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Staircase graph words

Sela Fried[a,*], Toufik Mansour[b]

[a] *Department of Computer Science, Israel Academic College, 52275 Ramat Gan, Israel*
[b] *Department of Mathematics, University of Haifa, 3103301 Haifa, Israel*

Abstract. Generalizing the notion of staircase words, introduced by Knopfmacher et al., we define staircase graph words. These are functions w from the vertex set V of a graph into the set $\{1,2,\dots,k\}$, such that $|w(x)-w(y)|\le 1$, for every adjacent $x,y\in V$. We find the explicit generating functions of the number of staircase graph words for the grid graph, the rectangle-triangular graph and the king's graph, all of size $2\times n$.

1. Introduction

Let k and n be two positive integers and let $[k]=\{1,2,\dots,k\}$ be an alphabet. A word over $[k]$ of length n is an element of $[k]^n$. Restricted words are words that do not contain certain subwords. Following the work of Burstein [1], that may be regarded as the first systematic study of restricted words, much research has been devoted to the study of this subject (e.g., [5–9, 11, 12]).

In this work we concentrate on a specific kind of restricted words, that was introduced by Knopfmacher et al. [6], namely *staircase words*. These are words $x=x_1\cdots x_n\in[k]^n$ such that $|x_i-x_{i+1}|\le 1$, for every $1\le i\le n-1$. We propose the following generalization.

Definition 1.1. *Let G be a graph with vertex set V. A (G,k)*-word *is any function $w\colon V\to[k]$. A (G,k)-word w is called* staircase *if $|w(x)-w(y)|\le 1$, for every adjacent $x,y\in V$. The number of staircase (G,k)-words is denoted by $s_k(G)$.*

Example 1.2. *Let P_n be the path graph with n vertices. A staircase word of length n, in the sense of [6], is a staircase (P_n,k)-word, in the sense of Definition 1.1. Knopfmacher et al. have shown in [6, Theorem 2.2] that the generating function of the number of staircase words of length n is given by*

$$1+\frac{x(k-(3k+2)x)}{(1-3x)^2}+\frac{2x^2}{(1-3x)^2}\frac{1+U_{k-1}\left(\frac{1-x}{2x}\right)}{U_k\left(\frac{1-x}{2x}\right)},$$

where $U_k(x)$ is the Chebyshev polynomial of the second kind (of degree k).

Moreover, Knopfmacher et al. [6] have also considered staircase-cyclic words. *These are staircase words $x=x_1\cdots x_n$ that additionally satisfy $|x_1-x_n|\le 1$. In our terminology, these are (C_n,k)-words, where C_n is the cycle graph with n vertices.*

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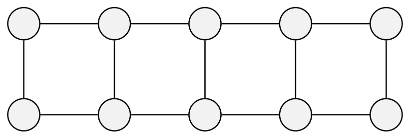
* Corresponding author: Sela Fried
Email addresses: friedsela@gmail.com (Sela Fried), tmansour@univ.haifa.ac.il (Toufik Mansour)

In this work we concentrate on the grid graph, the rectangle-triangular graph, and the king's graph, all of size $2 \times n$, defined as follows (see Figure 1 below for a visualization).

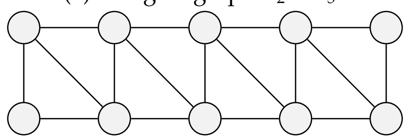
Definition 1.3. *Let* $V = \{(i, j) \ : \ i = 1, 2 \text{ and } 1 \leq j \leq n\}$. *The* grid graph of size $2 \times n$, *denoted by* $P_2 \times P_n$, *is the graph whose vertex set is* V *and two vertices* $(i_1, j_1), (i_2, j_2) \in V$ *are adjacent if* $|i_1 - i_2| + |j_1 - j_2| = 1$. *The* rectangle-triangular graph of size $2 \times n$, *denoted by* $RT_{2,n}$, *is the graph whose vertex set is* V *and two vertices* $(i_1, j_1), (i_2, j_2) \in V$ *are adjacent if (a)* $|i_1 - i_2| + |j_1 - j_2| = 1$, *or (b)* $i_1 = 1, i_2 = 2$ *and* $j_2 = j_1 + 1$, *or (c)* $i_1 = 2, i_2 = 1$ *and* $j_2 = j_1 - 1$. *The* king's graph of size $2 \times n$, *denoted by* $KG_{2,n}$ *(cf. [2, p. 223]), is the graph whose vertex set is* V *and two vertices* $(i_1, j_1), (i_2, j_2) \in V$ *are adjacent if (a)* $|i_1 - i_2| + |j_1 - j_2| = 1$, *or (b)* $|i_1 - i_2| = |j_1 - j_2| = 1$.

Remark 1.4. *(1) Let* G *be one of the graphs defined above with the vertex set* V. *It will be convenient to think of* V *as the index set of a* $2 \times n$ *matrix. In particular,* $(1, 1)$ *corresponds to the position of the upper left entry of the matrix. With this interpretation of* V, *every* (G, k)*-word corresponds to a* $2 \times n$ *matrix, whose entries belong to* $[k]$.

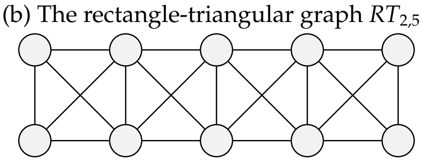
(2) We shall make extensive use of the following refinement of $s_k(G)$: *For* $i, j \in [k]$, *we denote by* $s_k(G, i, j)$ *the number of staircase* (G, k)*-words whose first column is* $(i, j)^T$, *where* v^T *stands for the transpose of the column vector* v.



(a) The grid graph $P_2 \times P_5$.



(b) The rectangle-triangular graph $RT_{2,5}$.



(c) The king's graph $KG_{2,5}$.

Figure 1: Examples of the three graph families considered in this work.

Example 1.5. *Table 1 below shows the numbers of staircase* $(G_n, 3)$*-words for* $n = 1, 2, \ldots, 7$, *where* G_n *is either* $P_2 \times P_n$, $RT_{2,n}$ *or* $KG_{2,n}$. *The last column refers to the On-Line Encyclopedia of Integer Sequences (OEIS) [13].*

n	1	2	3	4	5	6	7	OEIS
$s_3(P_2 \times P_n)$	7	35	181	933	4811	24807	127913	A051926
$s_3(RT_{2,n})$	7	33	161	783	3809	18529	90135	not registered
$s_3(KG_{2,n})$	7	31	145	673	3127	14527	67489	A086901

Table 1: Number of staircase graph-words corresponding to the graphs considered in this work, over an alphabet of size 3.

