Filomat 38:18 (2024), 6549–6561
https://doi.org/10.2298/FIL2418549C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Higher-order viability result for Carathéodory non-Lipschitz differential inclusion in Banach spaces

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Abstract. This paper deals with the construction of approximants and the existence of solutions to the following higher-order viability problem :

$x^{(k)}(t) \in F(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t))$ *a.e. on* $[0, T[$ and $x(t) \in D$ for all $t \in [0, T]$, where $F : [0, T] \times D \times \prod_{i=1}^{k-1} \Omega_i \to 2^E$ is a non-convex and non-compact multifunction and D is a closed subset of a separable Banach space E. It extends our result established in the first-order case [6].

1. Introduction

Let E be a separable Banach space with a norm $\|.\|$, D a nonempty closed subset of E, $\Omega_1, ...\Omega_{k-1}$ $(k \geq 2)$ are nonempty open subsets of E, T a strictly positive real. Put $I := [0, T]$ and denote $W^{k,1}(I, E)$ the space of functions possessing absolutely continuous derivatives up to order k. Let $F : I \times D \times \prod_{i=1}^{k-1} \Omega_i \to 2^E$ be a multifunction. The aim of this work is to study, for any fixed $(x_0, y_0^1, ..., y_0^{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$, the existence of solutions and the construction of approximants to the following problem :

$$\begin{cases} x^{(k)}(t) \in F(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t)) \quad a.e. \text{ on } [0, T[; \\ (x(0), x^{(1)}(0), ..., x^{(k-1)}(0)) = (x_0, y_0^1, ..., y_0^{k-1}); \\ x(t) \in D, \quad \forall\, t \in I. \end{cases} \tag{1}$$

By a solution to (1), we mean $x(.) \in W^{k,1}(I, E)$ satisfying (1). Here F is a separately measurable on I and separately upper semi-continuous multifunction on $D \times \prod_{i=1}^{k-1} \Omega_i$ with non-convex and non-compact values

2020 *Mathematics Subject Classification*. 34A60; 28B20.
Keywords. viability, measurable multifunction, selection, non-Lipschitz multifunction.
Received: 02 October 2023; Revised: 16 February 2024; Accepted: 04 March 2024
Communicated by Maria Alessandra Ragusa
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in E, uniformly continuous with respect to the last argument.
This result is an extension of our paper [6], where it has been proved the existence of solutions to the following first-order viability problem :

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \quad a.e.\ on\ [0, T[; \\ x(0) = x_0; \\ x(t) \in D, \ \ \forall\, t \in I. \end{cases} \tag{2}$$

assuming that the right-hand side $(t, x) \to F(t, x)$ is measurable on t and uniformly continuous on x in the following sense :

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0,\ \forall (t, x, y) \in I \times D \times D,\ \|x - y\| \le \delta(\varepsilon) \Rightarrow d_H(F(t, x), F(t, y)) \le \varepsilon,$$

where d_H stands for the Hausdorff distance. Solution to (2) is obtained under the following tangency condition :

For all $t \in [0, T[$ and $x \in D$, for all measurable selection $\sigma(.)$ of the multifunction $t \to F(t, x)$

$$\liminf_{h \to 0^+} \frac{1}{h} d_D\Big(x + \int_t^{t+h} \sigma(s)ds\Big) = 0.$$

As mentioned in [6], this result extends those of Larrieu [8] and Duc Ha [7] where these authors have studied problem (2) with Carathéodory Lipschitz single-valued map for the first author, while the second author gives a multivalued version of Larrieu's result.
Similar problem of (1) in the case of non-covex Carathéodory Lipschitz right-hand side where proved by Aitalioubrahim and Sajid [1].
In this paper, we prove the existence of solutions to problem (1) where the right- hand side is a Carathéodory-upper semi-continuous multifunction, uniformly continuous with respect to the last variable whose values are not necessarily convex not compact in separable Banach spaces and satisfying the following condition :

For all $(t, x, y^1, ..., y^{k-1}) \in [0, T[\times D \times \prod_{i=1}^{k-1} \Omega_i$, for all measurable selection $\sigma(.)$ of the multifunction $t \to F(t, x, y^1, ..., y^{k-1})$

$$\liminf_{h \to 0^+} \frac{k!}{h^k} d\Big(x + \sum_{i=1}^{k-1} \frac{h^i}{i!} y^i + \frac{h^{k-1}}{k!} \int_t^{t+h} \sigma(s)ds, D\Big) = 0. \tag{3}$$

As far as we know, higher-order viability problem was first investigated by Marco and Murillo [10]. It has been proved a necessary and sufficient condition for the problem (1), to have a solution. More precisely, they assume the following tangency condition :

$$\forall\ (x, x_1, ..., x_{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i, F(x, x_1, ..., x_{k-1}) \cap A_D^k(x, x_1, ..., x_{k-1}) \neq \emptyset$$

where $A_D^k(x_0, x_1, ..., x_{k-1})$ is the tangent set of k-th-order defined by

$$A_D^k(x_0, x_1, ..., x_{k-1}) = \Big\{ y \in E : \liminf_{h \to 0^+} \frac{k!}{h^k} d\Big(\sum_{i=0}^{k-1} \frac{h^i}{i!} x_i + \frac{h^k}{k!} y, D\Big) = 0 \Big\}.$$

Though under very strong assumptions, namely the multifunction F does not depend on the time with convex and compact values in finite dimensional space and the graph of the multifunction $(x_0, x_1, ..., x_{k-1}) \to$

$A_D^k(x_0, x_1, ..., x_{k-1})$ is locally compact.
In this paper, when F does not depend on the time ($F(t,x) = F(x)$), the tangency condition (3) becomes :

For all $(x, y_1, ..., y_{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$, $F(x, y_1, ..., y_{k-1}) \subset A_D^k(x, y_1, ..., y_{k-1})$.

