19]: Then the iteration is defined by

$$\begin{cases} \omega_n = (1-\gamma_n)\mu_n + \gamma_n \mathcal{H}\mu_n, \\ \nu_n = (1-\beta_n)\omega_n + \beta_n \mathcal{H}\omega_n, \\ \mu_{n+1} = (1-\alpha_n)\mathcal{H}\omega_n + \alpha_n \mathcal{H}\nu_n. \end{cases} \tag{1.6}$$

- Ullah and Arshad, 2018, [20] : Then the iteration is defined by

$$\begin{cases} \omega_n = (1-\beta_n)\mu_n + \beta_n \mathcal{H}\mu_n, \\ \nu_n = \mathcal{H}((1-\alpha_n)\omega_n + \alpha_n \mathcal{H}\omega_n), \\ \mu_{n+1} = \mathcal{H}\nu_n. \end{cases} \tag{1.7}$$

We designed the iteration by

$$\begin{cases} \omega_n = (1-\gamma_n)\mu_n + \gamma_n \mathcal{H}\mu_n, \\ \nu_n = \mathcal{H}((1-\beta_n)\omega_n + \beta_n \mathcal{H}\omega_n), \\ \mu_{n+1} = \mathcal{H}((1-\alpha_n)\nu_n + \alpha_n \mathcal{H}\nu_n). \end{cases} \tag{1.8}$$

It is shown that the iteration scheme (1.8) converges at a fixed point faster than all the Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas et al., Thakur et al. and Ullah and Arshad. We are also provide a numerical example in order to correlate the convergence of (1.8) with the previous ones.

2. Preliminaries

We recall some of the definitions and results to be used.

Definition 2.1. *[7] Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. If for each $\lambda \in (0,2]$ there exists $\theta > 0$ such that for $f, g \in \mathcal{X}$,*

$$\begin{cases} \|f\| \leq 1 \\ \|g\| \leq 1 \\ \|f-g\| \geq \lambda \end{cases} \text{implies } \left\| \frac{f+g}{2} \right\| \leq 1 - \theta, \tag{2.1}$$

then the space $(\mathcal{X}, \|\cdot\|)$ is called uniformly convex space.

Remark 2.2. *The $\mathcal{F}(\mathcal{H})$ denotes the set of all fixed points of the self-map $\mathcal{H}$ on $\mathcal{X}$.*

Definition 2.3. *[13] Let $\mathcal{X}$ be a Banach space, then it is said to satisfy Opial's property if for every sequence $\{f_n\}$ in $\mathcal{X}$, that converges weakly to $f \in \mathcal{X}$, we have*

$$\limsup_{n\to\infty} \|f_n - f\| < \limsup_{n\to\infty} \|f_n - g\| \tag{2.2}$$

for all $g \in \mathcal{X}$ such that $f \neq g$.

Definition 2.4. *Let $\mathcal{G}$ be a non-empty subset of a Banach space $\mathcal{X}$. A mapping $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ is called:*

(i) Contraction, if

$$\|\mathcal{H}\mu - \mathcal{H}\nu\| \leq \zeta\|\mu - \nu\|, \tag{2.3}$$

for all $\mu, \nu \in \mathcal{G}$, where $\zeta \in (0, 1)$;

(ii) Generalized contraction [5], if

$$\|\mathcal{H}f - \mathcal{H}g\| \leq \xi(\|f - \mathcal{H}f\|) + b\|f - g\|, \tag{2.4}$$

for all $f, g \in \mathcal{X}$, where $b \in [0, 1)$ and $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\xi(0) = 0$.

(iii) Nonexpansive, if

$$\|\mathcal{H}\mu - \mathcal{H}\nu\| \leq \|\mu - \nu\|, \tag{2.5}$$

for all $\mu, \nu \in \mathcal{G}$;

(iv) Quasi-nonexpansive, if

$$\|\mathcal{H}\mu - \tau\| \leq \zeta|\mu - \tau\|, \tag{2.6}$$

for all $\mu \in \mathcal{G}$ and $\tau \in \mathcal{F}(\mathcal{H})$, where $\zeta \in (0, 1)$;

(v) Generalized nonexpansive mapping [17], if

$$\frac{1}{2}\|f - \mathcal{H}f\| \leq \zeta\|f - g\| \text{ implies } \|\mathcal{H}f - \mathcal{H}g\| \leq \zeta\|f - g\|, \tag{2.7}$$

for all $f, g \in \mathcal{G}$, where $\zeta \in (0, 1)$.