As an illustration, of the four staircase $(P_2 \times P_2, 3)$*-words depicted in Figure 2, only the two on the left are staircase* $(RT_{2,2}, 3)$*-words and neither is a staircase* $(KG_{2,2}, 3)$*-word.*

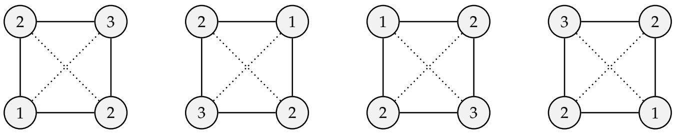


Figure 2: These four staircase $(P_2 \times P_2, 3)$-words illustrate the difference in the numbers appearing in the second column of Table 1.

2. Main results

We apply the kernel method (e.g., [10]) and make extensive use of the mathematics software Maple. The generating functions that we obtain are to be understood as defined in a small enough environment of 0 (e.g., [4, Chapter IV]). The order in which the graphs are studied is according to the complexity of the analysis, beginning with the easiest graph, namely, the king's graph.

2.1. The king's graph

Let $S_k(x) = \sum_{n\geq 1} s_k(KG_{2,n})x^n$ be the generating function of the number of staircase $(KG_{2,n}, k)$-words and, for $i, j \in [k]$, we define $S_k(x, i, j) = \sum_{n\geq 1} s_k(KG_{2,n}, i, j)x^n$ to be the generating function of the number of staircase $(KG_{2,n}, k)$-words whose first column is $(i, j)^T$. We set $S_k(x, i, j) = 0$ if either $i \notin [k]$ or $j \notin [k]$.

Lemma 2.1. 1. *We have*

$$S_k(x) = \sum_{i=1}^{k} S_k(x, i, i) + 2\sum_{i=1}^{k-1} S_k(x, i+1, i). \tag{1}$$

2. *For every $i \in [k]$, the generating function $S_k(x, i, i)$ satisfies*

$$\begin{aligned} S_k(x, i, i) = x &+ xS_k(x, i-1, i-1) + 2xS_k(x, i, i-1) \\ &+ xS_k(x, i, i) + 2xS_k(x, i+1, i) + xS_k(x, i+1, i+1). \end{aligned} \tag{2}$$

3. *For every $i \in [k]$, the generating function $S_k(x, i+1, i)$ satisfies*

$$S_k(x, i+1, i) = x + xS_k(x, i, i) + 2xS_k(x, i+1, i) + xS_k(x, i+1, i+1). \tag{3}$$

Proof. The leftmost column of any staircase $(KG_{2,n}, k)$-word is of the form $(i, j)^T$, for some $i, j \in [k]$ satisfying $|i - j| \leq 1$. Due to symmetry, $s_k(KG_{2,n}, i, j) = s_k(KG_{2,n}, j, i)$. It follows that

$$S_k(x) = \sum_{\substack{i,j\in[k],\\ |i-j|\leq 1}} S_k(x, i, j) = \sum_{i=1}^{k} S_k(x, i, i) + 2\sum_{i=1}^{k-1} S_k(x, i+1, i).$$

Let w be a staircase $(KG_{2,n}, k)$-word whose first column is $(i, i)^T$. If $n = 1$, there is only one such word. Suppose that $n \geq 2$. Then the second column of w must be one of

$$(i-1, i-1)^T, (i, i-1)^T, (i-1, i)^T, (i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T$$

(see Figure 3). Writing this in terms of generating functions, we obtain (2).

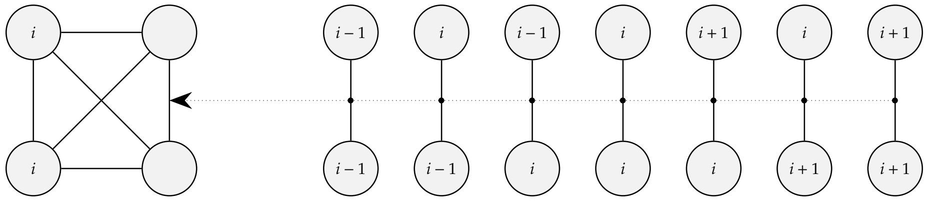


Figure 3: If the leftmost column of a staircase $(KG_{2,n}, k)$-word is $(i,i)^T$, there are at most seven possibilities for the next column.

The proof is identical to the previous one, except that the second column of w must be one of

$$(i,i)^T, (i+1,i)^T, (i,i+1)^T, (i+1,i+1)^T.$$

□

Equation (1) motivates the definition of two additional generating functions (in the variables x and t):

$$A_k(x,t) = \sum_{i=1}^{k} S_k(x,i,i)t^{i-1} \text{ and } B_k(x,t) = \sum_{i=1}^{k-1} S_k(x,i+1,i)t^{i-1}.$$

Lemma 2.2. *We have*

$$A_k(x,1) = \frac{x((2x+1)(A_k(x,0)+S_k(x,k,k)-k)+4x)}{2x^2+5x-1}, \tag{4}$$

$$B_k(x,1) = -\frac{x((x-1)(A_k(x,0)+S_k(x,k,k)-k)+3x-1)}{2x^2+5x-1}. \tag{5}$$

Furthermore,

$$S_k(x) = \frac{x(3(A_k(x,0)+S_k(x,k,k)-k)-2x+2)}{2x^2+5x-1}. \tag{6}$$

Proof. Multiplying (2) and (3) by t^{i-1} and summing over $i \in [k]$ and $i \in [k-1]$, respectively, we obtain

$$A_k(x,t) = \frac{x(1-t^k)}{1-t} + xt(A_k(x,t) - S_k(x,k,k)t^{k-1}) + 2xtB_k(x,t) + xA_k(x,t) + 2xB_k(x,t) + \frac{x}{t}(A_k(x,t) - A_k(x,0)), \tag{7}$$

$$B_k(x,t) = \frac{x(1-t^{k-1})}{1-t} + x\left(A_k(x,t) - S_k(x,k,k)t^{k-1}\right) + 2xB_k(x,t) + \frac{x}{t}\left(A_k(x,t) - A_k(x,0)\right). \tag{8}$$

Taking $\lim_{t\to 1}$ in (7) and (8), we obtain a linear system of two equations in the variables $A_k(x,1)$ and $B_k(x,1)$. Solving this system gives (4) and (5).