Clearly this tangency condition is rather strong than the one of Marco and Murillo. However, it is counterbalanced in this paper by weaker hypotheses, in particular the right-hand side is non-convex and non-compact not only in Euclidien spaces but in Banach spaces and the graph of the multifunction $(x, y_1, ..., y_{k-1}) \rightarrow A_D^k(x, y_1, ..., y_{k-1})$ is not necessarily locally compact.

2. Notations, definitions and main result

In all the paper, E is a separable Banach space with the norm $\|.\|$. We denote by $W^{k,1}(I, E)$ the space of functions possessing absolutely continuous derivatives up to order $k-1$. For $x \in E$ and $r > 0$, let $B(x, r) := \{y \in E : \|y - x\| < r\}$ be the open ball centered at x with radius r and $\overline{B}(x, r)$ be its closure and put $B = B(0, 1)$. For $x \in E$ and for nonempty bounded subsets A, B of E, we denote by $d_A(x)$ or $d(x, A)$ the real $\inf\{\|x - y\| : y \in A\}$; $e(A, B) := \sup\{d_B(x); x \in A\}$ and $d_H(A, B) = \max(e(A, B), e(B, A))$. Let $\mathcal{L}(I)$ the σ-algebra of Lebesgue measurable subsets of I, and $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E for the strong topology. A multifunction is said to be measurable if its graph (is measurable) belongs to $\mathcal{L}(I) \otimes \mathcal{B}(E)$. For more details on measurability theory, we refer the reader to the book by Castaing-Valadier [5].

Let $F : I \times D \times \prod_{i=1}^{k-1} \Omega_i \rightarrow 2^E$ be a multifunction with nonempty closed values in E. On F we make the following assumptions :

($\mathbf{A}_1$) For each $(x, y_1, ..., y_{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$, $\quad t \rightarrow F(t, x, y_1, ..., y_{k-1})$ is measurable.

($\mathbf{A}_2$) For any $t \in I$, $(x, y_1, ..., y_{k-1}) \rightarrow F(t, x, y_1, ..., y_{k-1})$ is upper semi-continuous :
$\forall\, \varepsilon > 0, \forall\, t \in I, \forall\, (x, y_1, ..., y_{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i, \exists\, \alpha > 0, \forall\, (x', z_1, ..., z_{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i,$

$$\max_{1 \le i \le k-1} \{\|x - x'\|, \|y_i - z_i\|\} < \alpha \Rightarrow F(t, x', z_1, ..., z_{k-1}) \subset F(t, x, y_1, ..., y_{k-1}) + B(0, \varepsilon)$$

($\mathbf{A}_3$) $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$, $\forall\, t \in I$, $\forall\, x, x' \in D$, and $(y_1, ..., y_{k-1}), (z_1, ..., z_{k-1}) \in \prod_{i=1}^{k-1} \Omega_i$

$$\|y_{k-1} - z_{k-1}\| \le \delta(\varepsilon) \Rightarrow d_H\Big(F(t, x, y_1, ..., y_{k-1}), F(t, x', z_1, ..., z_{k-1})\Big) \le \varepsilon$$

($\mathbf{A}_4$) There exists $M > 0$, for all $(t, x, y_1, ..., y_{k-1}) \in I \times D \times \prod_{i=1}^{k-1} \Omega_i$

$$\sup_{z \in F(t, x, y_1, ..., y_{k-1})} \|z\| \le M.$$

(**A**$_5$) For all $t \in [0, T[$ and $(x, y^1, ..., y^{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$, for all measurable selection $\sigma(.)$ of the multifunction $t \to F(x, y^1, ..., y^{k-1})$

$$\liminf_{h \to 0^+} \frac{k!}{h^k} d\Big(x + \sum_{i=1}^{k-1} \frac{h^i}{i!} y^i + \frac{h^{k-1}}{k!} \int_t^{t+h} \sigma(s) ds, D\Big) = 0.$$

Let $(x_0, y_0^1, ..., y_0^{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$. Under hypotheses (A$_1$)-(A$_5$) we shall prove the following main result :

Theorem 2.1. *There exists $x(.) \in W^{k,1}(I, E)$, such that*

$$\begin{cases} x^{(k)}(t) \in F(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t)) \quad a.e. \text{ on } [0, T[; \\ (x(0), x^{(1)}(0), ..., x^{(k-1)}(0)) = (x_0, y_0^1, ..., y_0^{k-1}); \\ x(t) \in D, \quad \forall t \in I. \end{cases}$$

3. Preliminary results

We will need the following lemmas which deal with measurability results.

Lemma 3.1. *[3] Let Ω be a nonempty set in E. Let $G : \big[a, b] \times \Omega \to 2^E$ be a multifunction with nonempty closed values satisfying :*
- For every $x \in \Omega$, $G(., x)$ is measurable on $[a, b]$.
- For every $t \in [a, b]$, $G(t, .)$ is (Hausdorff) continuous on Ω.
Then for any measurable function $x(.) : [a, b] \to \Omega$ the multifunction $G(., x(.))$ is measurable on $[a, b]$.

