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Statistical convergence with respect to power series method on product time scales

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Abstract. In this paper, we will give some new notions on arbitrary product time scales using statistical convergence in the sense of the power series method. We will then use these new concepts to present developments in the literature.

1. Introduction

The concept of statistical convergence introduced by Fast [7] has an important place in summability theory and this concept was extended to the double sequences by Moricz [16]. In recent years, double sequences and the concept of statistical convergence have gained popularity among mathematicians, resulting in numerous studies exploring this concept (see, for instance, [6, 15]). The main feature of the time scale calculation introduced by Hilger [14], which is an effective modeling method, is the combination of discrete and continuous states. This method was generally used in nature and some engineering problems. Turan and Duman [20] extended this approach to summability theory and provided a new perspective to the field via the concept of statistical convergence on time scales, which is an extension of the concept of statistical convergence. It is also known that double sequences can be considered a generalization of single sequences because they have some specific properties. Therefore, the concept of statistical convergence on product time scales for double sequences is presented as one generalization in [8].

More recently, the concept of statistical convergence with respect to the power series method was introduced in [22]. This new type of convergence and statistical convergence are incomparable so meaningful results are obtained. Recent studies provide application areas for extending this new convergence to different spaces or sequences with different properties ([5, 9–12, 18, 21]).

This study aims to introduce statistical convergence with respect to power series method on arbitrary product time scales and to present new developments by examining its fundamental characteristics.

2. Preliminary results on convergence methods and product time scales

We now recall some basic definitions and notations used in the paper.

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A double sequence $x = (x_{mn})$ is said to be convergent in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}_0$, belonging to the set of all natural numbers, such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N$, where L is called the Pringsheim limit of x and denoted by $P - \lim_{m,n} x_{mn} = L$ (see [17]). We shall call such an x, briefly, $P-$convergent. A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

In his work ([16]), Moricz introduced the concept of statistical convergence for double sequences, which has significantly contributed to the field.

Let $A \subset \mathbb{N}_0^2$ is a two-dimensional subset of positive integers and $|.|$ denote the cardinality of the set. The double natural density of A is given by

$$\delta_2(A) := P - \lim_{k,l} \frac{|\{m \leq k,\ n \leq l :\ (m, n) \in A\}|}{kl}$$

if it exists. The number sequence $x = (x_{mn})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$A := A_{kl}(\varepsilon) := \{m \leq k,\ n \leq l : |x_{mn} - L| \geq \varepsilon\}$$

has natural density zero, that is $\delta_2(A) = 0$, then we write $st_2\text{-}\lim x_{mn} = L$. Clearly, a $P-$convergent double sequence is statistically convergent to the same value but its converse is not always true (see for details, [16]).

Now we will remind some terminology and definitions in [1].

Let (p_{mn}) be a double sequence with $p_{00} > 0$, $p_{mn} \geq 0$ for every $m, n \geq 1$ such that the corresponding power series

$$p(u, v) := \sum_{m,n=0}^{\infty} p_{mn} u^m v^n$$

has radius of convergence $0 < R \leq \infty$ and $u, v \in (0, R)$. If for all $u, v \in (0, R)$,

$$\lim_{u,v \to R^-} \frac{1}{p(u, v)} \sum_{m,n=0}^{\infty} p_{mn} u^m v^n x_{mn} = L$$

then we say that the double sequence $x = (x_{mn})$ is convergent to L in the sense of power series method and denoted by $P_p^2 - \lim x_{mn} = L$ ([1]).

Note here this method is regular if and only if

$$\lim_{u,v \to R^-} \frac{\sum_{m=0}^{\infty} p_{m\nu} u^m}{p(u, v)} = 0 \text{ and } \lim_{u,v \to R^-} \frac{\sum_{n=0}^{\infty} p_{\mu n} v^n}{p(u, v)} = 0\ ,\ \text{for any } \mu, \nu,$$

hold (see, [1]).