To obtain (6), notice that equation (1) may be rewritten as $S_k(x) = A_k(x,1) + 2B_k(x,1)$. The assertion follows now from the previous part. □

Theorem 2.3. *We have*

$$S_k(x) = \frac{x(t_1+2x)(3kt_1+2t_1x-3k-2t_1-2x-4)t_1^k}{(2x^2+5x-1)(t_1^{k+1}+2t_1^kx+2t_1x+1)(1-t_1)} + \frac{(2t_1x+1)(3kt_1+2t_1x-3k+4t_1-2x+2)}{(2x^2+5x-1)(t_1^{k+1}+2t_1^kx+2t_1x+1)(1-t_1)},$$

where $t_1 = \frac{1-3x-2x^2-\sqrt{(1-3x-2x^2)^2-4x^2}}{2x}$.

Proof. First, we solve (8) for $B_k(x,t)$ and substitute it into (7). This gives us the equation

$$\frac{K(t)}{t}A_k(x,t) = \frac{(2tx+1)x}{t}A_k(x,0) + (t+2x)xt^{k-1}S_k(x,k,k) + \frac{2x^2(1-t^{k-1})(t+1)+(x-2x^2)(1-t^k)}{t-1}, \quad (9)$$

where $K(t) = xt^2 + (2x^2+3x-1)t + x$. The two roots of the kernel equation $K(t) = 0$ are given by t_1 and $1/t_1$. By substituting t_1 and $1/t_1$ into (9), we obtain a linear system of two equations in the variables $A_k(x,0)$ and $S_k(x,k,k)$. Solving this system we obtain

$$A_k(x,0) = S_k(x,k,k) = \frac{4\left(t_1^{2k} - t_1^{k+1}(1+t_1) + t_1^3\right)}{T}x^2 + \frac{2\left(2t_1^{2k+1} - t_1^k(1+t_1)(1+t_1^2) + 2t_1^2\right)}{T}x + \frac{t_1^{2k+2} - t_1^{k+1}(1+t_1) + t_1}{T},$$

where $T = (1-t_1)\left(1 - t_1^{2k+2} - 4(t_1^{2k+1} - t_1)x - 4(t_1^{2k} - t_1^2)x^2\right)$. Substituting these in (6), the assertion follows. □

Example 2.4. *The generating function of the number of staircase $(KG_{2,n}, 3)$-words is $x(3x+7)/(1-4x+3x^2)$. Thus,*

$$s_3(KG_{2,n}) = \frac{7+5\sqrt{7}}{14}(2+\sqrt{7})^n + \frac{7-5\sqrt{7}}{14}(2-\sqrt{7})^n.$$

In Table 2 we list the generating functions of the number of staircase $(KG_{2,n}, k)$-words for additional values of k.

k	Generating function
3	$\frac{x(3x+7)}{1-4x+3x^2}$
4	$\frac{2x(5-12x-3x^2)}{1-7x+9x^2+6x^3}$
5	$\frac{x(13-30x-42x^2-6x^3)}{1-7x+6x^2+18x^3+6x^4}$
6	$\frac{2x(8-42x+30x^2+51x^3+6x^4)}{1-10x+27x^2-3x^3-36x^4-12x^5}$

Table 2: The generating functions of the number of staircase $(KG_{2,n}, k)$-words, for $k = 3, 4, 5$, and 6.

2.2. *The grid graph*

Let $S_k(x) = \sum_{n\geq 1} s_k(P_2 \times P_n)x^n$ be the generating function of the number of staircase $(P_2 \times P_n, k)$-words and, for $i, j \in [k]$, we define $S_k(x,i,j) = \sum_{n\geq 1} s_k(P_2 \times P_n, i, j)x^n$ to be the generating function of the number of staircase $(P_2 \times P_n, k)$-words whose first column is $(i,j)^T$. We set $S_k(x,i,j) = 0$ if either $i \notin [k]$ or $j \notin [k]$.

Lemma 2.5. 1. *We have*

$$S_k(x) = \sum_{i=1}^{k} S_k(x,i,i) + 2\sum_{i=1}^{k-1} S_k(x,i+1,i). \quad (10)$$

2. *For every $i \in [k]$, the generating function $S_k(x,i,i)$ satisfies*

$$S_k(x,i,i) = x + xS_k(x,i-1,i-1) + 2xS_k(x,i,i-1) + xS_k(x,i,i) + 2xS_k(x,i+1,i) + xS_k(x,i+1,i+1). \quad (11)$$

3. *For every $i \in [k]$, the generating function $S_k(x,i+1,i)$ satisfies*

$$S_k(x,i+1,i) = x + xS_k(x,i,i-1) + xS_k(x,i,i) + 2xS_k(x,i+1,i) + xS_k(x,i+1,i+1) + xS_k(x,i+2,i+1). \quad (12)$$

Proof. The proof is identical to the proof of Lemma 2.1, with the following exceptions: Let w be a staircase $(P_2 \times P_n, k)$-word and assume that $n \geq 2$. If the first column of w is $(i, i)^T$, the second column of w must be one of

$$(i-1, i-1)^T, (i, i-1)^T, (i-1, i)^T, (i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T.$$

Similarly, if the first column of w is $(i+1, i)^T$, the second column of w must be one of

$$(i, i-1)^T, (i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T, (i+2, i+1)^T.$$

□

We now define two additional generating functions (in the variables x and t):

$$A_k(x,t) = \sum_{i=1}^{k} S_k(x,i,i)t^{i-1} \text{ and } B_k(x,t) = \sum_{i=1}^{k-1} S_k(x,i+1,i)t^{i-1}.$$

Lemma 2.6. *We have*

$$A_k(x,1) = -\frac{x(A_k(x,0) + S_k(x,k,k) - k + 4x(B_k(x,0) + S_k(x,k,k-1) + 1))}{4x^2 - 7x + 1}, \tag{13}$$

$$B_k(x,1) = \frac{x((x-1)(A_k(x,0) + S_k(x,k,k) - k) + (3x-1)(B_k(x,0) + S_k(x,k,k-1) + 1))}{4x^2 - 7x + 1}. \tag{14}$$