For the proof, see Lemma 8.2.3

Lemma 3.2. *[5] Let $R : I \to 2^E$ be a measurable multifunction with nonempty closed values in E. Then R admits a measurable selection : there exists a measurable function $r : I \to E$ that is $r(t) \in R(t)$ for all $t \in I$.*

Lemma 3.3. *[6] Let $G : I \to 2^E$ be a measurable multifunction with nonempty closed values and $z(.) : I \to E$ a measurable function. Then for any positive measurable function $r(.) : I \to \mathbb{R}^+$, there exists a measurable selection $g(.)$ of G such that for all $t \in I$,*

$$\|g(t) - z(t)\| \leq d(z(t), G(t)) + r(t).$$

4. Proof of the main result

The approach is based on two steps, it consists of the construction of a sequence of approximate solutions in the first one; while in the second step, we prove the convergence of such approximate solutions.

***Step* 1** *Construction of approximants.*

For any $i = 1, ..., k-1$, Ω_i is nonempty open subsets of E, then there exists $\eta_i > 0$ such that $\overline{B}(y_0^i, \eta_i) \subset \Omega_i$.
Put $\eta = \min_{1 \leq i \leq k-1} \eta_i$, then $\prod_{i=1}^{k-1} \overline{B}(y_0^i, \eta) \subset \prod_{i=1}^{k-1} \Omega_i$.
Let us define the sequence $(c_p)_{p \in \mathbb{N}}$ as following :

$$\begin{cases} c_0 = \max_{1 \leq i \leq k-1} \|y_0^i\|, \\ c_p = k c_{p-1} + M + 1, \quad \forall\, p \geq 1. \end{cases} \tag{4}$$

For each integer $n > \max(T; 1)$, put $\tau_n := \dfrac{T}{n}$ and consider the following partition of the interval I with the points : $t_i^n = i\tau_n$, $i = 0, 1, ..., n$. Remark that $I = \bigcup_{i=0}^{n-1} [t_i^n, t_{i+1}^n]$.

Since $t \to F(t, x_0, y_0^1, ..., y_0^{k-1})$ is measurable with closed values, then by Lemma 3.2, there exists a measurable function $f_0(.)$ such that for all $t \in I$, $f_0(t) \in F(t, x_0, y_0^1, ..., y_0^{k-1})$. Note that by (A_4), $f_0(.) \in L^1(I, E)$.

For any $n \in \mathbb{N}^*$, put $f_0^n(.) = f_0(.)$. We shall prove the following theorem :

Theorem 4.1. *For all $n \in \mathbb{N}^*$, there exist $\varphi_0(n) \in \mathbb{N}^*$, $(x_1^n, y_{1,n}^1, ..., y_{1,n}^{k-1}) \in D \times \prod_{i=1}^{k-1} \Omega_i$, $u_0^n(.)$, $f_1^n(.) \in L^1(I, E)$ such that :*

(*i*) $x_1^n := x_0 + \sum_{i=1}^{k-1} \dfrac{\tau_{\varphi_0(n)}^i}{i!} y_0^i + \dfrac{\tau_{\varphi_0(n)}^k}{k!} u_0^n(0) \in D,$

(*ii*) $y_{1,n}^i = \sum_{j=i}^{k-1} \dfrac{\tau_{\varphi_0(n)}^{j-i}}{(j-i)!} y_0^j + \dfrac{\tau_{\varphi_0(n)}^{k-i}}{(k-i)!} u_0^n(0), \quad i \in \{1, ..., k-1\},$

(*iii*) $(y_{1,n}^1, ..., y_{1,n}^{k-1}) \in \prod_{i=1}^{k-1} \overline{B}(y_0^i, \eta),$

(*iv*) $u_0^n(t) \in F(t, x_0, y_0^1, ..., y_0^{k-1}) + \dfrac{1}{2^n}\overline{B}$, $\|u_0^n(t) - f_0^n(t)\| \leq \dfrac{1}{2^n}$, *a.e. on* $[t_0^n, t_1^n[$,

(*v*) $f_1^n(t) \in F(t, x_1^n, y_{1,n}^1, ..., y_{1,n}^{k-1})$, $\|f_1^n(t) - f_0^n(t)\| \leq \dfrac{1}{2^{n+1}}$, *for all* $t \in I$.

Proof. By (A_5) for all $t \in [0, T[$,

$$\liminf_{n\to+\infty} \frac{k!}{\tau_n^k} d_D\Big(x_0 + \sum_{i=1}^{k-1} \frac{\tau_n^i}{i!} y_0^i + \frac{\tau_n^{k-1}}{k!} \int_t^{t+\tau_n} f_0^n(s)ds\Big) = 0.$$

Then for all $t \in [0, T[$, there exists an integer $\varphi_t(n) > n$ such that

$$\frac{k!}{\tau_{\varphi_t(n)}^k} d_D\Big(x_0 + \sum_{i=1}^{k-1} \frac{\tau_{\varphi_t(n)}^i}{i!} y_0^i + \frac{\tau_{\varphi_t(n)}^{k-1}}{k!} \int_t^{t+\tau_{\varphi_t(n)}} f_0^n(s)ds\Big) \leq \frac{1}{2^{n+2}}.$$