Remark 2.1. Note that in the case of $R = 1$, if $p_{mn} = 1$ and $p_{mn} = \frac{1}{(m+1)(n+1)}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of $R = \infty$ and $p_{mn} = \frac{1}{m!n!}$, the power series method coincides with Borel summability method.

Here and throughout the article, this method is always assumed to be regular.

The concepts of P_p^2-density and P_p^2-statistical convergence for double sequences have been introduced by Yıldız, Demirci and Dirik in [23] as follows:

Definition 2.2. Let $A \subset \mathbb{N}_0^2$. If the limit

$$\delta_{P_p}^2(A) := \lim_{u,v \to R^-} \frac{1}{p(u,v)} \sum_{(m,n)\in A} p_{mn} u^m v^n$$

exists, then $\delta_{P_p}^2(A)$ is said to be the P_p^2-density of A. It is evident that whenever it exists, $0 \le \delta_{P_p}^2(A) \le 1$ according to the definition of a power series method and P_p^2-density.

Definition 2.3. Let $x = (x_{mn})$ be a double sequence. Then, x is called statistically convergent with respect to power series method (P_p^2-statistically convergent) to L if for any $\varepsilon > 0$

$$\lim_{u,v \to R^-} \frac{1}{p(u,v)} \sum_{(m,n)\in A(\varepsilon)} p_{mn} u^m v^n = 0$$

where $A(\varepsilon) = \left\{(m,n) \in \mathbb{N}_0^2 : |x_{mn} - L| \ge \varepsilon\right\}$, in other words $\delta_{P_p}^2(A(\varepsilon)) = 0$ for any $\varepsilon > 0$. So, we may denote $st_{p_p}^2 - \lim x_{mn} = L$.

Definition 2.4. A sequence of real numbers $x = (x_{mn})$ is said to be P_p^2-statistically bounded if for some $M > 0$ such that $\delta_{P_p}^2\left(\left\{(m,n) \in \mathbb{N}_0^2 : |x_{mn}| > M\right\}\right) = 0$.

Let us now recall the concepts related to time scales used in the paper.

A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, the set of real numbers. Here and in the sequel, we study on a time scales such that $\inf \mathbb{T} = i_0$ $(i_0 > 0)$ and $\sup \mathbb{T} = \infty$. For $i \in \mathbb{T}$, the forward and backward jump operators are defined as follows, respectively:

$$\sigma \ : \ \mathbb{T} \to \mathbb{T}, \ \sigma(i) := \inf\{s \in \mathbb{T} : s > i\},$$
$$\rho \ : \ \mathbb{T} \to \mathbb{T}, \ \rho(i) := \sup\{s \in \mathbb{T} : s < i\},$$

and the graininess function μ is defined by

$$\mu : \mathbb{T} \to [0,\infty), \ \mu(i) = \sigma(i) - i.$$

A closed interval in a time scale $\mathbb{T}$ is given by the notation $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T} = \{j \in \mathbb{T} : a \le j \le b\}$. Thus, open intervals and half-open intervals can be given similarly.

Also, the Lebesgue $\Delta-$measure which given by Guseinov [13], denoted by μ_Δ and it is known that if $a,b \in \mathbb{T}$ and $a \le b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b - a$, $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$, $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$, and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$.

Let $i = 1,2$, and let $\mathbb{T}$ be a time scale. Set

$$\mathbb{T}^2 = \mathbb{T} \times \mathbb{T} = \{t = (t_1,t_2) : t_i \in \mathbb{T} \text{ for all } i = 1,2\}.$$