Furthermore,

$$S_k(x) = \frac{x((2x-3)(A_k(x,0) + S_k(x,k,k) - k) + (2x-2)(B_k(x,0) + S_k(x,k,k-1) + 1))}{4x^2 - 7x + 1}. \tag{15}$$

Proof. Multiplying (11) and (12) by t^{i-1} and summing over $i \in [k]$ and $i \in [k-1]$, respectively, we obtain

$$\begin{aligned} A_k(x,t) = {} & \frac{x(1-t^k)}{1-t} + xt(A_k(x,t) - S_k(x,k,k)t^{k-1}) + 2xtB_k(x,t) \\ & + xA_k(x,t) + 2xB_k(x,t) + \frac{x}{t}(A_k(x,t) - A_k(x,0)), \end{aligned} \tag{16}$$

$$\begin{aligned} B_k(x,t) = {} & \frac{x(1-t^{k-1})}{1-t} + xt(B_k(x,t) - S_k(x,k,k-1)t^{k-2}) + x(A_k(x,t) - S_k(x,k,k)t^{k-1}) \\ & + 2xB_k(x,t) + \frac{x}{t}(A_k(x,t) + B_k(x,t) - A_k(x,0) - B_k(x,0)). \end{aligned} \tag{17}$$

Taking $\lim_{t \to 1}$ in (16) and (17), we obtain a linear system of two equations in the variables $A_k(x,1)$ and $, B_k(x,1)$. Solving this system gives (13) and (14).

To obtain (15), notice that equation (10) is equivalent to $S_k(x) = A_k(x,1) + 2B_k(x,1)$ and the assertion follows from the previous part. □

Theorem 2.7. *We have*

$$S_k(x) = \frac{a_1(x,t_1,t_2)t_1^k t_2^k + a_2(x,t_1,t_2)t_1^k - a_2(x,t_2,t_1)t_2^k + a_3(x,t_1,t_2)}{(t_1-1)(t_2-1)(4x^2-7x+1)(b_1(x,t_1,t_2)t_1^k t_2^k + b_2(x,t_1,t_2)t_1^k - b_2(x,t_2,t_1)t_2^k + b_3(x,t_1,t_2))},$$

where

$$
\begin{aligned}
a_1(x,t_1,t_2) &= t_1t_2(t_2-t_1)(2k(t_2^2-1)(t_1^2-1)x^3 \\
&\quad + (-3kt_1^2t_2^2+3kt_1^2-2kt_1t_2+3kt_2^2-2t_1^2t_2-2t_1t_2^2+2kt_1+2kt_2-2t_1t_2-5k-2)x^2 \\
&\quad + (3kt_1t_2+2t_1^2t_2+2t_1t_2^2-3kt_1-3kt_2+6t_1t_2+3k+4)x-2t_2t_1), \\
a_2(x,t_1,t_2) &= t_1(1-t_1t_2)(2k(t_1-1)(t_2+1)(t_2-1)(t_1+1)x^3 \\
&\quad + (-3kt_1^2t_2^2-2kt_1t_2^2+3kt_1^2+2kt_1t_2+5kt_2^2+2t_1^2t_2-2kt_2+2t_1t_2+2t_2^2-3k+2t_1)x^2 \\
&\quad + (3kt_1t_2^2-3kt_1t_2-3kt_2^2-2t_1^2t_2+3kt_2-6t_1t_2-4t_2^2-2t_1)x+2t_2t_1), \\
a_3(x,t_1,t_2) &= (t_1-t_2)(2k(t_1^2-1)(t_2^2-1)x^3 \\
&\quad + (-5kt_1^2t_2^2+2kt_1^2t_2+2kt_1t_2^2-2t_1^2t_2^2+3kt_1^2-2kt_1t_2+3kt_2^2-2t_1t_2-3k-2t_1-2t_2)x^2 \\
&\quad + (3kt_1^2t_2^2-3kt_1^2t_2-3kt_1t_2^2+4t_1^2t_2^2+3kt_1t_2+6t_1t_2+2t_1+2t_2)x-2t_1t_2), \\
b_1(x,t_1,t_2) &= t_1t_2(t_1-t_2)((t_1+1)(t_2+1)x-1), \\
b_2(x,t_1,t_2) &= t_1(t_1t_2-1)((t_1+1)(t_2+1)x-t_2), \\
b_3(x,t_1,t_2) &= (t_2-t_1)((t_1+1)(t_2+1)x-t_1t_2),
\end{aligned}
$$

and

$$
t_{1,2} = \frac{2-x\pm\sqrt{x(9x+8)}+\sqrt{\left(2-x\pm\sqrt{x(9x+8)}\right)^2-16x^2}}{4x}.
$$

Proof. First, we solve (17) for $B_k(x,t)$ and substitute it into (16). This gives us the equation

$$
\begin{aligned}
\frac{K(t)}{t}A_k(x,t) &= -\frac{x(xt^2+t-x)}{t}A_k(x,0)-2x^2(t+1)B_k(x,0)+x(xt^2-t-x)t^kS_k(x,k,k) \\
&\quad -2x^2(t+1)t^kS_k(x,k,k-1)+\frac{x(xt^{k+2}-t^{k+1}-xt^k+xt^2+t-x)}{1-t}. \qquad (18)
\end{aligned}
$$

where $K(t)=x^2t^4+x(x-2)t^3+(1-3x)t^2+x(x-2)t+x^2$. The four roots of the kernel equation $K(t)=0$ are given by $t_1, 1/t_1, t_2$, and $1/t_2$. By substituting these four roots into (18), we obtain a linear system of four equations in the variables $A_k(x,0), B_k(x,0), S_k(x,k,k-1)$, and $S_k(x,k,k)$. Solving this system we obtain