Hence, by the characterization of the lower bound, there exists $\xi_1(t) \in D$ such that

$$\frac{k!}{\tau_{\varphi_t(n)}^k} \|\xi_1(t) - x_0 - \sum_{i=1}^{k-1} \frac{\tau_{\varphi_t(n)}^i}{i!} y_0^i - \frac{\tau_{\varphi_t(n)}^{k-1}}{k!} \int_t^{t+\tau_{\varphi_t(n)}} f_0^n(s)ds\| \quad \leq \quad \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}},$$

then

$$\|\frac{\xi_1(t) - x_0 - \sum_{i=1}^{k-1} \frac{\tau_{\varphi_t(n)}^i}{i!} y_0^i}{\frac{\tau_{\varphi_t(n)}^k}{k!}} - \frac{1}{\tau_{\varphi_t(n)}} \int_t^{t+\tau_{\varphi_t(n)}} f_0^n(s)ds\| \quad \leq \quad \frac{1}{2^{n+1}}.$$

On the other hand, in view of Lebesgue's Differentiation Theorem, we can suppose

$$\|\frac{1}{\tau_{\varphi_t(n)}}\int_t^{t+\tau_{\varphi_t(n)}} f_0^n(s)ds - f_0^n(t)\| \quad \leq \quad \frac{1}{2^{n+1}} \ \ a.e. \text{ on } I,$$

therefore

$$\|\frac{\xi_1(t) - x_0 - \sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_t(n)}}{i!}y_0^i}{\frac{\tau^k_{\varphi_t(n)}}{k!}} - f_0^n(t)\| \quad \leq \quad \frac{1}{2^n} \ \ a.e. \text{ on } I.$$

Set

$$u_0^n(t) = \frac{\xi_1(t) - x_0 - \sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_t(n)}}{i!}y_0^i}{\frac{\tau^k_{\varphi_t(n)}}{k!}},$$

then for all $t \in [0, T[$ $\quad \xi_1(t) = x_0 + \sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_t(n)}}{i!}y_0^i + \frac{\tau^k_{\varphi_t(n)}}{k!}u_0^n(t) \ \ \in D,$

and

$$\|u_0^n(t) - f_0^n(t)\| \leq \frac{1}{2^n} \quad a.e. \text{ on } I. \tag{5}$$

Then

$$u_0^n(t) \in F(t, x_0, y_0^1, ..., y_0^{k-1}) + \frac{1}{2^n}\overline{B}.$$

Particularly

$$x_0 + \sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_t(n)}}{i!}y_0^i + \frac{\tau^k_{\varphi_t(n)}}{k!}u_0^n(t) \ \ \in D, \qquad \forall\, t \in [t_0^n, t_1^n[;$$

and

$$u_0^n(t) \in F(t, x_0, y_0^1, ..., y_0^{k-1}) + \frac{1}{2^n}\overline{B}, \ \ a.e. \text{ on } [t_0^n, t_1^n[.$$

Let $\delta_n = \delta(\frac{1}{2^{n+2}})$ be the real given by (A_3) and for every $n \in \mathbb{N}^*$, choose an integer $\varphi_0(n)$ that is

$$\varphi_0(n) > \max(\frac{T(M+1)}{\delta_n}, \frac{4^1 T c_1}{\eta}) \tag{6}$$

and set

$$x_1^n := \xi_1(t_0^n) = x_0 + \sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_0(n)}}{i!}y_0^i + \frac{\tau^k_{\varphi_0(n)}}{k!}u_0^n(0) \in D.$$

For all $n \in \mathbb{N}^*$ and for $i = 1, ..., k-1$, denote

$$y_{1,n}^i = \sum_{j=i}^{k-1}\frac{\tau^{j-i}_{\varphi_0(n)}}{(j-i)!}y_0^j + \frac{\tau^{k-i}_{\varphi_0(n)}}{(k-i)!}u_0^n(0). \tag{7}$$

thus

$$\|y_{1,n}^i - y_0^i\| \quad \leq \quad \sum_{j=i+1}^{k-1}\frac{\tau^{j-i}_{\varphi_0(n)}}{(j-i)!}\|y_0^j\| + \frac{\tau^{k-i}_{\varphi_0(n)}}{(k-i)!}\|u_0^n(0)\|,$$

For all $j \in \mathbb{N}^*$, since

$$0 < \tau^j_{\varphi_0(n)} < \tau_{\varphi_0(n)} \text{ and } \frac{\tau^j_{\varphi_0(n)}}{j!} < 1,$$

we deduce, according relations (A_4), (5) and (6) that

$$\begin{aligned} \|y^i_{1,n} - y^i_0\| &\leq \Big((k-1)\max_{1\leq i\leq k-1}\|y^i_0\| + M + 1\Big)\tau_{\varphi_0(n)} \\ &\leq \Big((k-1)c_0 + M + 1\Big)\tau_{\varphi_0(n)} \\ &\leq c_1\tau_{\varphi_0(n)} \\ &\leq \frac{\eta}{4^1}, \end{aligned}$$

then

$$(y^1_{1,n}, ..., y^{k-1}_{1,n}) \in \prod_{i=1}^{k-1}\overline{B}(y^i_0, \eta).$$

By relation (7), for $i = k-1$

$$\begin{aligned} \|y^{k-1}_{1,n} - y^{k-1}_0\| &= \frac{T}{\varphi_0(n)}\|u^n_0(0)\| \\ &\leq \frac{T}{\varphi_0(n)}(M+1), \\ &< \delta_n, \end{aligned}$$

then by (A_3)

$$d_H(F(t, x^n_1, y^1_{1,n}, ..., y^{k-1}_{1,n}), F(t, x_0, y^1_0, ..., y^{k-1}_0)) \leq \frac{1}{2^{n+2}} \quad \forall\, t \in I,$$

thus

$$d(f^n_0(t), F(t, x^n_1, y^1_{1,n}, ..., y^{k-1}_{1,n})) \leq \frac{1}{2^{n+2}}, \quad \forall\, t \in I.$$