Proposition 2.5. *[17] Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a mapping. Then:*

(i) If $\mathcal{H}$ is nonexpansive than $\mathcal{H}$ is a generalized non-expansive mapping;

(ii) If $\mathcal{H}$ is a generalized nonexpansive mapping and has a fixed point, then $\mathcal{H}$ is a quasi-nonexpansive mapping;

(iii) If $\mathcal{H}$ is a generalized nonexpansive mapping, then

$$\|f - \mathcal{H}g\| \leq 3\|\mathcal{H}f - f\| + \|f - g\|,$$

for all $f, g \in \mathcal{G}$.

Lemma 2.6. *[17] Let $\mathcal{G}$ be a nonempty subset of $\mathcal{X}$. Also, let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive mapping with the Opial property. If $\{\mu_n\}$ weakly converges to μ and $\lim_{n\to\infty} \|\mathcal{H}\mu_n - \mu_n\| = 0$ then $\mathcal{H}\mu = \mu$.*

Lemma 2.7. *[17] Let $\mathcal{G}$ be a weakly convex compact subset of Banach space $\mathcal{X}$ which is uniformly convex. Then every generalized nonexpansive self-mapping on $\mathcal{G}$ has a fixed point.*

Lemma 2.8. *[15] Let $(\mathcal{X}, \|\cdot\|)$ be a uniformly convex Banach space. Let $0 < \alpha < \beta < 1$ and $\gamma \geq 0$ and $\{e_n\}$ be any real sequence such that $e_n \in [\alpha, \beta]$ for all $n \geq 1$. Also, let $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathcal{X}$ such that:*
1. $\limsup_{n\to\infty} \|f_n\| \leq \gamma$ and $\limsup_{n\to\infty} \|g_n\| \leq \gamma$,
2. $\limsup_{n\to\infty} \|e_n f_n + (1 - e_n) g_n\| = \gamma$.
Then $\limsup_{n\to\infty} \|f_n - g_n\| = 0$.

Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$ and $\{\mu_n\}$ be a bounded sequence in $\mathcal{X}$. Also, let $\mu \in \mathcal{X}$, and

$$\rho(\mu, \{\mu_n\}) = \limsup_{n\to\infty} \|\mu_n - u\|.$$

Then:

(i) Asymptotic radius of $\{\mu_n\}$ relative to $\mathcal{G}$ is given by
$\rho(\mathcal{G}, \{\mu_n\}) = \inf\{\rho(\mu, \{\mu_n\}) : \mu \in \mathcal{G}\}$;

(ii) Asymptotic center of $\{\mu_n\}$ relative $\mathcal{G}$ is given by
$\mathcal{A}(\mathcal{G}, \{\mu_n\}) = \{\mu \in \mathcal{G} : \rho(\mu, \{\mu_n\}) = \rho(\mathcal{G}, \{\mu_n\})\}$.

There is exactly one point of a uniformly convex Banach space, $\mathcal{A}(\mathcal{G}, \{\mu_n\})$ (see [20]).

Definition 2.9. *[8] Let $\mathcal{H}$ be a self mapping on a Banach space $\mathcal{X}$ and suppose that $\omega_0 \in \mathcal{X}$ and $\omega_{n+1} = f(\mathcal{H}, \omega_n)$ defines an iteration procedure such the sequence $\{\omega_n\}$ in $\mathcal{X}$. Assume that $\{\omega_n\}$ converges to the fixed point $\mathcal{L}$. Let $\{\nu_n\}$ be a sequences in $\mathcal{X}$ and a sequence $\{\lambda_n\}$ defined by $\lambda_n = \|\nu_{n+1} - f(\mathcal{H}, \nu_n)\|$. Then the iteration procedure $\omega_{n+1} = f(\mathcal{H}, \omega_n)$ is said to be $\mathcal{H}$-stable if*

$$\lim_{n\to\infty} \lambda_n = 0 \text{ if and only if } \lim_{n\to\infty} \nu_n = \mathcal{L}. \tag{2.8}$$

The following Lemma of Berinde (see [3]) shall be used to prove the stability result of our scheme.

