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# Sufficient conditions for $k$-connected graphs and $k$-leaf-connected graphs

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**Abstract.** A connected graph $G$ is said to be $k$-connected if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are deleted. In this paper, we present a sufficient condition in terms of the number of $r$-cliques to guarantee the a graph with minimum degree at least $\delta$ to be $k$-connected, which extends the result of Feng et al. [Linear Algebra Appl. 524 (2017) 182–198]. For any integer $k \geq 2$, a graph $G$ is called $k$-leaf-connected, if $|V(G)| \geq k+1$ and given any subset $S \subseteq V(G)$ with $|S| = k$, $G$ always has a spanning tree $T$ such that $S$ is precisely the set of leaves of $T$. The forgotten index of a graph is the sum of degree cube of all the vertices in graph. Motivated by the degree sequence condition of Gurgel and Wakabayashi [J. Combin. Theory Ser. B 41 (1986) 1–16], we provide a sufficient condition for a connected graph to be $k$-leaf-connected in terms of the forgotten index of $G$, which improve and extend the result of Su et al. [Australas. J. Combin. 77 (2020) 269–284].

## 1. Introduction

Throughout this paper we only consider simple, undirected and connected graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = e(G)$. The degree of vertex $v$ in $G$, denoted by $d_G(v)$, is the number of edges of $G$ containing $v$. The number of cliques of size $r$ in $G$ is denoted by $N_r(G)$. Let $K_n$ and $R(n,t)$ denote a complete graph of order $n$ and a $t$-regular graph with $n$ vertices, respectively. Let $G_1$ and $G_2$ be two vertex-disjoint graphs. We use $G_1 + G_2$ to denote the disjoint union of $G_1$ and $G_2$. The join $G_1 \vee G_2$ is the graph obtained from $G_1 + G_2$ by adding all possible edges between them.

Let $G$ be a graph of order $n$, $P$ a property defined on $G$, and $l$ a positive integer. A property $P$ is said to be $l$-stable, if whenever $G+uv$ has the property $P$ and $d_G(u)+d_G(v) \geq l$, then $G$ itself has the property $P$. The $l$-closure $C_l(G)$ [3, 18] of a graph $G$ is the graph obtained from $G$ by successively joining pairs of nonadjacent vertices whose degree sum is at least $l$ until no such pair exists. Then we have

$$d_{C_l(G)}(u) + d_{C_l(G)}(v) \leq l-1$$

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for every pair of nonadjacent vertices $u$ and $v$ of $C_l(G)$.

Füredi et al. [9] considered the number of cliques in $k$-hamiltonian graphs. Moreover, they listed some classes of graphs whose cliques number condition can be studied, such as $G$ contains $C_k$ ($l = 2n - k$), $G$ contains a path $P_k$ ($l = n - 1$), $G$ contains a matching $kK_2$ ($l = 2k - 1$), $G$ contains a $k$-factor ($l = n + 2k - 4$), $G$ is $k$-connected ($l = n + k - 2$), $G$ is $k$-wise hamiltonian (i.e., every $n - k$ vertices span a $C_{n-k}$) ($l = n + k - 2$). The corresponding question for the property containing long cycles (or Hamiltonian cycle) is well-studied [8, 14, 16]. Subsequently, Duan et al. [4] studied the $G$ contains a matching $kK_2$ according to the number of cliques of size $r$ in $G$.

A connected graph $G$ is said to be $k$-connected if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are deleted. Feng et al. [7] proved sufficient conditions based upon the size and spectral radius for a graph to be $k$-connected. Zhou et al. [23] further proposed some sufficient conditions for a graph to be $k$-connected in terms of signless Laplacian spectral radius, distance spectral radius and distance signless Laplacian spectral radius of $G$. Bondy and Chvátal [3] presented a closure theorem to guarantee that a graph to be $k$-connected.