If A is a $\Delta-$measurable subset of $\mathbb{T}^2$, then the density of A is defined by Çınar et al. [8] and given by

$$\delta_{\mathbb{T}^2}(A) := \lim_{(i,j)\to\infty} \frac{\mu_\Delta\left(\left\{(s,u) \in [i_0,i]_{\mathbb{T}} \times [j_0,j]_{\mathbb{T}} : (s,u) \in A\right\}\right)}{\mu_\Delta\left([i_0,i]_{\mathbb{T}} \times [j_0,j]_{\mathbb{T}}\right)}$$

if this limit exists. Now let $f : \mathbb{T}^2 \to \mathbb{R}$ be a $\Delta-$measurable function. Then, f is said to be statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta_{\mathbb{T}^2}\left(\left\{(i,j) \in \mathbb{T}^2 : \left|f(i,j) - L\right| \ge \varepsilon\right\}\right) = 0.$$

In this case, we write $st_{\mathbb{T}^2} - \lim_{(i,j)\to\infty} f(i,j) = L$. This definition can also be written as

$$\lim_{(i,j)\to\infty} \frac{\mu_\Delta\left(\left\{(s,u) \in [i_0,i]_{\mathbb{T}} \times [j_0,j]_{\mathbb{T}} : \left|f(s,u) - L\right| \ge \varepsilon\right\}\right)}{\mu_\Delta\left([i_0,i]_{\mathbb{T}} \times [j_0,j]_{\mathbb{T}}\right)} = 0.$$

3. Power series method and statistical type convergence on product time scales

In this section, we aim to present the notion of statistical convergence with respect to the power series method on arbitrary product time scales. Furthermore, we intend to provide some generalizations.

Here and in the sequel, we assume that $p:\mathbb{T}^2\to\mathbb{R}$ is non-negative Δ-measurable, also for every $(i,j)\in\mathbb{T}^2$, $p(i,j)$ is a Lebesgue Δ-integrable function on $\mathbb{T}^2$ and $\sup_{u,v\in(0,R)} p_\Delta(u,v)<\infty$ where $p_\Delta(u,v):=\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^j\Delta_i\Delta_j$, $R\in(0,+\infty]$.

We now present the following definition.

Definition 3.1. If for a given $\Delta-$measurable and Lebesgue $\Delta-$integrable function f on $\mathbb{T}^2$

$$\lim_{u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^jf(i,j)\,\Delta i\Delta j=L$$

then we say that f is convergent to L in the sense of power series method and denoted $P_{\Delta_{\mathbb{T}^2}}-\lim f=L$.

Lemma 3.2. *The power series method is regular for bounded functions on a product time scales $\mathbb{T}^2$ provided that, for every finite $M>0$,*

$$\lim_{0<u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{[i_0,M]_{\mathbb{T}}\times[j_0,M]_{\mathbb{T}}} p(i,j)\,u^iv^jf(i,j)\,\Delta i\Delta j=0. \tag{1}$$

Proof. Assume that $\lim_{(i,j)\to\infty} f(i,j)=L$, we can write that, for every $\varepsilon>0$, there exists a $M>0$ such that $\left|f(i,j)-L\right|<\varepsilon$ for all $i,j>M$ with $(i,j)\in\mathbb{T}^2$. Also, since f is bounded on $\mathbb{T}^2$, there exists a number $M_1>0$ such that $\left|f(i,j)-L\right|\le M_1$. Then

$$\begin{aligned}
&\left|\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^jf(i,j)\,\Delta i\Delta j-L\right|\\
\le\;&\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^j\left|f(i,j)-L\right|\Delta i\Delta j\\
=\;&\frac{1}{p_\Delta(u,v)}\left[\int\int_{[i_0,M]_{\mathbb{T}}\times[j_0,M]_{\mathbb{T}}} p(i,j)\,u^iv^j\left|f(i,j)-L\right|\Delta i\Delta j+\int\int_{\mathbb{T}^2/[i_0,M]_{\mathbb{T}}\times[j_0,M]_{\mathbb{T}}} p(i,j)\,u^iv^j\left|f(i,j)-L\right|\Delta i\Delta j\right]\\
\le\;&\frac{1}{p_\Delta(u,v)}\left[\int\int_{[i_0,M]_{\mathbb{T}}\times[j_0,M]_{\mathbb{T}}} p(i,j)\,u^iv^j\left|f(i,j)-L\right|\Delta i\Delta j+\varepsilon\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^j\Delta i\Delta j\right]\\
=\;&\frac{1}{p_\Delta(u,v)}\int\int_{[i_0,M]_{\mathbb{T}}\times[j_0,M]_{\mathbb{T}}} p(i,j)\,u^iv^j\left|f(i,j)-L\right|\Delta i\Delta j+\varepsilon\\
\le\;&M_1\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^j\Delta i\Delta j+\varepsilon.
\end{aligned}$$