Lemma 2.10. *[3] Let $\eta \in \mathbb{R}$ be such that $0 \leq \eta < 1$. Also, let $\{\upsilon_n\}$ be a sequence in $(0, +\infty)$ such that $\lim_{n\to\infty} \upsilon_n = 0$. Then $\lim_{n\to\infty} \nu_n = 0$ for any sequence of positive numbers $\{\nu_n\}$ satisfying*

$$\nu_{n+1} \leq \eta\nu_n + \upsilon_n \quad \text{for } n = 1, 2, \ldots$$

3. Stability and convergence results

First, we establish convergence result for iteration process (1.8).

Theorem 3.1. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a nonexpansive map, $\{\mu_n\}$ be a sequence defined by (1.8) and $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Then $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$.*

Proof. Let $\tau \in \mathcal{F}(\mathcal{H})$. From (1.8) we have

$$\begin{aligned} \|\omega_n - \tau\| &= \|(1 - \gamma_n)\mu_n + \gamma_n \mathcal{H}\mu_n - \tau\| \\ &\leq (1 - \gamma_n)\|\mu_n - \tau\| + \gamma_n \|\mathcal{H}\mu_n - \tau\| \\ &\leq (1 - \gamma_n)\|\mu_n - \tau\| + \gamma_n \|\mu_n - \tau\| \\ &= \|\mu_n - \tau\|. \end{aligned} \tag{3.1}$$

We get similar

$$\begin{aligned} \|\nu_n - \tau\| &= \|\mathcal{H}((1 - \beta_n)\omega_n + \beta_n \mathcal{H}\omega_n) - \tau\| \\ &\leq \|(1 - \beta_n)\omega_n + \beta_n \mathcal{H}\omega_n - \tau\| \\ &\leq (1 - \beta_n)\|\omega_n - \tau\| + \beta_n \|\mathcal{H}\omega_n - \tau\| \\ &\leq (1 - \beta_n)\|\omega_n - \tau\| + \beta_n \|\omega_n - \tau\| \\ &= \|\omega_n - \tau\|, \end{aligned}$$

now, from (3.1) we obtain

$$\|\nu_n - \tau\| \le \|\mu_n - \tau\|. \tag{3.2}$$

Now, using (1.8) and (3.2) we obtain

$$\begin{aligned} \|\mu_{n+1} - \tau\| &= \|\mathcal{H}((1-\alpha_n)\nu_n + \alpha_n \mathcal{H}\nu_n) - \tau\| \\ &\le \|(1-\alpha_n)\nu_n + \alpha_n \mathcal{H}\nu_n - \tau\| \\ &\le (1-\alpha_n)\|\nu_n - \tau\| + \alpha_n\|\mathcal{H}\nu_n - \tau\| \\ &\le (1-\alpha_n)\|\mu_n - \tau\| + \alpha_n\|\nu_n - \tau\| \\ &\le (1-\alpha_n)\|\mu_n - \tau\| + \alpha_n\|\mu_n - \tau\| \\ &= \|\mu_n - \tau\|. \end{aligned} \tag{3.3}$$

Hence $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$. □

Theorem 3.2. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a nonexpansive map. Suppose that $\{\mu_n\}$ is generated by (1.8) and $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Then*

$$\{\mu_n\} \text{ converges to } \tau \in \mathcal{F}(\mathcal{H}) \text{ if and only if } \liminf_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0, \tag{3.4}$$

where

$$\sigma(\mu, \mathcal{F}(\mathcal{H})) = \inf\{\|\mu - \tau\| : \tau \in \mathcal{F}(\mathcal{H})\}.$$

Proof. Firstly, suppose $\{\mu_n\}$ converges to $\tau \in \mathcal{F}(\mathcal{H})$. Then

$$\liminf_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0.$$

Conversely, assume that

$$\liminf_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0. \tag{3.5}$$

From Theorem 3.1 we obtain that $\lim_{n\to\infty} \|\mu_n - \mu\|$ exists for every $\mu \in \mathcal{F}(\mathcal{H})$, so $\lim_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H}))$ exists. Now, from (3.5), we conclude

$$\lim_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0. \tag{3.6}$$

Now, for given $\xi > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$,

$$\sigma(\mu_n, \mathcal{F}(\mathcal{H})) < \frac{\xi}{2}.$$

Particularly,

$$\inf\{\|\mu_{N_1} - \tau\| : \tau \in \mathcal{F}(\mathcal{H})\} < \frac{\xi}{2}.$$

