Filomat 38:18 (2024), 6609–6619
https://doi.org/10.2298/FIL2418609W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new perspective on constructing 2-uninorms on bounded lattices

Ya-Ming Wang[a,*], Yi-Qun Zhang[b], Hua-Wen Liu[b]

[a]*School of Mathematics, Southwest Jiaotong University, Chengdu 611756, PR China*
[b]*School of Mathematics, Shandong University, Jinan 250100, PR China*

Abstract. Recently, Ertuğrul provided a way to obtain a 2-uninorm on a bounded lattice L by using a disjunctive uninorm and a conjunctive uninorm. Later, Xie and Yi proposed two methods for constructing 2-uninorms on L by using two uninorms U_1 on $[0_L, k]$ and U_2 on $[k, 1_L]$, and showed that the function constructed by the first method is a 2-uninorm iff U_2 is conjunctive and the function constructed by the second method is a 2-uninorm iff U_1 is disjunctive. Motivated by the three methods, we present two approaches to construct a 2-uninorm on L via a uni-nullnorm and a t-conorm (a t-norm and a null-uninorm). By the first new one, we can obtain a 2-uninorm on L such that the uninorm on $[0_L, k]$ is not necessarily disjunctive and the uninorm on $[k, 1_L]$ is not necessarily conjunctive. The second approach is different from all existing construction ways for 2-uninorms on L.

1. Introduction

In 1996, Yager et al. [22] introduced the notion of uninorms on the real unit interval. Fodor et al. [9] studied the structures of uninorms extensively in 1997. By allowing the neutral element to be any number in $[0, 1]$, uninorms generalize and unify the concepts of t-norms and t-conorms. Nullnorms as another generalizations of t-norms and t-conorms were introduced by Calvo et al. [3]. It has been proved that uninorms and nullnorms are widely used in many fields like fuzzy system modeling, neural networks, fuzzy logic, aggregation of information, decision making and so on [6, 14, 23, 24]. In order to generalize the definition of nullnorms, Akella [1] introduced the concept of 2-uninorms. A 2-uninorm has an ordinal sum like structure made up of two uninorms and has been proved to be a generalization of uninorms. After that, Sun et al. [13] showed the definitions of null-uninorms and uni-nullnorms, which are two special cases of 2-uninorms. In recent years, 2-uninorms have attracted some research interest [5, 10, 15, 16, 19, 25, 27, 28] since they cover uninorms, uni-nullnorms, nullnorms and null-uninorms.

In the framework of fuzzy sets, Ertuğrul [7] generalized the notion of 2-uninorms from $[0, 1]$ to more general algebraic structure – bounded lattices. In [7], Ertuğrul provided a way to obtain a 2-uninorm on L by using a disjunctive uninorm and a conjunctive uninorm. Recently, Xie and Yi [21] presented two methods for constructing 2-uninorms on L by using two uninorms U_1 and U_2, and showed that the

2020 *Mathematics Subject Classification*. Primary 03B52; Secondary 06F05, 94D05
Keywords. bounded lattices, 2-uninorms, uni-nullnorms, null-uninorms, order-preserving mapping.
Received: 25 September 2023; Revised: 18 February 2024; Accepted: 27 April 2024
Communicated by Miodrag Spalević
Research supported by the China Postdoctoral Science Foundation (No. 2021M691950), the Natural Science Foundation of Shandong Province (No. ZR2021QA007), the Postdoctoral Innovation Project of Shandong Province and the National Natural Science Foundation of China (No. 12071259).
* Corresponding author: Ya-Ming Wang
Email addresses: 623073044@qq.com (Ya-Ming Wang), 2272052751@qq.com (Yi-Qun Zhang), hw.liu@sdu.edu.cn (Hua-Wen Liu)

function constructed by the first method is a 2-uninorm iff U_2 is conjunctive and the function constructed by the second method is a 2-uninorm iff U_1 is disjunctive. Furthermore, they proved that the 2-uninorm constructed by these two methods is, respectively, the weakest and the strongest one among all 2-uninorms. Subsequently, Wang [18] introduced two other ways to obtain a 2-uninorm on L via a disjunctive uninorm U_1 on $[0_L, k]$ or a conjunctive uninorm U_2 on $[k, 1_L]$. In this work, we provide two approaches to obtain a 2-uninorm on L via a uni-nullnorm and a t-conorm (a t-norm and a null-uninorm). By the first approach, we can obtain a 2-uninorm on L such that the uninorm on $[0_L, k]$ is not necessarily disjunctive and the uninorm on $[k, 1_L]$ is not necessarily conjunctive. The second new one is different from all known construction ways for 2-uninorms on L.