We get immediately from the (1) that

$$\lim_{0<u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^iv^jf(i,j)\,\Delta i\Delta j=0$$

this completes the proof. □

Definition 3.3. For a given $\Delta-$measurable and Lebesgue $\Delta-$integrable function f on $\mathbb{T}^2$

$$\lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \int\int_{\mathbb{T}^2} p(i,j) u^i v^j \left| f(i,j) - L \right| \Delta i \Delta j = 0$$

then we say that f is strongly convergent to L in the sense of power series method ($P_{\Delta_{\mathbb{T}^2}}-$strongly convergent).

Definition 3.4. Let A be a $\Delta-$measurable subset of $\mathbb{T}^2$. $P_{\Delta_{\mathbb{T}^2}}-$density of A in $\mathbb{T}^2$ is defined by

$$\delta_{P_{\Delta_{\mathbb{T}^2}}}(A) := \lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \int\int_A p(i,j) u^i v^j \Delta i \Delta j$$

if the limit exists.

We should note that the case of $\mathbb{T}^2 = \mathbb{N}_0 \times \mathbb{N}_0$, Definition 3.4 reduces to the concept of P-density in Definition 2.2.

Lemma 3.5. (i) $\delta_{P_{\Delta_{\mathbb{T}^2}}}\left(\mathbb{T}^2\right) = 1$.

(ii) For any $\Delta-$measurable subset A of $\mathbb{T}^2$, $0 \le \delta_{P_{\Delta_{\mathbb{T}^2}}}(A) \le 1$.

(iii) If A is $\Delta-$measurable subset of $\mathbb{T}^2$ and $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A)$ exists, then $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A^c)$ exists and $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A) + \delta_{P_{\Delta_{\mathbb{T}^2}}}(A^c) = 1$, since A is $\Delta-$measurable, A^c is $\Delta-$measurable. On the other hand $A \cup A^c = \mathbb{T}^2$ and

$$\begin{aligned} 1 &= \lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \int\int_{\mathbb{T}^2} p(i,j) u^i v^j \Delta i \Delta j \\ &= \lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \left[\int\int_{\mathbb{A}} p(i,j) u^i v^j \Delta_i \Delta_j + \int\int_{\mathbb{A}^c} p(i,j) u^i v^j \Delta i \Delta j \right]. \end{aligned}$$

Assume that A and B are $\Delta-$measurable subsets of $\mathbb{T}^2$ and $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A)$ and $\delta_{P_{\Delta_{\mathbb{T}^2}}}(B)$ exists. We have the following features:

(iv) If $A \subseteq B$, then $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A) \le \delta_{P_{\Delta_{\mathbb{T}^2}}}(B)$.

(v) $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A \cup B) \le \delta_{P_{\Delta_{\mathbb{T}^2}}}(A) + \delta_{P_{\Delta_{\mathbb{T}^2}}}(B)$.

(vi) If A is bounded, then $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A) = 0$,

for a sufficiently large $M \in \mathbb{T}$ we can write $A \subseteq [i_0, M]_{\mathbb{T}} \times [j_0, M]_{\mathbb{T}}$, then using the condition (1), we get

$$0 \le \lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \int\int_{\mathbb{A}} p(i,j) u^i v^j \Delta i \Delta j \le \lim_{0<u,v\to R^-} \frac{1}{p_\Delta(u,v)} \int\int_{[i_0,M]_{\mathbb{T}} \times [j_0,M]_{\mathbb{T}}} p(i,j) u^i v^j \Delta i \Delta j = 0.$$

(vii) If $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A) = \delta_{P_{\Delta_{\mathbb{T}^2}}}(B) = 1$, then $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A \cup B) = \delta_{P_{\Delta_{\mathbb{T}^2}}}(A \cap B) = 1$.