$$
\begin{aligned}
A_k(x,0) = S_k(x,k,k) &= \frac{f_1(x,t_1,t_2)t_1^kt_2^k+f_2(x,t_1,t_2)t_1^k-f_2(x,t_2,t_1)t_2^k+f_3(x,t_1,t_2)}{g_1(x,t_1,t_2)t_1^kt_2^k+g_2(x,t_1,t_2)t_1^k-g_2(x,t_2,t_1)t_2^k+g_3(x,t_1,t_2)}, \\
B_k(x,0) = S_k(x,k-1,k) &= \frac{q_1(x,t_1,t_2)t_1^kt_2^k+q_2(x,t_1,t_2)t_1^k-q_2(x,t_2,t_1)t_2^k+q_3(x,t_1,t_2)}{2x\left(g_1(x,t_1,t_2)t_1^kt_2^k+g_2(x,t_1,t_2)t_1^k-g_2(x,t_2,t_1)t_2^k+g_3(x,t_1,t_2)\right)}, \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(x,t_1,t_2) &= t_1t_2(t_1-t_2)(t_1t_2+1), \\
f_2(x,t_1,t_2) &= t_1t_2(t_1+t_2)(1-t_1t_2), \\
f_3(x,t_1,t_2) &= t_1t_2(t_1t_2+1)(t_2-t_1), \\
g_1(x,t_1,t_2) &= t_2t_1(t_1-1)(t_2-1)(t_1-t_2)(t_1t_2x+t_1x+t_2x+x-1), \\
g_2(x,t_1,t_2) &= t_1(t_1-1)(t_2-1)(t_1t_2-1)(t_1t_2x+t_1x+t_2x-t_2+x), \\
g_3(x,t_1,t_2) &= (t_1-1)(t_2-1)(t_2-t_1)(t_1t_2x-t_1t_2+t_1x+t_2x+x), \\
q_1(x,t_1,t_2) &= t_1t_2(t_2-t_1)(t_2^2x-t_2-x)(t_1^2x-t_1-x), \\
q_2(x,t_1,t_2) &= t_1(1-t_1t_2)(t_2^2x+t_2-x)(t_1^2x-t_1-x), \\
q_3(x,t_1,t_2) &= (t_1-t_2)(t_2^2x+t_2-x)(t_1^2x+t_1-x).
\end{aligned}
$$

Substituting these in (15), the assertion follows. □

Example 2.8. *The generating function of the number of staircase $(P_2 \times P_n, 3)$-words is*

$$\frac{x(7-x^2)}{1-5x-x^2+x^3}.$$

Thus,

$$s_3(P_2 \times P_n) = \frac{(7-a_1^2)a_2a_3}{(a_1-a_2)(a_1-a_3)a_1^{n-1}} + \frac{(7-a_2^2)a_1a_3}{(a_2-a_1)(a_2-a_3)a_2^{n-1}} + \frac{(7-a_3^2)a_1a_2}{(a_3-a_1)(a_3-a_2)a_3^{n-1}},$$

where

$$a_j = \frac{1}{3} + \frac{8}{3}\cos\left(\frac{1}{3}\arctan\left(\frac{3\sqrt{111}}{5}\right) - \frac{2(j-2)\pi}{3}\right),\ j = 1,2,3.$$

In Table 3 we list the generating functions of the number of staircase $(P_2 \times P_n, k)$-words for additional values of k.

k	Generating function
3	$\frac{x(7-x^2)}{1-5x-x^2+x^3}$
4	$\frac{2x(5-8x-3x^2+2x^3)}{1-7x+7x^2+4x^3-2x^4}$
5	$\frac{x(13-31x-31x^2+12x^3-4x^4)}{1-8x+10x^2+15x^3-4x^4-2x^5}$
6	$\frac{2x(8-34x+4x^2+29x^3-6x^4-3x^5)}{1-10x+24x^2+3x^3-21x^4-3x^5+2x^6}$

Table 3: The generating functions of the number of staircase $(P_2 \times P_n, k)$-words, for $k = 3, 4, 5$, and 6.

2.3. The rectangle-triangular graph

Let $S_k(x) = \sum_{n\geq 1} s_k(RT_{2,n})x^n$ be the generating function of the number of staircase $(RT_{2,n}, k)$-words and, for $i, j \in [k]$, we define $S_k(x, i, j) = \sum_{n\geq 1} s_k(RT_{2,n}, i, j)x^n$ to be the generating function of the number of staircase $(RT_{2,n}, k)$-words whose first column is $(i, j)^T$. We set $S_k(x, i, j) = 0$ if either $i \notin [k]$ or $j \notin [k]$.

Lemma 2.9. 1. *We have*

$$S_k(x) = \sum_{i=1}^{k} S_k(x,i,i) + \sum_{i=1}^{k-1} S_k(x,i+1,i) + \sum_{i=1}^{k-1} S_k(x,i,i+1). \tag{20}$$

2. *The generating function $S_k(x, i, i)$ satisfies*

$$\begin{aligned} S_k(x,i,i) = {} & x + xS_k(x,i-1,i-1) + xS_k(x,i,i-1) + xS_k(x,i-1,i) + xS_k(x,i,i) \\ & + xS_k(x,i+1,i) + xS_k(x,i,i+1) + xS_k(x,i+1,i+1). \end{aligned} \tag{21}$$

3. *The generating function $S_k(x, i+1, i)$ satisfies*

$$\begin{aligned} S_k(x,i+1,i) = {} & x + xS_k(x,i,i) + xS_k(x,i+1,i) + xS_k(x,i,i+1) \\ & + xS_k(x,i+1,i+1) + xS_k(x,i+2,i+1). \end{aligned} \tag{22}$$

4. *The generating function $S_k(x, i, i+1)$ satisfies*

$$\begin{aligned} S_k(x,i,i+1) = {} & x + xS_k(x,i-1,i) + xS_k(x,i,i) + xS_k(x,i+1,i) \\ & + xS_k(x,i,i+1) + xS_k(x,i+1,i+1). \end{aligned} \tag{23}$$

Proof. The proof is identical to the proof of Lemma 2.1, with the following exceptions: First, symmetry does not hold and we cannot claim that if $i, j \in [k]$, where $i \neq j$, then $s_k(RT_{2,n}, i, j) = s_k(RT_{2,n}, j, i)$. Now, let w be a staircase $(RT_{2,n}, k)$-word and assume that $n \geq 2$. If the first column of w is $(i, i)^T$, the second column of w must be one of

$$(i-1, i-1)^T, (i, i-1)^T, (i-1, i)^T, (i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T.$$