In view of Lemma 3.3, there exists a function $f^n_1(.) \in L^1(I, E)$ such that
$f^n_1(t) \in F(t, x^n_1, y^1_{1,n}, ..., y^{k-1}_{1,n})$ and for all $t \in I$

$$\begin{aligned} \|f^n_1(t) - f^n_0(t)\| &\leq d(f^n_0(t), F(t, x^n_1, y^1_{1,n}, ..., y^{k-1}_{1,n})) + \frac{1}{2^{n+2}} \\ &\leq \frac{1}{2^{n+1}}. \end{aligned}$$

□

By induction, for $p \in \{2, ..., n\}$, assume that have been constructed
$\varphi_{p-2}(n) \in \mathbb{N}^*$, $x^n_{p-1} \in D$, $y^i_{p-1,n} \in \Omega_i$, $i \in \{1, ..., k-1\}$, $u^n_{p-2}(.) : [t^n_{p-2}, t^n_{p-1}[\to E$, and $f^n_{p-1}(t) \in F(t, x^n_{p-1}, y^1_{p-1,n}, ..., y^{k-1}_{p-1,n})$
satisfying the following relations :

(*i*) For all $j \in \{0, ..., p-2\}$, $\varphi_j(n) > \dfrac{4^{j+1}Tc_{j+1}}{\eta}$,

(*ii*) $x^n_{p-1} := \xi_p(t^n_{p-2}) = x^n_{p-2} + \displaystyle\sum_{i=1}^{k-1}\frac{\tau^i_{\varphi_{p-2}(n)}}{i!}y^i_{p-2,n} + \frac{\tau^k_{\varphi_{p-2}(n)}}{k!}u^n_{p-2}(t^n_{p-2}) \in D$,

(*iii*) $y^i_{p-1,n} = \sum_{j=i}^{k-1} \frac{\tau^{j-i}_{\varphi_{p-2}(n)}}{(j-i)!} y^j_{p-2,n} + \frac{\tau^{k-i}_{\varphi_{p-2}(n)}}{(k-i)!} u^n_{p-2}(t^n_{p-2}),$

(*iv*) For all $j \in \{1, ..., p-1\}$, $\|y^i_{j,n}\| \le c_j$,

(*v*) For all $i \in \{1, ..., k-1\}$, and for $j \in \{1, ..., p-1\}$, $\|y^i_{j,n} - y^i_{j-1,n}\| \le \frac{\eta}{4^j}$,

(*vi*) $(y^1_{p-1,n}, ..., y^{k-1}_{p-1,n}) \in \prod_{i=1}^{k-1} \overline{B}(y^i_0, \eta)$,

(*vii*) $\|u^n_{p-2}(t) - f^n_{p-2}(t)\| \le \frac{1}{2^n}$ *a.e. on* $[t^n_{p-2}, t^n_{p-1}[$,

(*iix*) $u^n_{p-2}(t) \in F(t, x^n_{p-2}, y^1_{p-2,n}, ..., y^{k-1}_{p-2,n}) + \frac{1}{2^n}\overline{B}$ *a.e. on* $[t^n_{p-2}, t^n_{p-1}[$,

(*ix*) $\|f^n_{p-1}(t) - f^n_{p-2}(t)\| \le \frac{1}{2^{n+1}}$ for all $t \in I$.

Let us define x^n_p, $(y^i_{p,n})_{i=1,...,k-1}$, $f^n_p(.)$, $u^n_{p-1}(.)$ and $\varphi_{p-1}(n)$ that is $\varphi_{p-1}(n) > \varphi_{p-2}(n)$.

Indeed, for all $t \in [0, T[$, by applying (A$_5$) for the measurable selection $f^n_{p-1}(t) \in F(t, x^n_{p-1}, y^1_{p-1,n}, ..., y^{k-1}_{p-1,n})$, we have

$$\liminf_{n\to+\infty} \frac{k!}{\tau^k_n} d_D\Big(x^n_{p-1} + \sum_{i=1}^{k-1} \frac{\tau^i_n}{i!} y^i_{p-1,n} + \frac{\tau^{k-1}_n}{k!} \int_t^{t+\tau_n} f^n_{p-1}(s)ds\Big) = 0.$$

Then for all $t \in [0, T[$, there exists $\varphi^{p-1}_t(n) \in \mathbb{N}$ such that $\varphi^{p-1}_t(n) > \varphi^{p-2}_t(n)$,

$$\frac{k!}{\tau^k_{\varphi^{p-1}_t(n)}} d_D\Big(x^n_{p-1} + \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi^{p-1}_t(n)}}{i!} y^i_{p-1,n} + \frac{\tau^{k-1}_{\varphi^{p-1}_t(n)}}{k!} \int_t^{t+\tau_{\varphi^{p-1}_t(n)}} f^n_{p-1}(s)ds\Big) \le \frac{1}{2^{n+2}},$$

hence, by the characterization of the lower bound, there exists $\xi_p(t) \in D$ such that