**Theorem 1.1 (Bondy and Chvátal [3]).** *Let $G$ be an graph of order $n$, and let $1 \le k \le n-2$ be an integer. Then $G$ is $k$-connected if and only if $C_{n+k-2}(G)$ is $k$-connected.*

Inspired by the works of [4, 9], and using Pósa property, we prove a sufficient condition in terms of the number of $r$-cliques to guarantee a graph with minimum degree at least $\delta$ to be $k$-connected.

Let $n, r, k$ and $q$ be integers. Define

$$\theta(n, r, k, q) = \binom{n-q+k-2}{r} + (q-k+2)\binom{q}{r-1}.$$

**Theorem 1.2.** *Let $n, r, k$ and $\delta$ be integers with $r \ge 2$ and $1 \le k \le n-2$. Suppose that $G$ is a graph of order $n$ with minimum degree at least $\delta$ and $k \le \delta \le \left\lfloor \frac{n+k-3}{2} \right\rfloor$. If*

$$N_r(G) > \max\left\{\theta(n, r, k, \delta+1), \theta\left(n, r, k, \left\lfloor \frac{n+k-3}{2} \right\rfloor\right)\right\},$$

*then $G$ is $k$-connected unless $C_{n+k-2}(G) \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$.*

By maple, $\theta(n, 2, k, \delta+1) \ge \theta\left(n, 2, k, \left\lfloor \frac{n+k-3}{2} \right\rfloor\right)$ for $n \ge 7\delta + 10k - 1$. The following corollary results from putting $r = 2$ in Theorem 1.2.

**Corollary 1.1.** *Let $G$ be a graph of order $n \ge 7\delta + 10k - 1$ with minimum degree at least $\delta$ and $k \le \delta \le \left\lfloor \frac{n+k-3}{2} \right\rfloor$. If*

$$e(G) > \theta(n, 2, k, \delta+1),$$

*then $G$ is $k$-connected unless $C_{n+k-2}(G) \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$.*

Feng et al. [7] presented a sufficient condition in terms of $e(G)$ for the graph to be $k$-connected.

**Theorem 1.3 (Feng et al. [7]).** *Let $G$ be a graph of order $n \ge k+1$. If $e(G) \ge \binom{n-1}{2} + k - 1$, then $G$ is $k$-connected unless $G \cong K_{k-1} \vee (K_{n-k} + K_1)$.*

It is easy to verify that $\binom{n-1}{2} + k - 1 \ge \theta(n, 2, k, \delta+1)$ for $n \ge \frac{1}{2}(3\delta - k + 9)$. Hence our result improves Theorem 1.3 for $n \ge 7\delta + 10k - 1$.

**Theorem 1.4.** *Let $G$ be a graph of order $n \ge 7\delta + 10k - 1$ with minimum degree at least $\delta$ and $k \le \delta \le \left\lfloor \frac{n+k-3}{2} \right\rfloor$. If*

$$\rho(G) > \frac{\delta-1}{2} + \sqrt{n^2 - (3\delta - 2k + 7)n + \frac{13}{4}\delta^2 - \left(4k - \frac{31}{2}\right)\delta + k^2 - 9k + \frac{73}{4}},$$

*then $G$ is $k$-connected unless $C_{n+k-2}(G) \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$.*

For any integer $k \geq 2$, a graph $G$ is called $k$-leaf-connected if $|V(G)| \geq k + 1$ and given any subset $S \subseteq V(G)$ with $|S| = k$, $G$ always has a spanning tree $T$ such that $S$ is precisely the set of leaves of $T$. Note that a graph is 2-leaf-connected if and only if it is Hamilton-connected. Therefore, as a generalization of Hamilton-connectedness, the $k$-leaf-connectedness of a graph $G$ is an $\mathcal{NP}$-hard problem.