Similarly, if the first column of w is $(i+1, i)^T$, the second column of w must be one of

$$(i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T, (i+2, i+1)^T.$$

Finally, if the first column of w is $(i, i+1)^T$, the second column of w must be one of

$$(i-1, i)^T, (i, i)^T, (i+1, i)^T, (i, i+1)^T, (i+1, i+1)^T.$$

□

We now define three additional generating functions (in the variables x and t):

$$A_k(x,t) = \sum_{i=1}^{k} S_k(x, i, i)t^{i-1}, \qquad B_k(x,t) = \sum_{i=1}^{k-1} S_k(x, i+1, i)t^{i-1}, \qquad C_k(x,t) = \sum_{i=1}^{k-1} S_k(x, i, i+1)t^{i-1}.$$

Lemma 2.10. *We have*

$$A_k(x,1) = -\frac{x\left((x+1)(A_k(x,0) + S_k(x,k,k) - k) + 2x(B_k(x,0) + S_k(x,k-1,k) + 2)\right)}{x^2 - 6x + 1}, \tag{24}$$

$$B_k(x,1) = \frac{x\left((x-1)^2(A_k(x,0) + S_k(x,k,k) - k) + (2x^2 - 5x + 1)B_k(x,0)\right)}{x^3 - 7x^2 + 7x - 1} + \frac{x\left(x(x+1)S_k(x,k-1,k) + 3x^2 - 4x + 1\right)}{x^3 - 7x^2 + 7x - 1}, \tag{25}$$

$$C_k(x,1) = \frac{x\left((x-1)^2(A_k(x,0) + S_k(x,k,k) - k) + (2x^2 - 5x + 1)S_k(x,k-1,k)\right)}{x^3 - 7x^2 + 7x - 1} + \frac{x\left(x(x+1)B_k(x,0) + 3x^2 - 4x + 1\right)}{x^3 - 7x^2 + 7x - 1}. \tag{26}$$

Furthermore,

$$S_k(x) = \frac{x\left((x-3)(A_k(x,0) + S_k(x,k,k) - k) + (x-1)(B_k(x,0) + S_k(x,k-1,k) + 2)\right)}{x^2 - 6x + 1}. \tag{27}$$

Proof. Multiplying (21), (22), and (23) by t^{i-1} and summing over $i \in [k]$, $i \in [k-1]$, and $i \in [k-1]$, respectively, we obtain

$$A_k(x,t) = \frac{x(1-t^k)}{1-t} + xt\left(A_k(x,t) - S_k(x,k,k)t^{k-1}\right) + xtB_k(x,t) + xtC_k(x,t) + xA_k(x,t) + xB_k(x,t) + xC_k(x,t) + \frac{x}{t}\left(A_k(x,t) - A_k(x,0)\right), \tag{28}$$

$$B_k(x,t) = \frac{x(1-t^{k-1})}{1-t} + x\left(A_k(x,t) - S_k(x,k,k)t^{k-1}\right) + xB_k(x,t) + xC_k(x,t) + \frac{x}{t}\left(A_k(x,t) - A_k(x,0)\right) + \frac{x}{t}\left(B_k(x,t) - B_k(x,0)\right), \tag{29}$$

$$C_k(x,t) = \frac{x(1-t^{k-1})}{1-t} + xt\left(C_k(x,t) - S_k(x,k-1,k)t^{k-2}\right) + x\left(A_k(x,t) - S_k(x,k,k)t^{k-1}\right) + xB_k(x,t) + xC_k(x,t) + \frac{x}{t}\left(A_k(x,t) - A_k(x,0)\right). \tag{30}$$

Taking $\lim_{t\to 1}$ in (28), (29), and (30), we obtain a linear system of three equations in the variables $A_k(x,1), B_k(x,1)$, and $C_k(x,1)$. Solving this system gives (24), (25), and (26).

To obtain (27), notice that equation (20) is equivalent to $S_k(x) = A_k(x,1) + B_k(x,1) + C_k(x,1)$ and the assertion follows from the previous part. □

Theorem 2.11. *We have*

$$S_k(x) = \frac{a_1(x,t_1,t_2)t_1^kt_2^k + a_2(x,t_1,t_2)t_1^k - a_2(x,t_2,t_1)t_2^k + a_3(x,t_1,t_2)}{(t_1-1)(t_2-1)(x^2-6x+1)(b_1(x,t_1,t_2)t_1^kt_2^k - b_2(x,t_1,t_2)t_1^k + b_2(x,t_2,t_1)t_2^k - b_3(x,t_1,t_2))},$$