$$\frac{k!}{\tau^k_{\varphi^{p-1}_t(n)}} \|\xi_p(t) - x^n_{p-1} - \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi^{p-1}_t(n)}}{i!} y^i_{p-1,n} - \frac{\tau^{k-1}_{\varphi^{p-1}_t(n)}}{k!} \int_t^{t+\tau_{\varphi^{p-1}_t(n)}} f^n_{p-1}(s)ds\| \quad \le \quad \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}},$$

then

$$\|\frac{\xi_p(t) - x^n_{p-1} - \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi^{p-1}_t(n)}}{i!} y^i_{p-1,n}}{\frac{\tau^k_{\varphi^{p-1}_t(n)}}{k!}} - \frac{1}{\tau_{\varphi^{p-1}_t(n)}} \int_t^{t+\tau_{\varphi^{p-1}_t(n)}} f^n_{p-1}(s)ds\| \quad \le \quad \frac{1}{2^{n+1}}.$$

On the other hand, in view of Lebesgue's Differentiation Theorem, we can suppose

$$\|\frac{1}{\tau_{\varphi^{p-1}_t(n)}} \int_t^{t+\tau_{\varphi^{p-1}_t(n)}} f^n_{p-1}(s)ds - f^n_{p-1}(t)\| \quad \le \quad \frac{1}{2^{n+1}} \ \textit{a.e.} \text{ on } I,$$

therefore

$$\|\frac{\xi_p(t) - x^n_{p-1} - \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi_t^{p-1}(n)}}{i!} y^i_{p-1,n}}{\frac{\tau^k_{\varphi_t^{p-1}(n)}}{k!}} - f^n_{p-1}(t)\| \leq \frac{1}{2^n} \; a.e. \text{ on } I.$$

Set

$$u^n_{p-1}(t) = \frac{\xi_p(t) - x^n_{p-1} - \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi_t^{p-1}(n)}}{i!} y^i_{p-1,n}}{\frac{\tau^k_{\varphi_t^{p-1}(n)}}{k!}},$$

then for all $t \in [0, T[$ $\quad \xi_p(t) = x^n_{p-1} + \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi_t^{p-1}(n)}}{i!} y^i_{p-1,n} + \frac{\tau^k_{\varphi_t^{p-1}(n)}}{k!} u^n_{p-1}(t) \in D,$
and

$$\|u^n_{p-1}(t) - f^n_{p-1}(t)\| \leq \frac{1}{2^n} \quad a.e. \text{ on } I,$$

from which, we deduce that

$$u^n_{p-1}(t) \in F(t, x^n_{p-1}, y^1_{p-1,n}, ..., y^{k-1}_{p-1,n}) + \frac{1}{2^n}\overline{B}.$$

Then we have

$$x^n_{p-1} + \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi_t^{p-1}(n)}}{i!} y^i_{p-1,n} + \frac{\tau^k_{\varphi_t^{p-1}(n)}}{k!} u^n_{p-1}(t) \in D, \quad \forall\, t \in [t^n_{p-1}, t^n_p[,$$

and

$$u^n_{p-1}(t) \in F(t, x^n_{p-1}, y^1_{p-1,n}, ..., y^{k-1}_{p-1,n}) + \frac{1}{2^n}\overline{B}, \; a.e. \text{ on } [t^n_{p-1}, t^n_p[.$$

Choose

$$\varphi_{p-1}(n) > \max(\varphi^{p-1}_{t^n_{p-1}}(n); \frac{4^p T c_p}{\eta})$$

Put

$$x^n_p := \xi_p(t^n_{p-1}) = x^n_{p-1} + \sum_{i=1}^{k-1} \frac{\tau^i_{\varphi_{p-1}(n)}}{i!} y^i_{p-1,n} + \frac{\tau^k_{\varphi_{p-1}(n)}}{k!} u^n_{p-1}(t^n_{p-1}) \in D,$$

for all $n \in \mathbb{N}^*$ and for $i = 1, ..., k-1$, denote

$$y^i_{p,n} = \sum_{j=i}^{k-1} \frac{\tau^{j-i}_{\varphi_{p-1}(n)}}{(j-i)!} y^j_{p-1,n} + \frac{\tau^{k-i}_{\varphi_{p-1}(n)}}{(k-i)!} u^n_{p-1}(t^n_{p-1}), \tag{8}$$

Fix $i \in \{1, ..., k-1\}$, by the same previous reasoning

$$\begin{aligned} \|y^i_{p,n} - y^i_{p-1,n}\| &\leq \sum_{j=i+1}^{k-1} \frac{\tau^{j-i}_{\varphi_{p-1}(n)}}{(j-i)!} \|y^j_{p-1,n}\| + \frac{\tau^{k-i}_{\varphi_{p-1}(n)}}{(k-i)!} \|u^n_{p-1}(t^n_{p-1})\| \\ &\leq \Big((k-1)c_{p-1} + M + 1\Big)\tau_{\varphi_{p-1}(n)} \\ &\leq c_p \tau_{\varphi_{p-1}(n)} \\ &\leq \frac{\eta}{4^p}. \end{aligned}$$

So that

$$\begin{aligned} \|y^i_{p,n} - y^i_{0,n}\| &\le \sum_{j=0}^{p-1} \|y^i_{j+1,n} - y^i_{j,n}\| \\ &\le \sum_{j=1}^{p} \frac{\eta}{4^j} \\ &\le \frac{\eta}{2}, \end{aligned}$$

and

$$\begin{aligned} \|y^i_{p,n}\| &\le \|y^i_{p,n} - y^i_{p-1,n}\| + \|y^i_{p-1,n}\| \\ &\le \Big((k-1)c_{p-1} + M + 1\Big) + c_{p-1} \\ &\le kc_{p-1} + M + 1 = c_p. \end{aligned}$$