Up to now, there have been lots of research works to seek the sufficient conditions for a graph to be $k$-leaf-connected. Gurgel and Wakabayashi [11] presented that if $G$ is a $k$-leaf-connected graph, then $G$ is $(k + 1)$-connected. Hence $\delta \geq k + 1$ is a trivial necessary condition for a graph to be $k$-leaf-connected. In the same paper, they also proposed sufficient conditions based on the minimum degree, the degree sum and the size to assure a graph to be $k$-leaf-connected, respectively. Egawa et al. [5] improved the degree sum condition of Gurgel and Wakabayashi [11]. Maezawa et al. [13] provided a Fan-type condition for a graph to be $k$-leaf-connected. Ao et al. [1] presented a new sufficient condition based on the size for a graph to be $k$-leaf-connected. Subsequently, Wu et al. [21] proved a sufficient condition for a graph to be $k$-leaf-connected in terms of the number of $r$-cliques, which generalized the result of Ao et al. [1]. For a graph to be $k$-leaf-connected, one can refer to [2, 15, 20].

The forgotten index [6, 10] of a graph $G$ is defined as

$$F(G) = \sum_{u \in V(G)} d(u)^3 = \sum_{uv \in E(G)} \left(d(u)^2 + d(v)^2\right).$$

Su, Li and Shi [19] presented a sufficient condition for a graph to be Hamilton-connected in terms of the forgotten index of $G$.

**Theorem 1.5 (Su, Li and Shi [19]).** *Let $G$ be a connected graph of order $n \geq 3$. If*

$$F(G) > n^4 - 7n^3 + 24n^2 - 38n + 30,$$

*then $G$ is Hamilton-connected unless $G \cong K_3 \vee 3K_1$.*

Using the forgotten index $F(G)$, we provide a sufficient condition for a graph to be $k$-leaf-connected graphs, which extends and improves the above result.

**Theorem 1.6.** *Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq k + 1$, where $2 \leq k \leq n - 3$. If*

$$F(G) \geq n^4 - 11n^3 + (6k + 51)n^2 - (24k + 105)n + 2k^3 + 6k^2 + 32k + 82,$$

*then $G$ is $k$-leaf-connected unless $G \in \{K_3 \vee (K_{n-5} + 2K_1), K_6 \vee 6K_1, K_5 \vee 5K_1, K_4 \vee (K_{1,4} + K_1), K_4 \vee (K_2 + 3K_1), K_4 \vee 4K_1, K_3 \vee (K_{1,3} + K_1)\}$.*

## 2. Proof of Theorems 1.2 and 1.4

Let $G$ be a graph on $n$ vertices. If there are at least $s$ vertices in $V(G)$ with degree at most $q$, then we say $G$ has $(s, q)$-*Pósa property*.

**Lemma 2.1 (Xue, Liu and Kang [22]).** *Let property $P$ is $l$-stable and the complete graph $K_n$ has the property $P$. Suppose that $G$ is a graph of order $n$ with minimum degree at least $\delta$. If $G$ does not have property $P$, then there exists an integer $q$ with $\delta \leq q \leq \left\lfloor \frac{l-1}{2} \right\rfloor$ such that $G$ has $(n - l + q, q)$-Pósa property.*

**Fact 2.1 (Füredi, Kostochka and Luo[9]).** If $G$ has $(s, q)$-Pósa property and $n \geq s + q$, then

$$N_r(G) \leq \binom{n-s}{r} + s\binom{q}{r-1}.$$

**Lemma 2.2 (Duan et al. [4]).** *Suppose that $G$ has $n$ vertices and is stable under taking $l$-closure. Let $q$ be the maximum integer such that $G$ has $(n - l + q, q)$-Pósa property and $q \leq \left\lfloor \frac{l-1}{2} \right\rfloor$. If $U$ is the set of vertices in $V(G)$ with degree greater than $q$, then $G[U]$ is a complete graph.*

**Proof of Theorem 1.2.** Suppose that $G$ is not $k$-connected. Let $H = C_{n+k-2}(G)$. By Theorem 1.1, $H$ is not $k$-connected. By Lemma 2.1, there exists an integer $q$ with $\delta \leq q \leq \left\lfloor \frac{n+k-3}{2} \right\rfloor$ such that $H$ has $(q-k+2, q)$-Pósa property. Let $q$ be the maximum one with the above Pósa property. First, we will prove the following claim.