where

$$\begin{aligned}
a_1(x,t_1,t_2) &= (t_2-t_1)(((t_2-1)(t_1-1)k - 2t_2t_1)x^5\\
&\quad + ((t_1-1)(1-t_2)(t_1^2t_2 + t_1t_2^2 + t_1^2 + t_1t_2 + t_2^2 + t_1 + t_2 + 3)k + 2t_1^3t_2^2 + 2t_1^2t_2^3 + 2t_2t_1 + 4t_1 + 4t_2)x^4\\
&\quad + ((t_1-1)(t_2-1)(t_1^2t_2^2 + 4t_1^2t_2 + 4t_1t_2^2 + 3t_1^2 + 5t_1t_2 + 3t_2^2 + 4t_1 + 4t_2)k - 2t_2t_1(t_1^2t_2 + t_1t_2^2 + 3t_1t_2 + 1))x^3\\
&\quad + ((t_1-1)(1-t_2)(3t_1^2t_2^2 + 3t_1^2t_2 + 3t_1t_2^2 + 7t_1t_2 + 3t_1 + 3t_2)k - 2t_1^3t_2^2 - 2t_1^2t_2^3 + 2t_1^2t_2^2 - 2t_2t_1 - 4t_1 - 4t_2)x^2\\
&\quad + (3t_2t_1(t_2-1)(t_1-1)k + 2t_2t_1(t_1^2t_2 + t_1t_2^2 + 3t_1t_2 + 2))x - 2t_1^2t_2^2),\\
a_2(x,t_1,t_2) &= (t_1t_2-1)((-t_2^2(t_2-1)(t_1-1)k - 2t_2^2t_1)x^5\\
&\quad + ((t_1-1)(t_2-1)(t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + t_1t_2 + 3t_2^2 + t_1 + t_2 + 1)k + 2t_1^3t_2 + 4t_1t_2^3 + 2t_2^2t_1 + 2t_1^2 + 4t_2^2)x^4\\
&\quad + ((t_1-1)(1-t_2)(3t_1^2t_2^2 + 4t_1^2t_2 + 4t_1t_2^2 + t_1^2 + 5t_1t_2 + 4t_1 + 4t_2 + 3)k - 2t_1(t_1^2t_2 + 3t_1t_2 + t_2^2 + t_1))x^3\\
&\quad + ((t_1-1)(t_2-1)(3t_1^2t_2 + 3t_1t_2^2 + 3t_1^2 + 7t_1t_2 + 3t_1 + 3t_2)k - 2t_1^3t_2 - 4t_1t_2^3 + 2t_1^2t_2 - 2t_2^2t_1 - 2t_1^2 - 4t_2^2)x^2\\
&\quad + (3t_1t_2(t_1-1)(1-t_2)k + 2t_1(t_1^2t_2 + 3t_1t_2 + 2t_2^2 + t_1))x - 2t_1^2t_2),\\
a_3(x,t_1,t_2) &= (t_1-t_2)((t_1^2t_2^2(t_1-1)(t_2-1)k - 2t_1^2t_2^2)x^5\\
&\quad + ((t_1-1)(1-t_2)(3t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + t_1^2 + t_1t_2 + t_2^2 + t_1 + t_2)k + 4t_1^3t_2^2 + 4t_1^2t_2^3 + 2t_1^2t_2^2 + 2t_1 + 2t_2)x^4\\
&\quad + ((t_1-1)(t_2-1)(4t_1^2t_2 + 4t_1t_2^2 + 3t_1^2 + 5t_1t_2 + 3t_2^2 + 4t_1 + 4t_2 + 1)k - 2t_1^2t_2^2 - 6t_1t_2 - 2t_1 - 2t_2)x^3\\
&\quad + ((t_1-1)(1-t_2)(3t_1^2t_2 + 3t_1t_2^2 + 7t_1t_2 + 3t_1 + 3t_2 + 3)k - 4t_1^3t_2^2 - 4t_1^2t_2^3 - 2t_1^2t_2^2 + 2t_1t_2 - 2t_1 - 2t_2)x^2\\
&\quad + (3t_1t_2(t_1-1)(t_2-1)k + 4t_1^2t_2^2 + 6t_1t_2 + 2t_1 + 2t_2)x - 2t_1t_2),\\
b_1(x,t_1,t_2) &= (t_1-t_2)((-t_1^2t_2 - t_1t_2^2 - t_1^2 - t_1t_2 - t_2^2 - t_1 - t_2)x^2 + (t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + 2t_1t_2 + t_1 + t_2)x - t_2t_1),\\
b_2(x,t_1,t_2) &= (t_1t_2-1)((t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + t_1t_2 + t_1 + t_2 + 1)x^2 + (-t_1^2t_2 - t_1t_2^2 - t_1^2 - 2t_1t_2 - t_1 - t_2)x + t_2t_1),\\
b_3(x,t_1,t_2) &= (t_1-t_2)((-t_1^2t_2 - t_1t_2^2 - t_1^2 - t_1t_2 - t_2^2 - t_1 - t_2)x^2 + (t_1^2t_2 + t_1t_2^2 + 2t_1t_2 + t_1 + t_2 + 1)x - t_2t_1),
\end{aligned}$$

and

$$t_{1,2} = \frac{x(1-x) - x(1+x)\sqrt{x}}{2x^2} + \frac{\sqrt{x(1+x)(1-x)^2 \pm 2x(x^2-1)\sqrt{x}}}{2x\sqrt{x}}.$$

Proof. First, we solve (30) for $C_k(x,t)$ and substitute it into (29). Then we solve the result for $B_k(x,t)$ and substitute it into (28). This gives us the equation

$$\begin{aligned}
\frac{K(t)}{t}A_k(x,t) &= \frac{x(x^2t^3 - xt^2 - t + x)}{t}A_k(x,0) + x^2(t+1)(xt-1)B_k(x,0)\\
&\quad - x^2t^{k-1}(t+1)(t-x)S_k(x,k-1,k) + xt^{k-1}(t^3x - t^2 - xt + x^2)S_k(x,k,k)\\
&\quad + \frac{x(-xt^{k+2} + t^{k+1} + xt^k - x^2t^{k-1} + x^2t^3 - xt^2 - t + x)}{t-1},
\end{aligned} \tag{31}$$

where $K(t) = x^2t^4 + 2x(x-1)t^3 + (1 - 3x + x^2 - x^3)t^2 + 2x(x-1)t + x^2$. The four roots of the kernel equation $K(t) = 0$ are given by $t_1, 1/t_1, t_2$, and $1/t_2$. By substituting these four roots into (31), we obtain a linear system of four equations in the variables $A_k(x,0), B_k(x,0), S_k(x,k-1,k)$, and $S_k(x,k,k)$. Solving this system we obtain