Definition 3.6. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a $\Delta-$measurable function on $\mathbb{T}^2$. We say that f is $P_{\Delta_{\mathbb{T}^2}}-$statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta_{P_{\Delta_{\mathbb{T}^2}}}\left(\left\{(i,j) \in \mathbb{T}^2 : \left| f(i,j) - L \right| \ge \varepsilon\right\}\right) = 0,$$

holds, i.e.,

$$\lim_{0<u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{\{(i,j)\in\mathbb{T}^2:|f(i,j)-L|\geq\varepsilon\}} p(i,j)\,u^i v^j \Delta i\Delta j=0$$

and denoted by $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L$.

Remark 3.7. In the case $\mathbb{T}^2=\mathbb{N}_0^2$, statistical convergence with respect to power series method on arbitrary product time scales is reduced to the statistical convergence in the sense of power series method, which is given by Definition 2.3.

Definition 3.8. Let $f:\mathbb{T}^2\to\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}^2$. We say that f is $P_{\Delta_{\mathbb{T}^2}}$-statistical Cauchy if there exists $(I,J)\in\mathbb{T}^2$ such that, for every $\varepsilon>0$,

$$\lim_{0<u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{\{(i,j)\in\mathbb{T}^2:|f(i,j)-f(I,J)|\geq\varepsilon\}} p(i,j)\,u^j v^j \Delta i\Delta j=0.$$

Now we give some basic properties of $P_{\Delta_{\mathbb{T}^2}}-$ statistical convergence. Their proofs are similar to those of statistical ones. We will just Proposition 3.10.

Theorem 3.9. *Let $f:\mathbb{T}^2\to\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}^2$. Then the followings are equivalent:*
(*i*) $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L$,
(*ii*) *f is $P_{\Delta_{\mathbb{T}^2}}$-statistical Cauchy on $\mathbb{T}^2$,*
(*iii*) *there exists two Δ-measurable functions g and h such that* $\lim g(i,j)=L$ *and* $\delta_{P_{\Delta_{\mathbb{T}^2}}}\left(\left\{(i,j)\in\mathbb{T}^2:f(i,j)\neq 0\right\}\right)=0$ *such that f can be represented as the sum of these functions,*
(*iv*) *there exists a Δ-measurable subset A of $\mathbb{T}^2$ such that $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A)=1$ and $\lim f(i,j)=L$.*

Proposition 3.10. *The $st_{P_{\Delta_{\mathbb{T}^2}}}-$limit of a function $f:\mathbb{T}^2\to\mathbb{R}$ is unique.*

Proof. Let $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L_1$ and $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L_2$. Thanks to Theorem 3.9-(*iv*), we may write that, for every $\varepsilon>0$, there exist subsets A_1,A_2 of $\mathbb{T}^2$ such that $\delta_{P_{\Delta_{\mathbb{T}^2}}}(A_1)=\delta_{P_{\Delta_{\mathbb{T}^2}}}(A_2)=1$ and $\lim_{(i,j)\to\infty((i,j)\in A_1)} f(i,j)=L_1$, $\lim_{(i,j)\to\infty((i,j)\in A_2)} f(i,j)=L_2$. Hence, we get $|f(i,j)-L_1|<\frac{\varepsilon}{2}$, for $(i,j)\in A_1$ and $|f(i,j)-L_2|<\frac{\varepsilon}{2}$, for $(i,j)\in A_2$. Then, for every $(i,j)\in A_1\cap A_2$ one has

$$|L_1-L_2|\leq|f(i,j)-L_1|+|f(i,j)-L_2|<\varepsilon,$$

thus $L_1=L_2$. □

Proposition 3.11. *Let $f,g:\mathbb{T}^2\to\mathbb{R}$. If $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L_1$ and $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim g(i,j)=L_2$, then we get the followings:*
(*i*) $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim\{f(i,j)+g(i,j)\}=L_1+L_2$,
(*ii*) $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim cf(i,j)=cL_1\ (c\in\mathbb{R})$.