Therefore, exists $\tau^* \in \mathcal{F}(\mathcal{H})$, such that

$$\|\mu_{N_1} - \tau^*\| < \frac{\xi}{2}.$$

Now for all $l, n \ge N_1$, we obtain

$$\|\mu_{l+n} - x_n\| \le \|\mu_{l+n} - \tau^*\| + \|\mu_n - \tau^*\| \le 2\|\mu_{N_1} - \tau^*\| < \xi,$$

which shows that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{G}$. Also, as $\mathcal{G}$ is a closed subset of the Banach space $\mathcal{X}$, we conclude that there exists $\tau \in \mathcal{G}$ such that $\lim_{n\to\infty} \mu_n = \tau$. This implies $\lim_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0$. So, $\tau \in \mathcal{F}(\mathcal{H})$. □

In the next result, we shows that iteration (1.8) is a $\mathcal{H}$-stable.

Theorem 3.3. *Let $\mathcal{H}$ be a selfmap on a Banach space $\mathcal{X}$ satisfying* (2.4)*. Suppose that $\tau \in \mathcal{F}(\mathcal{H})$. Let $\mu_0 \in \mathcal{X}$ and assume that (1.8) holds where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* $[0,1]$ *such that* $0 < \alpha \leq \alpha_n$*,* $0 < \beta \leq \beta_n$*,* $0 < \gamma \leq \gamma_n$*, for all* $n \geq 0$*. Then iteration process* (1.8) *is a $\mathcal{H}$-stable.*

Proof. Let $\{\nu_n\}$ be a sequence in $\mathcal{X}$ and $\{\eta_n\}$ in $\mathbb{R}^+$ defined as,

$$\eta_n = \|\nu_{n+1} - \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n)\|, \tag{3.7}$$

where,

$$s_n = \mathcal{H}((1-\beta_n)\nu_n + \beta_n\mathcal{H}\nu_n). \tag{3.8}$$

Let $\lim_{n\to\infty}\eta_n = 0$ and consider

$$\begin{aligned}
\|\nu_{n+1} - \tau\| &= \|\nu_{n+1} - \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n) + \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n) - \tau\| \\
&\leq \|\nu_{n+1} - \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n)\| + \\
&\|\mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n) - \tau\| \\
&\leq \eta_n + [\xi\|\tau - \mathcal{H}\tau\| + b\|(1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n - \tau\|] \\
&\leq \eta_n + b(1-\alpha_n)\|s_n - \tau\| + b\alpha_n[\xi\|\tau - \mathcal{H}\tau\| + b\|s_n - \tau\|] \\
&= \eta_n + b[(1-\alpha_n) + b\alpha_n]\|s_n - \tau\| \\
&\leq \eta_n + b[(1-\alpha_n) + b\alpha_n][0 + b\|(1-\beta_n)\nu_n + \beta_n\mathcal{H}\nu_n - \tau\|] \\
&\leq \eta_n + b^2[(1-\alpha_n) + b\alpha_n](1-\beta_n)\|\nu_n - \tau\| + \\
&b^2[(1-\alpha_n) + b\alpha_n]\beta_n\|\mathcal{H}\nu_n - \tau\| \\
&\leq \eta_n + b^2[(1-\alpha_n) + b\alpha_n](1-\beta_n)\|\nu_n - \tau\| + \\
&b^2[(1-\alpha_n) + b\alpha_n]\beta_n[0 + b\|\nu_n - \tau\|] \\
&\leq \eta_n + b^2[1 - \alpha_n(1-b)][(1-\beta_n(1-b)]\|\nu_n - \tau\|
\end{aligned} \tag{3.9}$$

since $b \in [0,1)$, therefore $b^2[(1-\alpha_n) + b\alpha_n][(1-\beta_n) + b\beta_n] < 1$, hence by using Lemma 2.10 we have,