**Claim 2.1.** $q = \delta$.

*Proof.* Assume that $\delta + 1 \leq q \leq \left\lfloor \frac{n+k-3}{2} \right\rfloor$. By Fact 2.1, we have

$$N_r(H) \leq \binom{n-q+k-2}{r} + (q-k+2)\binom{q}{r-1} = \theta(n, r, k, q).$$

Note that $G \subseteq H$. It follows that $N_r(G) \leq \max\left\{\theta(n, r, k, \delta+1), \theta\left(n, r, k, \left\lfloor \frac{n+k-3}{2} \right\rfloor\right)\right\}$, which contradicts the assumption. □

By Claim 2.1, Pósa property of $H$ and the maximality of $q$, there are exactly $\delta - k + 2$ vertices of degree $\delta$ in $V(H)$. Let $X$ be the set of vertices with degree $\delta$ in $H$ and $C = V(H) \setminus X$. Then $|X| = \delta - k + 2$ and $|C| = n - \delta + k - 2$. By Lemma 2.2, $C$ forms a clique in $H$.

Let $Y = \{v : d_H(v) \geq n - \delta + k - 2\}$. Since $\delta \leq \left\lfloor \frac{n+k-3}{2} \right\rfloor$, then $Y \subseteq C$. For $u \in X$ and $v \in Y$, we have $d_H(u) + d_H(v) \geq \delta + (n - \delta + k - 2) = n + k - 2$. Note that $H$ is an $(n+k-2)$-closed graph. Then every vertex of $Y$ is adjacent to all vertices of $X$, and thus $H[X, Y]$ forms a complete bipartite graph.

**Claim 2.2.** $k - 1 \leq |Y| \leq \delta$.

*Proof.* If $|Y| \geq \delta + 1$, then $d_H(u) \geq \delta + 1$ for $u \in X$, a contradiction. Moreover, we have $N_H(u) \subseteq X \cup Y$ for $u \in X$. Then $|X \cup Y| \geq \delta + 1$, and thus $|Y| \geq \delta + 1 - |X| = \delta + 1 - (\delta - k + 2) = k - 1$. Hence $k - 1 \leq |Y| \leq \delta$. □

Let $|Y| = s$. By Claim 2.2, we have $k - 1 \leq s \leq \delta$.

**Case 1.** $s = k - 1$.

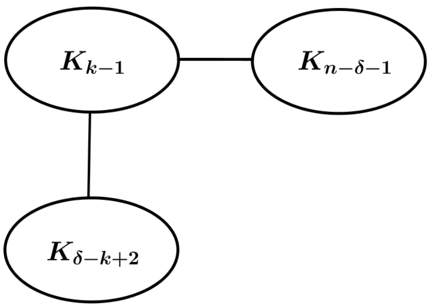


Figure 1: Graph $K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$.

Obviously, $H \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$ (see Fig. 1). Note that $H - V(K_{k-1})$ is not connected. By the definition of $k$-connected, we know that $H$ is not $k$-connected. Therefore, $H \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$.

**Case 2.** $k \leq s \leq \delta$.

Recall that $H[X, Y]$ forms a complete bipartite graph and $d_H(v) = \delta$ for $v \in X$. Then $H \cong K_s \vee (K_{n-s-\delta+k-2} + R(\delta-k+2, \delta-s))$ (see Fig. 2). Clearly, there exist at least $k$ internal disjoint paths for any two distinct vertices of $H$. Hence $H - S$ remains connected when $S \subseteq V(H)$ with $|S| \leq k - 1$. It follows that $K_s \vee (K_{n-s-\delta+k-2} + R(\delta - k + 2, \delta - s))$ is $k$-connected, a contradiction. □