Lemma 3.12. *Let $f:\mathbb{T}^2\to\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. If $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L$ and f is bounded, then we get*

$$\lim_{0<u,v\to R^-}\frac{1}{p_\Delta(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j f(i,j)\,\Delta i\Delta j=L.$$

Lemma 3.13. *Let $f:\mathbb{T}^2\rightarrow\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}^2$ and $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L$. If $g:\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function at L, then we get*

$$st_{P_{\Delta_{\mathbb{T}^2}}}-\lim g\left(f\left(i,j\right)\right)=g\left(L\right).$$

Now, considering the above terminology, we can give the following theorem.

Theorem 3.14. *For a Δ-measurable function $f:\mathbb{T}^2\rightarrow\mathbb{R}$,*

$$st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f\left(i,j\right)=L \tag{2}$$

a necessary and sufficient condition, for every $\kappa\in\mathbb{R}$,

$$\lim_{0<u,v\rightarrow R^-}\frac{1}{p_\Delta\left(u,v\right)}\int\int_{\mathbb{T}^2}p\left(i,j\right)u^iv^je^{r\kappa f\left(i,j\right)}\Delta i\Delta j=e^{r\kappa L}. \tag{3}$$

Proof. Let $st_{P_{\Delta_{\mathbb{T}^2}}}-\lim f(i,j)=L$. Since, for a fixed $\kappa\in\mathbb{R}$, $g(\kappa)=e^{i\kappa f(i,j)}$ is continuous, thanks to Lemma 3.13, we get

$$st_{P_{\Delta_{\mathbb{T}^2}}}-\lim e^{r\kappa f\left(i,j\right)}=e^{r\kappa L}$$

and since $g(\kappa)$ is bounded, thanks to Lemma 3.12, we immediately get the equality (3). Conversely, assume that the equality (3) holds. Following [19] (see also [4, 20]), let us define the following continuous function

$$H\left(t\right)=\begin{cases}0, & t\leq-1 \text{ or } t\geq1,\\ 1+t, & -1<t<0,\\ 1-t, & 0\leq t<1.\end{cases}$$

It follows immediately from the inverse Fourier transformation that we have

$$H\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa}{2}}{\frac{\kappa}{2}}\right)^2e^{r\kappa y}d\kappa \text{ for } t\in\mathbb{R}. \tag{4}$$

It is enough to prove equality (3) for the case in which $L=0$. Hence, by hypothesis, for every $\kappa\in\mathbb{R}$,

$$\lim_{0<u,v\rightarrow R^-}\frac{1}{p_\Delta\left(u,v\right)}\int\int_{\mathbb{T}^2}p\left(i,j\right)u^iv^je^{r\kappa f\left(i,j\right)}\Delta i\Delta j=1. \tag{5}$$

Let $\varepsilon>0$ and $E:=\left\{j\in\mathbb{T}^2:\left|f\left(i,j\right)\right|\geq\varepsilon\right\}$. Then, from equality (4), we can write

$$H\left(\frac{f\left(i,j\right)}{\varepsilon}\right)=\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2e^{r\kappa f\left(i,j\right)}d\kappa.$$

Thus

$$\begin{aligned}&\frac{1}{p_\Delta\left(u,v\right)}\int\int_{\mathbb{T}^2}p\left(i,j\right)u^iv^jH\left(\frac{f\left(i,j\right)}{\varepsilon}\right)\Delta i\Delta j\\ =\;&\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2\left\{\frac{1}{p_\Delta\left(u,v\right)}\int\int_{\mathbb{T}^2}p\left(i,j\right)u^iv^je^{r\kappa f\left(i,j\right)}\Delta i\Delta j\right\}d\kappa.\end{aligned}$$