$$\lim_{n\to\infty}\|\nu_n - \tau\| = 0. \tag{3.10}$$

Conversely, suppose that

$$\lim_{n\to\infty}\nu_n = \tau. \tag{3.11}$$

Then,

$$\begin{aligned}
\eta_n &= \|\nu_{n+1} - \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n)\| \\
&\leq \|\nu_{n+1} - \tau\| + \|\tau - \mathcal{H}((1-\alpha_n)s_n - \alpha_n\mathcal{H}s_n)\| \\
&\leq \|\nu_{n+1} - \tau\| + b(1-\alpha_n)\|s_n - \tau\| + b\alpha_n\|\mathcal{H}s_n - \tau\| \\
&\leq \|\nu_{n+1} - \tau\| + b(1-\alpha_n)\|s_n - \tau\| + b\alpha_n[0 + b\|s_n - \tau\|] \\
&= \|\nu_{n+1} - \tau\| + b[(1-\alpha_n) + b\alpha_n]\|s_n - \tau\| \\
&\leq \|\nu_{n+1} - \tau\| + b[(1-\alpha_n) + b\alpha_n][0 + b\|(1-\beta_n)\nu_n + \beta_n\mathcal{H}\nu_n - \tau\|] \\
&\leq \|\nu_{n+1} - \tau\| + b^2[(1-\alpha_n) + b\alpha_n](1-\beta_n)\|\nu_n - \tau\| + \\
&b^2[(1-\alpha_n) + b\alpha_n]\beta_n\|\mathcal{H}\nu_n - \tau\| \\
&\leq \|\nu_{n+1} - \tau\| + b^2[1 - \alpha_n(1-b)][(1-\beta_n(1-b)]\|\nu_n - \tau\|
\end{aligned} \tag{3.12}$$

Taking limit $n \to \infty$ in (3.12) we get,

$$\lim_{n\to\infty}\eta_n = 0. \tag{3.13}$$

□

4. Numerical Experiments

Next to compare the convergence rate of the iteration scheme (1.8) with other schemes.

Let $\mathcal{X} = \mathbb{R}$, $\mathcal{G} = [1, 40]$ and $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ by $\mathcal{H}y = \sqrt{y^2 - 9y + 54}$ for all $y \in \mathcal{G}$. For $y_1 = 30$ and $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, $n = 1, 2, \ldots$.

n	Picard	Ishikawa	Noor	Agarwal et al.	Abbas et al.	Thakur et al.	Ullah	(1.8)-Iter.
1	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000	30.00000000
2	26.15339366	25.01198240	23.48910332	24.05033082	22.61079008	21.30667585	17.14034293	14.70769244
3	22.41917610	20.25475590	17.46681907	18.43727194	15.82815627	13.58899597	7.920241534	6.461675585
4	18.83737965	15.85090878	12.32658573	13.39382036	10.25820641	8.112973955	6.038818684	6.002732886
5	15.46966242	12.01330515	8.727576617	9.372555587	7.001837925	6.225674626	6.000469229	6.000014314
6	12.41303724	9.068862033	6.958571160	6.993935718	6.119154210	6.015130221	6.000005614	6.000000076
7	9.816626625	7.282040026	6.310214626	6.186206786	6.011213258	6.000960494	6.000000067	**6.000000000**
8	7.875056741	6.466803146	6.097925567	6.028369366	6.001024303	6.000060749	6.000000001	6.000000000
9	6.718705828	6.160065238	6.030680843	6.004133882	6.000093304	6.000003841	**6.000000000**	6.000000000
10	6.218734240	6.053725040	6.009590308	6.000598188	6.000008497	6.000000242	6.000000000	6.000000000
11	6.058386534	6.017902837	6.002995608	6.000086472	6.000000774	6.000000016	6.000000000	6.000000000
12	6.014862308	6.005951431	6.000935492	6.000012498	6.000000071	6.000000001	6.000000000	6.000000000
13	6.003732823	6.001976848	6.000292122	6.000001806	6.000000005	**6.000000000**	6.000000000	6.000000000
14	6.00093429	6.000656462	6.000091217	6.000000261	6.000000001	6.000000000	6.000000000	6.000000000
15	6.000233641	6.000217976	6.000028483	6.000000037	**6.000000000**	6.000000000	6.000000000	6.000000000
16	6.000058415	6.000072376	6.000008894	6.000000005	6.000000000	6.000000000	6.000000000	6.000000000
17	6.000014603	6.000024032	6.000002778	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000
18	6.000003651	6.000007979	6.000000866	**6.000000000**	6.000000000	6.000000000	6.000000000	6.000000000
19	6.000000912	6.000002649	6.000000270	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
20	6.000000227	6.000000880	6.000000084	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
21	6.000000057	6.000000293	6.000000026	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
22	6.000000014	6.000000097	6.000000008	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
23	6.000000003	6.000000032	6.000000003	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
24	6.000000001	6.000000010	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
25	**6.000000000**	6.000000003	**6.000000000**	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
26	6.000000000	6.000000001	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
27	6.000000000	**6.000000000**	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000

Table 1 : Comparison of iteration scheme (1.8) with Picard, Ishikawa, Noor, Agarwal et al., Abbas et al., Thakur et al. and Ullah iterations

From the Table 1 we determine that the fixed point of the above equation is 6 and it is clear that our iteration scheme needs less iteration than Picard, the Mann, the Ishikawa [10], the Noor [12], the Agarwal et al. [2], the Abbas et al. [1], the Thakur et al. [19] and the Ullah and Arshad [20].