The rest of this work is organized as follows. In Section 2, we recall some definitions related to bounded lattices and uninorms, uni-nullnorms, 2-uninorms on them. In Section 3, we present two ways to obtain a 2-uninorm on L via a uni-nullnorm and a t-conorm (a t-norm and a null-uninorm). In addition, we show that the uninorm on $[0_L, k]$ is not necessarily disjunctive and the uninorm on $[k, 1_L]$ is not necessarily conjunctive if a 2-uninorm is constructed by the first approach. Section 4 contains our conclusions.

2. Preliminaries

If a lattice $(L, \leq, \wedge, \vee)$ has a top element (written as 1_L) as well as a bottom element (written as 0_L), that is, there exist two elements $1_L, 0_L \in L$ such that $0_L \leq x \leq 1_L$ for all $x \in L$, then we call it a bounded lattice and denote it as $(L, \leq, 0_L, 1_L)$. More details about lattices can be found in [2].

For convenience, we use L to denote a bounded lattice instead of $(L, \leq, 0_L, 1_L)$ in this work. The notation $u \parallel v$ is used for $u, v \in L$ such that they are *incomparable*, i.e., neither $u \leq v$ nor $u \geq v$. The notation $u \nparallel v$ denotes that u is *comparable* with v, that is, $u \leq v$ or $u \leq v$. The notation I_u is defined as $I_u = \{x \in L \mid x \parallel u\}$. Furthermore, $[u, v] = \{x \in L \mid u \leq x \leq v\}$, $]u, v] = \{x \in L \mid u < x \leq v\}$, $[u, v[= \{x \in L \mid u \leq x < v\}$ and $]u, v[= \{x \in L \mid u < x < v\}$ are defined as *subintervals* of L.

Definition 2.1 ([4]). *A function $T : L^2 \to L$ (resp. $S : L^2 \to L$) is called a t-norm (resp. t-conorm) on L if it is associative, increasing, commutative, and satisfies $T(x, 1_L) = x$ (resp. $S(x, 0_L) = x$) for all $x \in L$.*

Definition 2.2 ([11]). *A function $\widehat{T} : L^2 \to L$ (resp. $\widehat{S} : L^2 \to L$) is called a t-subnorm (resp. t-superconorm) on L if it is associative, increasing, commutative, and satisfies $\widehat{T}(x, y) \leq x \wedge y$ (resp. $\widehat{S}(x, y) \geq x \vee y$) for all $x, y \in L$.*

Definition 2.3 ([12]). *A function $U : L^2 \to L$ is called a uninorm on L if it is associative, increasing, commutative, and has a neutral element $e \in L$ such that $U(x, e) = x$ for all $x \in L$.*

In particular, a uninorm U with a neutral element $e = 1_L$ is a t-norm, a uninorm U with a neutral element $e = 0_L$ is a t-conorm. In addition, a uninorm U is called conjunctive if $U(0_L, 1_L) = 0_L$, a uninorm U is called disjunctive if $U(0_L, 1_L) = 1_L$.

Definition 2.4 ([26]). *Let $e \in L \backslash \{0_L, 1_L\}$. The notation $\mathcal{U}_{\min}$ denotes the class of all uninorms on L with a neutral element e satisfying $U(x, y) = y$ for $(x, y) \in (e, 1_L] \times \{L \backslash [e, 1_L]\}$. Similarly, the notation $\mathcal{U}_{\max}$ denotes the class of all uninorms on L with a neutral element e satisfying $U(x, y) = y$ for $(x, y) \in [0_L, e) \times \{L \backslash [0_L, e]\}$.*