**Lemma 2.3 (Hong, Shu and Fang [12], Nikiforov [17]).** *Let $G$ be a graph with minimum degree $\delta$. Then*

$$\rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta+1)^2}{4}}.$$

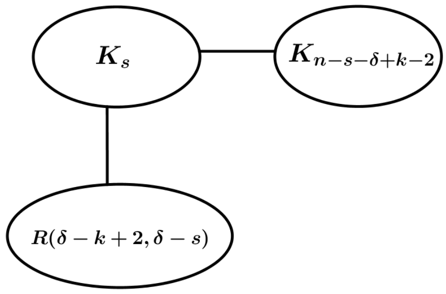


Figure 2: Graph $K_s \vee (K_{n-s-\delta+k-2} + R(\delta-k+2, \delta-s))$.

**Proof of Theorem 1.4.** By Lemma 2.3, we have

$$\rho(G) \leq \frac{\delta-1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta+1)^2}{4}}.$$

Since $\rho(G) > \frac{\delta-1}{2} + \sqrt{n^2 - (3\delta - 2k + 7)n + \frac{13}{4}\delta^2 - \left(4k - \frac{31}{2}\right)\delta + k^2 - 9k + \frac{73}{4}}$, then

$$e(G) > \frac{n^2}{2} - \left(\delta - k + \frac{7}{2}\right)n + \frac{3}{2}\delta^2 - \left(2k - \frac{15}{2}\right)\delta + \frac{1}{2}k^2 - \frac{9}{2}k + 9 = \theta(n, 2, k, \delta+1).$$

By Corollary 1.1, $G$ is $k$-connected unless $C_{n+k-2}(G) \cong K_{k-1} \vee (K_{n-\delta-1} + K_{\delta-k+2})$. □

## 3. Proof of Theorem 1.6

Gurgel and Wakabayashi [11] proved a sufficient condition in terms of the degree sequence for a graph to be $k$-leaf-connected.

**Lemma 3.1 (Gurgel and Wakabayashi [11]).** *Let $k$ and $n$ be such that $2 \leq k \leq n-3$. Let $G$ be a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose that there is no integer $k \leq i \leq \frac{n+k-2}{2}$ such that $d_{i-k+1} \leq i$ and $d_{n-i} \leq n-i+k-2$. Then $G$ is $k$-leaf-connected.*

**Proof of Theorem 1.6.** Suppose, to the contrary, that $G$ is not $k$-leaf-connected, where $2 \leq k \leq n-3$ and $\delta \geq k+1$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$ with $d_1 \leq d_2 \leq \cdots \leq d_n$. By Lemma 3.1, there exists an integer $i$ with $k \leq i \leq \frac{n+k-2}{2}$ such that $d_{i-k+1} \leq i$ and $d_{n-i} \leq n-i+k-2$. Then

$$\begin{aligned}
F(G) &= \sum_{u\in V(G)} d^3(u) = \sum_{j=1}^{i-k+1} d_j^3 + \sum_{j=i-k+2}^{n-i} d_j^3 + \sum_{j=n-i+1}^{n} d_j^3 \\
&\leq (i-k+1)i^3 + (n-2i+k-1)(n-i+k-2)^3 + i(n-1)^3 \\
&= n^4 - 11n^3 + (6k+51)n^2 - (24k+105)n + 2k^3 + 6k^2 + 32k + 82 \\
&\quad +(i-k-1)[3i^3 - (7n+5k-17)i^2 + (9n^2 - 40n + 11kn + 4k^2 - 21k + 47)i \\
&\quad -4n^3 - (6k-33)n^2 - (4k^2 - 25k + 85)n - k^3 + 10k^2 - 22k + 74].
\end{aligned}$$

By the assumptions $F(G) \geq n^4 - 11n^3 + (6k+51)n^2 - (24k+105)n + 2k^3 + 6k^2 + 32k + 82$, we have

$$\begin{aligned}
&(i-k-1)[3i^3 - (7n+5k-17)i^2 + (9n^2 - 40n + 11kn + 4k^2 - 21k + 47)i \\
&\quad -4n^3 - (6k-33)n^2 - (4k^2 - 25k + 85)n - k^3 + 10k^2 - 22k + 74] \geq 0.
\end{aligned}$$

Note that $i \geq d_{i-k+1} \geq \delta \geq k+1$. Next we will evaluate the value of $i$.