In view of relation (8), as $y^{k-1}_{p,n} = y^{k-1}_{p-1,n} + \tau_{\varphi_{p-1}(n)} u^n_{p-1}(t^n_{p-1})$, one has

$$\begin{aligned} \|y^{k-1}_{p,n} - y^{k-1}_{p-1,n}\| &= \frac{T}{\varphi_{p-1}(n)} \|u^n_{p-1}(t^n_{p-1})\| \\ &\le \frac{T}{\varphi_{p-1}(n)}(M+1) \\ &< \delta_n, \end{aligned}$$

hence, by (A_3)

$$d_H(F(t, x^n_p, y^1_{p,n}, ..., y^{k-1}_{p,n}), F(t, x^n_{p-1}, y^1_{p-1,n}, ..., y^{k-1}_{p-1,n})) \le \frac{1}{2^{n+2}} \quad \forall\, t \in I,$$

thus

$$d(f^n_{p-1}(t), F(t, x^n_p, y^1_{p,n}, ..., y^{k-1}_{p,n})) \le \frac{1}{2^{n+2}}, \quad \forall\, t \in I.$$

By Lemma 3.3, there exists a measurable function $f^n_p(.) \in L^1(I, E)$ such that
$f^n_p(t) \in F(t, x^n_p, y^1_{p,n}, ..., y^{k-1}_{p,n})$ and for all $t \in I$

$$\|f^n_p(t) - f^n_{p-1}(t)\| \le d(f^n_{p-1}(t), F(t, x^n_p, y^1_{p,n}, ..., y^{k-1}_{p,n})) + \frac{1}{2^{n+2}}.$$

Then

$$\|f^n_p(t) - f^n_{p-1}(t)\| \le \frac{1}{2^{n+1}}. \tag{9}$$

Put $q_n = \varphi_n(n)$. Remark that the previous relations are satisfied for q_n.

Now, let us define the step functions.

For all $n \ge 1$, for all $t \in [0, T[$, set $\theta_n(t) = t^n_{p-1}$, whenever $t \in [t^n_{p-1}, t^n_p[$, and consider the functions
$f_n(t) = \sum_{p=1}^{n} \chi_{[t^n_{p-1}, t^n_p[}(t) f^n_{p-1}(t)$ and $u_n(t) = \sum_{p=1}^{n} \chi_{[t^n_{p-1}, t^n_p[}(t) u^n_{p-1}(t)$,
when $\chi_J(.)$ denotes the characteristic function for any interval J.

On each interval $[t^n_{p-1}, t^n_p]$, define by induction

$$g_{1,n}(t) = \int_{t^n_{p-1}}^{t} u_n(s)ds.$$

and for all $i \in \{2, ..., k\}$

$$g_{i,n}(t) = \int_{t^n_{p-1}}^{t} g_{i-1,n}(s)ds,$$

and consider

$$x_n(t) = x^n_{p-1} + \sum_{i=1}^{k-1} \frac{(t - t^n_{p-1})^i}{i!} y^i_{p-1} + g_{k,n}(t).$$

It is clear that $x_n(.)$, $u_n(.)$ and $f_n(.)$ satisfy the following relations :

$$x_n(.) \in W^{k,1}(I, E), \quad x_n(\theta_n(t)) = x^n_{p-1} \in D, \quad \forall\, t \in [0, T[,$$

$$x^{(k)}_n(t) = u_n(t) \in F\Big(t, x_n(\theta_{q_n}(t)), x^{(1)}_n(\theta_{q_n}(t)), ..., x^{(k-1)}_n(\theta_{q_n}(t))\Big) + \frac{1}{2^n}\overline{B} \quad a.e. \text{ on } I, \tag{10}$$

and

$$\|u_n(t) - f_n(t)\| \leq \frac{1}{2^n} \quad a.e. \text{ on } I. \tag{11}$$

***Step* 2** *The convergence of* $(x_n(.))$

By construction for all $t \in I$

$$f_n(t) \in F\Big(t, x_n(\theta_{q_n}(t)), x^{(1)}_n(\theta_{q_n}(t)), ..., x^{(k-1)}_n(\theta_{q_n}(t))\Big).$$

On the other hand let $t \in I$ and $p = 1, 2, ..., n$, by relation (9)

$$\|f^n_p(t) - f^n_{p-1}(t)\| \leq \frac{1}{2^{n+1}},$$

then, by induction

$$\|f^n_p(t) - f^n_0(t)\| \leq \frac{p}{2^{n+1}},$$

from which we deduce that

$$\|f_n(t) - f_0(t)\| \leq \frac{n}{2^{n+1}},$$

then

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &\leq \|f_{n+1}(t) - f_0(t)\| + \|f_n(t) - f_0(t)\| \\ &\leq \frac{3(n+1)}{2^{n+2}}. \end{aligned}$$

Let $t \in I$ and $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ with $m > n$

$$\begin{aligned} \|f_m(t) - f_n(t)\| &\leq \|f_m(t) - f_{m-1}(t)\| + \|f_{m-1}(t) - f_{m-2}(t)\| .... \|f_{n+1}(t) - f_n(t)\| \\ &\leq \frac{3}{2}\Big(\frac{m}{2^m} + \frac{m-1}{2^{m-1}} + ... + \frac{n+1}{2^{n+1}}\Big). \end{aligned}$$