Theorem 2.5 ([26]). *Let $e \in L \backslash \{0_L, 1_L\}$ and $U : L^2 \to L$. Then $U \in \mathcal{U}_{\min}$ if and only if there is a t-subnorm $\widehat{T}$ on $L \backslash [e, 1_L]$ and a t-conorm S on $[e, 1_L]$ such that U is shown as Eq. (1).*

$$U(x, y) = \begin{cases} S(x, y) & \text{when } x, y \text{ in } [e, 1_L], \\ y & \text{when } x \text{ in } [e, 1_L] \text{ and } y \text{ in } \{L \backslash [e, 1_L]\}, \\ x & \text{when } x \text{ in } \{L \backslash [e, 1_L]\} \text{ and } y \text{ in } [e, 1_L], \\ \widehat{T}(x, y) & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 2.6 ([26]). *Let $e \in L\backslash\{0_L, 1_L\}$ and $U : L^2 \to L$. Then $U \in \mathcal{U}_{\max}$ if and only if there is a t-norm T on $[0_L, e]$ and a t-superconorm $\widehat{S}$ on $L\backslash[0_L, e]$ such that U is shown as Eq.* (2).

$$U(x,y) = \begin{cases} T(x,y) & \text{when } x, y \text{ in } [0_L, e], \\ y & \text{when } x \text{ in } [0_L, e] \text{ and } y \text{ in } \{L\backslash[0_L, e]\}, \\ x & \text{when } x \text{ in } \{L\backslash[0_L, e]\} \text{ and } y \text{ in } [0_L, e], \\ \widehat{S}(x,y) & \text{otherwise.} \end{cases} \tag{2}$$

Definition 2.7 ([17]). *A function $F : L^2 \to L$ is called a uni-nullnorm on L if it is increasing, commutative, associative, and has a neutral element $e \in L$ and an absorbing element $k \in L$ such that $0_L \leq e < k \leq 1_L$, $F(e, x) = x$ for all $x \in [0_L, k]$ and $F(x, 1_L) = x$ for all $x \in [k, 1_L]$.*

Definition 2.8 ([8]). *A function $G : L^2 \to L$ is called a null-uninorm on L if it is increasing, commutative, associative, and has a neutral element $f \in L$ and an absorbing element $k \in L$ such that $0_L \leq k < f \leq 1_L$, $G(0_L, x) = x$ for all $x \in [0_L, k]$ and $G(f, x) = x$ for all $x \in [k, 1_L]$.*

The ways to obtain a uni-nullnorm on L from Theorems 3 and 4 will be used in the next section.

Theorem 2.9 ([20]). *Let $e, k \in L$ and $0_L \leq e < k < 1_L$, $U : [0_L, k]^2 \to [0_L, k]$ be a uninorm with a neutral element e and $T : [k, 1_L]^2 \to [k, 1_L]$ be a t-norm. If $F_U : L^2 \to L$ is shown by Eq.*(3)*, then F_U is a uni-nullnorm.*

$$F_U(x,y) = \begin{cases} U(x,y) & \text{when } x, y \text{ in } [0_L, k], \\ T(x,y) & \text{when } x, y \text{ in } [k, 1_L], \\ U(x \wedge k, y \wedge k) & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 2.10 ([20]). *Let $e, k \in L$ and $0_L \leq e < k < 1_L$, $U : [0_L, k]^2 \to [0_L, k]$ be a uninorm with a neutral element e and $T : [k, 1_L]^2 \to [k, 1_L]$ be a t-norm. If $F_T : L^2 \to L$ is shown by Eq.*(4)*, then F_T is a uni-nullnorm if and only if U is disjunctive.*

$$F_T(x,y) = \begin{cases} U(x,y) & \text{when } x, y \text{ in } [0_L, k], \\ T(x,y) & \text{when } x, y \text{ in } [k, 1_L], \\ T(x \vee k, y \vee k) & \text{otherwise.} \end{cases} \tag{4}$$

Definition 2.11 ([7]). *A function $H : L^2 \to L$ is called a 2-uninorm on L if it is increasing, commutative, associative, and there exists $e, f \in L$ and $k \in L\backslash\{0_L, 1_L\}$ such that $0_L \leq e \leq k \leq f \leq 1_L$, $H(x, e) = x$ for all $x \in [0_L, k]$ and $H(x, f) = x$ for all $x \in [k, 1_L]$.*

From Definitions 2.7, 2.13 and 2.14, it is evident that a 2-uninorm with $f = 1_L$ is a uni-nullnorm, a 2-uninorm with $e = 0_L$ is a null-uninorm. Here we recall three ways for constructing 2-uninorms on L that will be compared with the new methods in Section 3.