**Case 1.** $i = k + 1$.

Then $F(G) = n^4 - 11n^3 + (6k + 51)n^2 - (24k + 105)n + 2k^3 + 6k^2 + 32k + 82$, and all inequalities in the above arguments must be equalities. Then the degree sequence of $G$ is

$$d_1 = d_2 = k + 1, d_3 = d_4 = \cdots = d_{n-k-1} = n - 3, d_{n-k} = d_{n-k+1} = \cdots = d_n = n - 1.$$

Hence $G \cong K_{k+1} \vee (K_{n-k-3} + 2K_1)$. By [1], we know that $K_{k+1} \vee (K_{n-k-3} + 2K_1)$ is $k$-leaf-connected for $k \geq 3$, a contradiction. However, it is easy to see that $K_3 \vee (K_{n-5} + 2K_1)$ is not Hamilton-connected, and thus $G \cong K_3 \vee (K_{n-5} + 2K_1)$.

**Case 2.** $i \neq k + 1$.

Note that $i \geq k + 1$. Then $i \geq k + 2$ and $f(i) = 3i^3 - (7n + 5k - 17)i^2 + (9n^2 - 40n + 11kn + 4k^2 - 21k + 47)i - 4n^3 - (6k - 33)n^2 - (4k^2 - 25k + 85)n - k^3 + 10k^2 - 22k + 74 \geq 0$. Since $k + 2 \leq i \leq \frac{n+k-2}{2}$, then $n \geq k + 6$. Then we shall divide the following six cases.

**Subcase 2.1.** $n \geq k + 11$.

We claim that $\max_{k+2 \leq i \leq \frac{n+i-2}{2}} f(i) = f\left(\frac{n+k-2}{2}\right)$. In fact,

$$f'(i) = 9i^2 - 2(7n + 5k - 17)i + 9n^2 - 40n + 11kn + 4k^2 - 21k + 47.$$

By maple, we can obtain that $\Delta = -4[32n^2 + (29k - 122)n + 11k^2 - 19k + 134] < 0$. Hence $f'(i) > 0$ for $k + 2 \leq i \leq \frac{n+k-2}{2}$. Then $f(i)$ is a strictly monotonically increasing function on $\left[k + 2, \frac{n+k-2}{2}\right]$, and hence $\max_{k+2 \leq i \leq \frac{n+k-2}{2}} f(i) = f\left(\frac{n+k-2}{2}\right)$.

Note that $i$ is an integer. If $n + k$ is even and $n \geq k + 11$, then

$$f\left(\frac{n+k-2}{2}\right) = -\frac{7}{8}n^3 + \left(\frac{3}{8}k + 13\right)n^2 + \left(\frac{3}{8}k^2 - \frac{1}{2}k - 41\right)n + \frac{1}{8}k^3 + \frac{5}{2}k^2 + 5k + 41 < 0.$$

If $n + k$ is odd and $n \geq k + 11$, then

$$f\left(\frac{n+k-3}{2}\right) = -\frac{7}{8}n^3 + \left(\frac{3}{8}k + \frac{87}{8}\right)n^2 + \left(\frac{3}{8}k^2 - \frac{9}{4}k - \frac{261}{8}\right)n + \frac{1}{8}k^3 + \frac{15}{8}k^2 + \frac{51}{8}k + \frac{253}{8} < 0.$$

Therefore, $\max_{k+2 \leq i \leq \frac{n+k-2}{2}} f(i) < 0$. It follows that $f(i) < 0$, a contradiction.