5. Weak and strong convergence results

In this section, we have shown some weak and strong convergence results for the iteration process (1.8).

Lemma 5.1. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive map. with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Let $\{\mu_n\}$ be formulated by iteration scheme (1.8), then $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$.*

Proof. The proof of this lemma follows from Theorem 3.1 and Proposition 2.5(*ii*). □

Lemma 5.2. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive map, with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Let $\{\mu_n\}$ be formulated by iteration scheme (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[b, c]$ such that $0 < b \leq c < 1$. Then $\mathcal{F}(\mathcal{H}) \neq \emptyset$ if and only if $\{\mu_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{H}\mu_n - \mu_n\| = 0$.*

Proof. Let $\mathcal{F}(\mathcal{H}) \neq \phi$ and $\tau \in \mathcal{G}$. Then by theorem 3.1, $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists and $\{\mu_n\}$ is bounded. Let

$$\lim_{n\to\infty} \|\mu_n - \tau\| = \rho, \tag{5.1}$$

from (3.1) and (5.1), we have

$$\limsup_{n\to\infty} \|w_n - \tau\| \leq \limsup_{n\to\infty} \|\mu_n - \tau\| = \rho, \tag{5.2}$$

$$
\begin{aligned}
\|\mu_{n+1}-\tau\| &= \|\mathcal{H}((1-\alpha_n)\nu_n+\alpha_n\mathcal{H}\nu_n)-\tau\| \\
&\leq \|(1-\alpha_n)\nu_n+\alpha_n\mathcal{H}\nu_n-\tau\| \\
&= \|(1-\alpha_n)(\nu_n-\tau)+\alpha_n(\mathcal{H}\nu_n-\tau)\| \\
&\leq (1-\alpha_n)\|\nu_n-\tau\|+\alpha_n\|\mathcal{H}\nu_n-\tau\| \\
&\leq (1-\alpha_n)\|\nu_n-\tau\|+\alpha_n\|\nu_n-\tau\| \\
&= \|\nu_n-\tau\| \\
&= \|\mathcal{H}((1-\beta_n)\omega_n+\beta_n\mathcal{H}\omega_n)\tau\| \\
&\leq \|(1-\beta_n)\omega_n+\beta_n\mathcal{H}\omega_n-\tau\| \\
&= \|(1-\beta_n)(\omega_n-\tau)+\beta_n(\mathcal{H}\omega_n-\tau)\| \\
&\leq (1-\beta_n)\|\omega_n-\tau\|+\beta_n\|\mathcal{H}\omega_n-\tau\| \\
&\leq (1-\beta_n)\|\mu_n-\tau\|+\beta_n\|\omega_n-\tau\| \\
&= \|\mu_n-\tau\|-\beta_n\|\mu_n-\tau\|+\beta_n\|\omega_n-\tau\|.
\end{aligned} \tag{5.3}
$$

Therefore,

$$
\|\mu_{n+1}-\tau\|-\|\mu_n-\tau\| \leq \frac{\|\mu_{n+1}-\tau\|-\|\mu_n-\tau\|}{\beta_n} \leq \|\omega_n-\tau\|-\|\mu_n-\tau\|, \tag{5.4}
$$

so,

$$
\|\mu_{n+1}-\tau\| \leq \|\omega_n-\tau\|,
$$

taking $n \to \infty$, we obtain

$$
\rho \leq \limsup_{n\to\infty} \|\omega_n-\tau\|, \tag{5.5}
$$

from (5.1) and (5.5),

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \|\omega_n-\tau\| \\
&= \lim_{n\to\infty} \|(1-\gamma_n)\mu_n+\gamma_n\mathcal{H}\mu_n-\tau\| \\
&= \lim_{n\to\infty} \|(1-\gamma_n)(\mu_n-\tau)+\gamma_n(\mathcal{H}\mu_n-\tau)\|,
\end{aligned} \tag{5.6}
$$