Note that (4) is an absolutely convergent integral. By the time scale version of Fubini theorem (see [2, 3]) we can get

$$\begin{aligned}
&\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j \\
=\;& \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2\left\{\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j e^{r\kappa f(i,j)}\Delta i\Delta j\right\}d\kappa.
\end{aligned}$$

Since $g(\kappa)$ is bounded, there exists a finite constants B such that, for every $\kappa\in\mathbb{R}$ and $(i,j)\in\mathbb{T}^2$,

$$\left|\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j e^{r\kappa f(i,j)}\Delta i\Delta j\right|\leq B\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j\Delta i\Delta j=B.$$

Hence, thanks to Lebesgue Dominated Convergence Theorem and if we consider equality (4) and (5), we get

$$\begin{aligned}
&\lim_{0<u,v\to R^-}\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j \\
=\;& \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2\left\{\lim_{0<u,v\to R^-}\frac{1}{p_{\Delta}(u,v)}\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j e^{r\kappa f(i,j)}\Delta i\Delta j\right\}d\kappa \\
=\;& \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2 d\kappa=H(0)=1, \qquad (6)
\end{aligned}$$

and it follows immediately from the definition of H that we may also have

$$\begin{aligned}
&\int\int_{\mathbb{T}^2} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j \\
=\;& \int\int_{\{(i,j)\in\mathbb{T}^2:|f(i,j)|<\varepsilon\}} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j \\
=\;& \int\int_{\left\{(i,j)\in\mathbb{T}^2:\,-1<\frac{f(i,j)}{\varepsilon}<0\right\}} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j+\int\int_{\left\{(i,j)\in\mathbb{T}^2:\,0<\frac{f(i,j)}{\varepsilon}<1\right\}} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i\Delta j \\
=\;& \int\int_{\left\{(i,j)\in\mathbb{T}^2:\,-1<\frac{f(i,j)}{\varepsilon}<0\right\}} p(i,j)\,u^i v^j\Delta i\Delta j+\int\int_{\left\{(i,j)\in\mathbb{T}^2:\,-1<\frac{f(i,j)}{\varepsilon}<0\right\}} p(i,j)\,u^i v^j\frac{f(i,j)}{\varepsilon}\Delta i\Delta j \\
&+\int\int_{\left\{(i,j)\in\mathbb{T}^2:\,0<\frac{f(i,j)}{\varepsilon}<1\right\}} p(i,j)\,u^i v^j\Delta i\Delta j-\int\int_{\left\{(i,j)\in\mathbb{T}^2:\,0<\frac{f(i,j)}{\varepsilon}<1\right\}} p(i,j)\,u^i v^j\frac{f(i,j)}{\varepsilon}\Delta i\Delta j \\
\leq\;& \int\int_{\{(i,j)\in\mathbb{T}^2:\,|f(i,j)|<\varepsilon\}} p(i,j)\,u^i v^j\Delta i\Delta j=1-\int\int_{\{(i,j)\in\mathbb{T}^2:\,|f(i,j)|\geq\varepsilon\}} p(i,j)\,u^i v^j\Delta i\Delta j.
\end{aligned}$$

Then,

$$\iint_{\{(i,j)\in\mathbb{T}^2:\,|f(i,j)|\geq\varepsilon\}} p(i,j)\,u^i v^j \Delta i \Delta j \leq 1 - \iint_{\mathbb{T}^2} p(i,j)\,u^i v^j H\left(\frac{f(i,j)}{\varepsilon}\right)\Delta i \Delta j.$$

Hence, using (6) and taking limit $0 < u, v \to R^-$, we get

$$\lim_{0<u,v\to R^-} \iint_{\{(i,j)\in\mathbb{T}^2:\,|f(i,j)|\geq\varepsilon\}} p(i,j)\,u^i v^j \Delta i \Delta j = 0.$$

This completes the proof. □

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