**Subcase 2.2.** $n = k + 10$.

Note that $k+2 \leq i \leq \frac{n+k-2}{2}$ is an integer. Then $k+2 \leq i \leq k+4$. If $i = k+2$, then $f(i) = -18k^2-216k-570 < 0$, a contradiction. If $i = k+3$, then $f(i) = -9k^2 - 108k - 231 < 0$, a contradiction. If $i = k+4$, then $f(i) = -6k+56$. For $k \geq 10$, we have $f(i) < 0$, a contradiction. For $2 \leq k \leq 9$, we have $f(i) = -6k+56 > 0$. Note that $d_5 \leq k+4$, $d_6 \leq k + 4$ and

$$k^4 + 37k^3 + 423k^2 + 2007k + 3132 \leq F(G) \leq k^4 + 37k^3 + 423k^2 + 1989k + 3300.$$

Then the degree sequence of the permissible graphs is

$$d_1 = d_2 = \cdots = d_6 = k + 4,\ d_7 = d_8 = \cdots = d_{k+10} = k + 9.$$

This implies that $G \cong K_{k+4} \vee 6K_1$. One can check that $K_{k+4} \vee 6K_1$ is $k$-leaf-connected for $k \geq 3$, a contradiction. But $K_6 \vee 6K_1$ is not Hamilton-connected. Hence $G \cong K_6 \vee 6K_1$.

**Subcase 2.3.** $n = k + 9$.

Note that $k+2 \leq i \leq \frac{n+k-2}{2}$ is an integer. Then $k+2 \leq i \leq k+3$. If $i = k+2$, then $f(i) = -12k^2-126k-262 < 0$, a contradiction. If $i = k + 3$, then $f(i) = -3k^2 - 33k - 19 < 0$, a contradiction.

**Subcase 2.4.** $n = k + 8$.

Then $k+2 \le i \le k+3$. If $i = k+2$, then $f(i) = -6k^2 - 54k - 68 < 0$, a contradiction. If $i = k+3$, then $d_4 \le k+3$ and $d_5 \le k+3$. Note that

$$k^4 + 29k^3 + 249k^2 + 871k + 970 \le F(G) = \sum_{u \in V(G)} d^3(u) \le k^4 + 29k^3 + 255k^2 + 919k + 1164.$$

By a simple calculation, we obtain that

$$\begin{aligned} \sum_{j=6}^{k+8} d_j^3 &= F(G) - \sum_{j=1}^{5} d_j^3 \\ &\ge k^4 + 29k^3 + 249k^2 + 871k + 970 - 5(k+3)^3 \\ &= k^4 + 24k^3 + 204k^2 + 736k + 835. \end{aligned}$$

We claim that $d_7 = d_8 = \cdots = d_{k+8} = k+7$. Otherwise,

$$\sum_{j=6}^{k+8} d_j^3 \le 2(k+6)^3 + (k+1)(k+7)^3 = k^4 + 24k^3 + 204k^2 + 706k + 775,$$

a contradiction. So we have $d_6^3 \ge k^4 + 24k^3 + 204k^2 + 736k + 835 - (k+2)(k+7)^3 = k^3 + 15k^2 + 99k + 149$. Then $d_6 = k+7$ or $d_6 = k+6$.

If $d_6 = k+7$, then the degree sequence of $G$ must be

$$d_1 = d_2 = \cdots = d_5 = k+3,\ d_6 = d_7 = \cdots = d_{k+8} = k+7.$$

This means that $G \cong K_{k+3} \vee 5K_1$. It is easy to check that $K_{k+3} \vee 5K_1$ is $k$-leaf-connected for $k \ge 3$, a contradiction. However, $K_5 \vee 5K_1$ is not Hamilton-connected, and hence $G \cong K_5 \vee 5K_1$.