by using (5.1), (5.2), (5.6) and (2.8),

$$
\lim_{n\to\infty} \|\mathcal{H}\mu_n-\mu_n\| = 0. \tag{5.7}
$$

Conversely, assume that $\{\mu_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{H}\mu_n-\mu_n\| = 0$. Suppose that $\tau \in \mathcal{A}(\mathcal{G}, \{\mu_n\})$. Using proposition 2.5(*iii*), we get

$$
\begin{aligned}
\rho(\mathcal{H}\tau, \{\mu_n\}) &= \limsup_{n\to\infty} \|\mu_n-\mathcal{H}\tau\| \\
&\leq \limsup_{n\to\infty} [3\|\mathcal{H}\mu_n-\mu_n\|+\|\mu_n-\tau\|] \\
&\leq \limsup_{n\to\infty} \|\mu_n-\tau\| \\
&= \rho(\tau, \{\mu_n\}).
\end{aligned} \tag{5.8}
$$

This shows that $\mathcal{H}\tau \in \mathcal{A}(\mathcal{G}, \{\mu_n\}$. Since $\mathcal{X}$ is uniformly convex, $\mathcal{A}(\mathcal{G}, \{\mu_n\}$ is singleton. Thus $\mathcal{F}(\mathcal{H}) \neq \phi$. □

Theorem 5.3. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$, with the Opial's property. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive map, with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Let $\{\mu_n\}$ be formulated by iteration scheme (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[b, c]$ such that $0 < b \leq c < 1$ with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Then $\{\mu_n\}$ converges weakly to a fixed point of $\mathcal{H}$.*

Proof. As $\mathcal{F}(\mathcal{H}) \neq \emptyset$, therefore from Theorem 5.2 the sequence $\{\mu_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{H}\mu_n - \mu_n\| = 0$. Also, $\mathcal{X}$ is uniformly convex so $\mathcal{X}$ is reflexive, thus by Eberlin's Theorem we have a subsequence of $\{\mu_n\}$ say $\{\mu_{n_i}\}$ converging weakly to some $l_1 \in \mathcal{X}$. Now, using Mazur's Theorem $l_1 \in \mathcal{G}$ and by Lemma 2.6 $l_1 \in \mathcal{F}(\mathcal{H})$. On contrary, if it is not true then there must have a subsequence of $\{\mu_n\}$ such that the subsequence say $\{\mu_{n_j}\}$ weakly converges to $l_2 \in \mathcal{G}$ and $l_1 \neq l_2$. By Lemma 2.6 we get $l_2 \in \mathcal{F}(\mathcal{H})$. Since, $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$. From Opial's property and Theorem 5.2,

$$\begin{aligned}
\lim_{n\to\infty} \|\mu_n - l_1\| &= \lim_{i\to\infty} \|\mu_{n_i} - l_1\| \\
&< \lim_{i\to\infty} \|\mu_{n_i} - l_2\| \\
&= \lim_{n\to\infty} \|\mu_n - l_2\| \\
&= \lim_{j\to\infty} \|\mu_{n_j} - l_2\| \\
&< \lim_{j\to\infty} \|\mu_{n_j} - l_1\| \\
&= \lim_{n\to\infty} \|\mu_n - l_1\|.
\end{aligned} \tag{5.9}$$

which is a contradiction and this contradiction arises due to our wrong supposition, hence $l_1 = l_2$. Hence proved that $\{\mu_n\}$ weakly converges to a fixed point of $\mathcal{H}$. □

Theorem 5.4. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive map, with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Let $\{\mu_n\}$ be formulated by iteration scheme (1.8)), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[b, c]$ such that $0 < b \leq c < 1$ with $\mathcal{F}(\mathcal{H}) \neq \phi$. Then $\{\mu_n\}$ converges strongly to a fixed point of $\mathcal{H}$.*

Proof. By Lemma 2.7, $\mathcal{F}(\mathcal{H}) \neq \phi$ and by Lemma 5.2 we get $\lim_{n\to\infty} \|\mathcal{H}\mu_n - \mu_n\| = 0$. Now, compactness of $\mathcal{G}$, $\{\mu_n\}$ implies that there exists a subsequence, $\{\mu_{n_i}\}$ converging strongly to $\tau \in \mathcal{G}$. Now,

$$\|\mu_{n_i} - \mathcal{H}\tau\| \leq \|\mathcal{H}\mu_{n_i} - \mu_{n_i}\| + \|\mu_{n_i} - \tau\|. \tag{5.10}$$