$$\begin{aligned}
&A_k(x,0) = S_k(x,k,k) = \\
&\frac{(x-1)(x+1)\left(f_1(x,t_1,t_2)t_1^kt_2^k + f_2(x,t_1,t_2)t_1^k - f_2(x,t_2,t_1)t_2^k + f_3(x,t_1,t_2)\right)}{(t_1-1)(t_2-1)\left(g_1(x,t_1,t_2)t_1^kt_2^k - g_2(x,t_1,t_2)t_1^k + g_2(x,t_2,t_1)t_2^k - g_3(x,t_1,t_2)\right)}, \\
&B_k(x,0) = S_k(x,k-1,k) = \\
&\frac{q_1(x,t_1,t_2)t_1^kt_2^k + q_2(x,t_1,t_2)t_1^k - q_2(x,t_2,t_1)t_2^k + q_3(x,t_1,t_2)}{x(t_1-1)(t_2-1)\left(g_1(x,t_1,t_2)t_1^kt_2^k - g_2(x,t_1,t_2)t_1^k + g_2(x,t_2,t_1)t_2^k - g_3(x,t_1,t_2)\right)},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
f_1(x,t_1,t_2) &= (t_2-t_1)((-t_1-t_2)x + t_2t_1(t_1t_2+1)), \\
f_2(x,t_1,t_2) &= t_2(t_1t_2-1)(-t_2(t_1t_2+1)x + t_1(t_1+t_2)), \\
f_3(x,t_1,t_2) &= t_2t_1(t_2-t_1)(t_1t_2(t_1+t_2)x - t_1t_2 - 1), \\
g_1(x,t_1,t_2) &= (t_1-t_2)((-t_1^2t_2 - t_1t_2^2 - t_1^2 - t_1t_2 - t_2^2 - t_1 - t_2)x^2 + (t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + 2t_1t_2 + t_1 + t_2)x - t_1t_2), \\
g_2(x,t_1,t_2) &= ((t_1t_2-1)((t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + t_1t_2 + t_1 + t_2 + 1)x^2 + (-t_1^2t_2 - t_1t_2^2 - t_1^2 - 2t_1t_2 - t_1 - t_2)x + t_1t_2), \\
g_3(x,t_1,t_2) &= (t_1-t_2)((t_1^2t_2^2 + t_1^2t_2 + t_1t_2^2 + t_1t_2 + t_1 + t_2 + 1)x^2 + (-t_1^2t_2 - t_1t_2^2 - t_1^2 - 2t_1t_2 - t_1 - t_2)x + t_1t_2), \\
q_1(x,t_1,t_2) &= (t_1-t_2)(x^2 + (t_2^3 - t_2)x - t_2^2)(x^2 + (t_1^3 - t_1)x - t_1^2), \\
q_2(x,t_1,t_2) &= (t_1t_2-1)(t_2^3x^2 + (1-t_2^2)x - t_2)(x^2 + (t_1^3 - t_1)x - t_1^2), \\
q_3(x,t_1,t_2) &= (t_1-t_2)(t_2^3x^2 + (1-t_2^2)x - t_2)(t_1^3x^2 + (1-t_1^2)x - t_1).
\end{aligned}$$

Substituting these in (27), the assertion follows. □

Example 2.12. *The generating function of the number of staircase $(RT_{2,n}, 3)$-words is*

$$\frac{x(7+5x+x^2)}{1-4x-4x^2-x^3}.$$

Thus, with $i^2 = -1$ and $\alpha = (172 + 12\sqrt{177})^{1/3}$, we have

$$s_3(RT_{2,n}) = \frac{(a_1^2+5a_1+7)a_2a_3}{(a_1-a_2)(a_1-a_3)a_1^{n-1}} + \frac{(a_2^2+5a_2+7)a_1a_3}{(a_2-a_1)(a_2-a_3)a_2^{n-1}} + \frac{(a_3^2+5a_3+7)a_1a_2}{(a_3-a_1)(a_3-a_2)a_3^{n-1}},$$

where

$$a_1 = \frac{(\alpha-4)^2}{6\alpha}, \quad a_2 = \frac{(\sqrt{3}i-1)\alpha^2 - 16(\sqrt{3}i+\alpha) - 16}{12\alpha}, \quad a_3 = \frac{-(\sqrt{3}i+1)\alpha^2 + 16(\sqrt{3}i-\alpha) - 16}{12\alpha}.$$

In Table 4 we list the generating functions of the number of staircase $(RT_{2,n}, k)$-words for additional values of k.

k	Generating function
3	$\frac{x(7+5x+x^2)}{1-4x-4x^2-x^3}$
4	$\frac{2x(5-10x-x^3)}{1-7x+9x^2+x^3+x^4}$
5	$\frac{x(13-24x-45x^2-9x^3+x^4-3x^5)}{1-7x+5x^2+18x^3+6x^4+x^6}$
6	$\frac{2x(8-38x+25x^2+20x^3-2x^4-3x^5+2x^6)}{1-10x+27x^2-10x^3-15x^4-x^5+2x^6-x^7}$

Table 4: The generating functions of the number of staircase $(RT_{2,n}, k)$-words, for $k = 3, 4, 5$, and 6.

Remark 2.13. *An alternative approach is the Transfer-matrix Method (e.g., [14, Section 4.7]), that we demonstrate now on the graph $P_2 \times P_n$. Let*

$$\mathcal{S} = \{ij \in [k]^2 \ : \ |i-j| \leq 1\},$$

ordered in some manner, say, lexicographically. Thus,

$$\mathcal{S} = \{11, 12, 21, 22, 23, \ldots, (k-1)k, kk\}.$$

Set $N = |\mathcal{S}|$. We construct an undirected graph G whose vertex set is V. Two vertices $i_1 j_1$ and $i_2 j_2$ of G are adjacent if $|i_1 - i_2| \leq 1$ and $|j_1 - j_2| \leq 1$. Let A be the adjacency matrix of G. For example, if $k = 3$ then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We define $F_{ij}(x) = \sum_{n\geq 0}(A^n)_{ij}x^n$. Clearly, $s_3(P_2 \times P_n) = \sum_{1\leq i,j\leq N}(A^{n-1})_{ij}$. By [14, Theorem 4.7.2], we have

$$F_{ij}(x) = \frac{(-1)^{i+j}\det(I - xA \ : \ i, j)}{\det(I - xA)},$$

where $(I - xA \ : \ i, j)$ denotes the matrix obtained by removing the jth row and ith column of $I - xA$.

Thus, the generating function of $s_3(P_2 \times P_n)$ is given by

$$\begin{aligned}
\sum_{n\geq 1} s_3(P_2 \times P_n)x^n &= x \sum_{1\leq i,j\leq N} \sum_{n\geq 1}(A^{n-1})_{ij}x^{n-1} \\
&= x \sum_{1\leq i,j\leq N} F_{ij}(x) \\
&= \frac{x\sum_{1\leq i,j\leq N}(-1)^{i+j}\det(I - xA \ : \ i, j)}{\det(I - xA)} \\
&= \frac{-x^6 - 2x^5 + 9x^4 + 16x^3 - 15x^2 - 14x + 7}{x^7 + x^6 - 9x^5 - 9x^4 + 15x^3 + 7x^2 - 7x + 1} \\
&= \frac{x(7 - x^2)}{1 - 5x - x^2 + x^3}.
\end{aligned}$$

This may be done for every k and for each of the three graph families that we study in this work. It follows that all the generating functions in this work are rational (see [3] for a possible extension of this approach).

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