Theorem 2.12 ([7]). *Let $k \in L\backslash\{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0_L, k]$ be a disjunctive uninorm with a neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a conjunctive uninorm with a neutral element f. Then the function $H : L^2 \to L$ given by Eq.* (5) *is a 2-uninorm on L.*

$$H(x,y) = \begin{cases} U_1(x,y) & \text{when } x, y \text{ in } [0_L, k], \\ U_2(x,y) & \text{when } x, y \text{ in } [k, 1_L], \\ k & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 2.13 ([21]). *Let $k \in L\backslash\{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0_L, k]$ be a uninorm with a neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a uninorm with a neutral element f. Then the function $H_W : L^2 \to L$ shown by Eq. (6) is a 2-uninorm on L if and only if U_2 is conjunctive.*

$$H_W(x, y) = \begin{cases} U_1(x, y) & \text{when } x, y \text{ in } [0_L, k], \\ U_2(x, y) & \text{when } x, y \text{ in } [k, 1_L], \\ U_1(x \wedge k, y \wedge k) & \text{otherwise.} \end{cases} \tag{6}$$

Theorem 2.14 ([21]). *Let $k \in L\backslash\{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0, k]$ be a uninorm with a neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a uninorm with a neutral element f. Then the function $H_S : L^2 \to L$ shown by Eq. (7) is a 2-uninorm on L if and only if U_1 is disjunctive.*

$$H_S(x, y) = \begin{cases} U_1(x, y) & \text{when } x, y \text{ in } [0_L, k], \\ U_2(x, y) & \text{when } x, y \text{ in } [k, 1_L], \\ U_2(x \vee k, y \vee k) & \text{otherwise.} \end{cases} \tag{7}$$

Xie and Yi have proved that H_W in Theorem 2.13 and H_S in Theorem 2.14 are, respectively, the weakest and the strongest 2-uninorm among all 2-uninorms with the given underlying uninorms U_1 and U_2.

3. Several methods to construct 2-uninorms on L

We provide two methods for obtaining a 2-uninorm on L via a uni-nullnorm and a t-conorm in the first subsection. There are also two examples in this subsection. The first example illustrates that one can use a conjunctive uninorm on $[0_L, k]$ and a disjunctive uninorm on $[k, 1_L]$ to construct a 2-uninorm on L. The second example shows that the method for obtaining 2-uninorms on bounded lattices in Theorem 3.2 differs from that ones in Theorems 2.12 and 2.13. Dually, we introduce two approaches to construct a 2-uninorms on L by using a t-norm and a null-uninorm in the second subsection.

3.1. The methods to obtain a 2-uninorm via a uni-nullnorm and a t-conorm on L

First we recall the definition of the order-preserving mapping. A mapping $h : L \to L'$ is called order-preserving if $x \leq y$ implies $h(x) \leq h(y)$ for all $x, y \in L$. Then we show a theorem for constructing 2-uninorms on bounded lattices by a order-preserving mapping, a uni-nullnorm and a t-conorm.

Theorem 3.1. *Let $f \in L\backslash\{0_L, 1_L\}$, F be a uni-nullnorm on $[0_L, f]$ with a neutral element e and an absorbing element k, S be a t-conorm on $[f, 1_L]$, $h : L \to [f, 1_L]$ be an order-preserving mapping such that $h(x) = x$ for any $x \in [f, 1_L]$. Then $H_{S_f} : L^2 \to L$ shown by Eq. (8) is a 2-uninorm,*

$$H_{S_f}(x, y) = \begin{cases} F(x, y) & \text{when } x, y \text{ in } [0_L, f], \\ S(x, y) & \text{when } x, y \text{ in } [f, 1_L], \\ S(h(x), h(y)) & \text{otherwise.} \end{cases} \tag{8}$$

if and only if one of the following conditions is satisfied.

(i) $I_f = \emptyset$;

(ii) $I_f \neq \emptyset$ *and* $x \parallel k$ *for any* $x \in I_f$.