If $d_6 = k+6$, then the degree sequence of $G$ must be

$$d_1 = k+2,\ d_2 = \cdots = d_5 = k+3,\ d_6 = k+6,\ d_7 = \cdots = d_{k+8} = k+7.$$

When $k \ge 9$, we have $F(G) = k^4 + 29k^3 + 249k^2 + 865k + 1018 < k^4 + 29k^3 + 249k^2 + 871k + 970$, a contradiction. When $2 \le k \le 8$, we have $G \cong K_{k+2} \vee (K_{1,4} + K_1)$. One can determine that $K_{k+2} \vee (K_{1,4} + K_1)$ is $k$-leaf-connected for $k \ge 3$, a contradiction. But $K_4 \vee (K_{1,4} + K_1)$ is not Hamilton-connected, and thus $G \cong K_4 \vee (K_{1,4} + K_1)$.

**Subcase 2.5.** $n = k+7$.

Then $i = k+2$, and hence $d_3 \le k+2, d_5 \le k+3$. Note that

$$k^4 + 25k^3 + 180k^2 + 522k + 474 \le F(G) \le k^4 + 25k^3 + 180k^2 + 522k + 510.$$

Then the degree sequence of $G$ is

$$d_1 = d_2 = d_3 = k+2,\ d_4 = d_5 = k+3,\ d_6 = d_7 = \cdots = d_{k+7} = k+6.$$

This implies that $G \cong K_{k+2} \vee (K_2 + 3K_1)$. It is easy to see that $K_{k+2} \vee (K_2 + 3K_1)$ is $k$-leaf-connected for $k \ge 3$, a contradiction. But $K_4 \vee (K_2 + 3K_1)$ is not Hamilton-connected, and hence $G \cong K_4 \vee (K_2 + 3K_1)$.

**Subcase 2.6.** $n = k+6$.

Then $i = k+2$, and hence $d_3 \le k+2, d_4 \le k+2$. Note that

$$k^4 + 21k^3 + 123k^2 + 287k + 208 \le F(G) \le k^4 + 21k^3 + 129k^2 + 323k + 282.$$

Then we have

$$\begin{aligned} \sum_{j=5}^{k+6} d_j^3 &= F(G) - \sum_{j=1}^{4} d_j^3 \\ &\ge k^4 + 21k^3 + 123k^2 + 287k + 208 - 4(k+2)^3 \\ &= k^4 + 17k^3 + 99k^2 + 239k + 176. \end{aligned}$$

We assert that $d_6 = d_7 = \cdots = d_{k+6} = k + 5$. Otherwise,

$$\sum_{j=5}^{k+6} d_j^3 \leq 2(k+4)^3 + k(k+5)^3 = k^4 + 17k^3 + 99k^2 + 221k + 128,$$

a contradiction. Hence $d_5^3 \geq k^4 + 17k^3 + 99k^2 + 239k + 176 - (k+1)(k+5)^3 = k^3 + 9k^2 + 39k + 51$. Then $d_5 = k + 5$ or $d_6 = k + 4$.

If $d_5 = k + 5$, then the degree sequence of $G$ is

$$d_1 = d_2 = d_3 = d_4 = k + 2,\ d_5 = d_6 = \cdots = d_{k+6} = k + 5.$$

Hence $G \cong K_{k+2} \vee 4K_1$. It is easy to check that $K_{k+2} \vee 4K_1$ is $k$-leaf-connected for $k \geq 3$, a contradiction. However, $K_4 \vee 4K_1$ is not Hamilton-connected, and thus $G \cong K_4 \vee 4K_1$.

If $d_5 = k + 4$, then the degree sequence of $G$ is

$$d_1 = k + 1, d_2 = d_3 = d_4 = k + 2, d_5 = k + 4, d_6 = \cdots = d_{k+6} = k + 5.$$

Then $G \cong K_{k+1} \vee (K_{1,3} + K_1)$. One can check that $K_{k+1} \vee (K_{1,3} + K_1)$ is $k$-leaf-connected for $k \geq 3$, a contradiction. But $K_3 \vee (K_{1,3} + K_1)$ is not Hamilton-connected. Hence $G \cong K_3 \vee (K_{1,3} + K_1)$. □

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