Taking limit $i \to \infty$, we get $\mathcal{H}\tau = \tau$, that is $\tau \in \mathcal{F}(\mathcal{H})$. By using Lemma 5.1, $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$, hence μ_n converges strongly to τ. □

Theorem 5.5. *Let $\mathcal{G} \neq \emptyset$ be a closed convex set contained in a Banach space $\mathcal{X}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a generalized nonexpansive map, with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. Let $\{\mu_n\}$ be formulated by iteration scheme (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[b, c]$ such that $0 < b \leq c < 1$ with $\mathcal{F}(\mathcal{H}) \neq \emptyset$. If $\mathcal{H}$ satisfy condition*

$$\sigma(\mu_n, \mathcal{F}(\mathcal{H})) \leq \|\mathcal{H}\mu_n - \mu_n\|, \tag{5.11}$$

for all $n \geq 0$, then $\{\mu_n\}$ converges strongly to a fixed point of $\mathcal{H}$.

Proof. Since, $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists for all $\tau \in \mathcal{F}(\mathcal{H})$, we obtain that $\lim_{n\to\infty} \sigma(\mu_n, \mathcal{F}(\mathcal{H}))$ exists. Let $\lim_{n\to\infty} \|\mu_n - \tau\| = \delta$ for some $\delta \geq 0$. Now if $\delta = 0$ then there is nothing to prove the result follows immediately. Suppose $\delta > 0$. As $\mathcal{F}(\mathcal{H}) \neq \emptyset$, from Theorem 5.3 we obtain $\lim_{n\to\infty} \|\mathcal{H}\mu_n - \mu_n\| = 0$. Hence by condition (5.11) we get $\lim_{n\to\infty}(\sigma(\mu_n, \mathcal{F}(\mathcal{H})) = 0$. Therefore, we conclude that exist subsequence $\{\mu_{n_\iota}\}$ of $\{\mu_n\}$ and a sequence $\{\nu_\iota\}$ in $\mathcal{F}(\mathcal{H})$ such that

$$\|\mu_{n_\iota} - \nu_\iota\| < \frac{1}{2^\iota}, \quad \text{for all} \quad \iota \in \mathbb{N}, \tag{5.12}$$

from (5.12) we have,

$$\|\mu_{n_{\iota+1}} - \nu_\iota\| \leq \|\mu_{n_\iota} - \nu_\iota\| < \frac{1}{2^\iota}. \tag{5.13}$$

Since,

$$\|\nu_{\iota+1} - \nu_\iota\| \leq \|\nu_{\iota+1} - \mu_{\iota+1}\| + \|\mu_{\iota+1} - \nu_\iota\|, \tag{5.14}$$

from (5.13) we obtain

$$\|\nu_{\iota+1} - \nu_\iota\| \leq \frac{1}{2^{\iota+1}} + \frac{1}{2^\iota} < \frac{1}{2^{\iota-1}}. \tag{5.15}$$

So, sequence $\{\nu_\iota\}$ is a Cauchy and $\lim_{\iota\to+\infty} \nu_\iota = \tau$. Since $\mathcal{F}(\mathcal{H})$ is closed, we have $\tau \in \mathcal{F}(\mathcal{H})$ and $\{\mu_{n_\iota}\}$ strongly converges to τ. Also, as $\lim_{n\to\infty} \|\mu_n - \tau\|$ exists we obtain $\mu_n \to \tau \in \mathcal{F}(\mathcal{H})$. □

Conclusion

We define new iteration schemes (1.8) and it is shown that this iteration scheme converges at a fixed point faster in some cases than all the Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas et al., Thakur et al. al. and Ullah and Arshad. We are also providing a numerical example in order to correlate the convergence of iteration scheme (1.8) with the previous ones. Finally we have shown some weak and strong convergence results for this iteration process.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Author's contributions

Investigation, Q. A., A.H., R.G., S.H., M. A. and Z.D.M.; Methodology, A. H., R.G. and Z.D.M.; Software, Q. A., A. H. and Z.D.M.; Supervision, A. H., R.G. and Z.D.M. All authors have read and agreed to the published version of the manuscript.

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