Put $v_n = \dfrac{n}{2^n}$, by a classical argument (the d'Alembert's criterion), the numerical series $\sum_{i=0}^{+\infty} v_i$ converges, hence $(S_n) = (\sum_{i=0}^{n} v_i)$ is a Cauchy sequence.
Since

$$\|f_m(t) - f_n(t)\| \leq \frac{3}{2}(S_m - S_n)$$

then $(f_n(.))_{n\geq 1}$ is a Cauchy sequence in $L^1(I, E)$, denotes $f(.)$ its limit.
Thus, by (11), the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $f(.)$ in $L^1(I, E)$, which implies, in view of (10), that the subsequence $(x_n^{(k)}(.))_n$ converges to $f(.)$ in $L^1(I, E)$.

Furthermore, by (10), we get

$$\|x_n^{(k)}(t)\| \leq M + 1,$$

again, by dominated convergence theorem, $(x_n^{(k-1)}(.))_n$ converges strongly in $L^1(I, E)$.
as

$$\|x_n^{(k-1)}(t)\| \leq \|y_0^{k-1}\| + (M + 1)T,$$

By induction, for all $i \in \{1, 2, 3, ..., k - 1\}$, we prove that for all $t \in I$,

$$\|x_n^{(k-i)}(t)\| \leq \sum_{p=1}^{i} \|y_0^{k-p}\| T^{i-p} + (M + 1)T^{i}.$$

Since

$$x_n^{(k-i-1)}(t) = x^{(k-i-1)}(0) + \int_0^t x_n^{(k-i)}(s)ds,$$

then by the dominated convergence theorem, we deduce that for all $i \in \{1, 2, 3, ..., k - 1\}$, the sequence $(x_n^{(i)}(.))_n$ converges strongly in $L^1(I, E)$. We prove easily that for each $i = 1, ..., k - 1$; $\lim_{n\to\infty} x_n^{(i)}(.) = x^{(i)}(.)$ where $x(.) = \lim_{n\to\infty} x_n(.)$ in $L^1(I, E)$.
Recall that

$$|\theta_{q_n}(t) - t| < \frac{T}{q_n}.$$

Since

$$\begin{aligned}
\|x_n^{(k-1)}(\theta_{q_n}(t)) - x^{(k-1)}(t)\| &\leq \|x_n^{(k-1)}(\theta_{q_n}(t)) - x_n^{(k-1)}(t)\| + \|x_n^{(k-1)}(t) - x^{(k-1)}(t)\| \\
&\leq \int_{\theta_{q_n}(t)}^{t} (M + 1)ds + \|x_n^{(k-1)}(t) - x^{(k-1)}(t)\|,
\end{aligned}$$

then $x_n^{(k-1)}(\theta_{q_n}(.))$ converges strongly to $x^{(k-1)}(.)$ in $L^1(I, E)$.

By the same reasoning, for $i \in \{1, ..., k - 2\}$, we have

$$\begin{aligned}
\|x_n^{(i)}(\theta_{q_n}(t)) - x^{(i)}(t)\| &\leq \|x_n^{(i)}(\theta_{q_n}(t)) - x_n^{(i)}(t)\| + \|x_n^{(i)}(t) - x^{(i)}(t)\| \\
&\leq \int_{\theta_{q_n}(t)}^{t} \|x_n^{(i+1)}(s)\|ds + \|x_n^{(i)}(t) - x^{(i)}(t)\|,
\end{aligned}$$

so that the subsequences $(x_n(\theta_{q_n}(.))_n$ and $(x_n^{(i)}(\theta_{q_n}(.))_n$ for $i \in \{1, ..., k - 1\}$, converge strongly to $x(.)$ and $x^{(i)}(.)$ respectively in $L^1(I, E)$.

We are able to finish the proof of the main result. For all $t \in I$

$$x^{(k-1)}(t) = \lim_{n\to\infty} x_n^{(k-1)}(t) = \lim_{n\to\infty}(y_0^{k-1} + \int_0^t u_n(s)ds)$$

Since $(u_n)_{n\in\mathbb{N}}$ converges to $f(.)$ in $L^1(I, E)$, then

$$x^{(k-1)}(t) = y_0^{k-1} + \int_0^t f(s)ds,$$

hence,

$$f(t) = x^{(k)}(t) \ a.e. \ on \ \ I.$$

On the other hand, it is easy to check that $x(0) = x_0$ and $x_0^{(i)}(0) = y_0^i$, $\forall i \in \{1, ..., k-1\}$.

In addition, for every $t \in [0, T[$ we have $x_n(\theta_{q_n}(t)) \in D$. Since D is closed, then $x(t) \in D$. Moreover, as $x(.)$ is $(M + 1)-$Lipschitz then $x(t) \in D, \ \forall \ t \in [0, T]$.

Since $F(t, ., ..., .)$ is upper semi-continuous at $\Big(x(t), x^{(1)}(t), ..., x^{(k-1)}(t)\Big)$, $x_n^{(k)}(\theta_{q_n}(.))$ converges strongly in $L^1(I, E)$ to $x^{(k)}(.)$ and F is closed values in E, then, $x^{(k)}(t) = f(t) \in F\Big(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t)\Big)$ for a.e. $t \in I$. This completes the proof of Theorem 2.1.

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