Proof. It is easy for us to get that $h(x) = f$ for any $x \in [0_L, f]$ from the definition of the order-preserving mapping and the fact $h(f) = f$.

Necessity. Assume that $I_f \neq \emptyset$ and there exists some $x_0 \in I_f$ such that $x_0 \nparallel k$. There are two cases: $x_0 \leq k$ and $x_0 > k$. If $x_0 \leq k$, then from the definition of uni-nullnorms on L it follows that $x_0 \leq k < f$, which contradicts with $x_0 \in I_f$. If $x_0 > k$, then $x_0 = H_{S_f}(x_0, f) = S(h(x_0), f) = h(x_0) \in [f, 1_L]$, which contradicts with $x_0 \in I_f$. Therefore, there are two possibilities: $I_f = \emptyset$; $I_f \neq \emptyset$ and $x \parallel k$ for any $x \in I_f$.

Sufficiency. It is clear that the commutativity of H_{S_f} holds. We can obtain that $H_{S_f}(x,e) = x$ for all $x \in [0_L,k]$, $H_{S_f}(x,f) = x$ for all $x \in [k,f]$ and $H_{S_f}(x,f) = x$ for all $[f,1_L]$. If $I_f = \emptyset$, then $[k,1_L] = [k,f] \cup [f,1_L]$. If $x \parallel k$ for any $x \in I_f$, that is, $I_f \subseteq I_k$, then $[k,1_L] = [k,f] \cup [f,1_L]$. Thus, $H_{S_f}(x,f) = x$ for all $x \in [k,1_L]$. The monotonicity of H_{S_f} can be easily verified from the inequality $F(f,f) = f = S(f,f) < S(x,f) \leq S(x,y)$ for any $x,y \in]f,1_L]$. Now let us prove that H_{S_f} satisfies the associativity, that is, $H_{S_f}(x,H_{S_f}(y,z)) = H_{S_f}(H_{S_f}(x,y),z)$ for all $x,y,z \in L$.

Case 1. If $x,y,z \in [0_L,f]$, then $H_{S_f}(x,H_{S_f}(y,z)) = H_{S_f}(x,F(y,z)) = F(x,F(y,z)) = F(F(x,y),z) = H_{S_f}(F(x,y),z) = H_{S_f}(H_{S_f}(x,y),z)$.

Case 2. If only one of x,y,z belongs to $L\backslash[0_L,f]$, and assume that $z \in L\backslash[0_L,f]$ without loss of generality, then $H_{S_f}(x,H_{S_f}(y,z)) = H_{S_f}(x,h(z)) = h(z) = H_{S_f}(F(x,y),z) = H_{S_f}(H_{S_f}(x,y),z)$.

Case 3. If only one of x,y,z belongs to $[0_L,f]$, and assume that $x \in [0_L,f]$ without loss of generality, then $H_{S_f}(x,H_{S_f}(y,z)) = H_{S_f}(x,S(h(y),h(z))) = h(S(h(y),h(z))) = S(h(y),h(z)) = H_{S_f}(h(y),z) = H_{S_f}(H_{S_f}(x,y),z)$.

Case 4. If $x,y,z \in L\backslash[0_L,f]$, then $H_{S_f}(x,H_{S_f}(y,z)) = H_{S_f}(x,S(h(y),h(z))) = S(h(x),S(h(y),h(z))) = S(S(h(x),h(y)),h(z)) = H_{S_f}(S(h(x),h(y)),z) = H_{S_f}(H_{S_f}(x,y),z)$.

In summary, H_{S_f} is a 2-uninorm on L. □

	$[0_L, f]$	$[f, 1_L]$	I_f
I_f	$h(y)$	$S(x,h(y))$	$S(h(x),h(y))$
$[f, 1_L]$	$h(y)$	$S(x,y)$	$S(h(x),y)$
$[0_L, f]$	$F(x,y)$	$h(x)$	$h(x)$

Figure 1: H_{S_f} on L

The method for constructing 2-uninorms on L in Theorem 3.1 differs from those in [7, 18, 21] which requiring U_1 on $[0_L,k]$ be disjunctive or U_2 on $[k,1_L]$ be conjunctive. Example 1 illustrates that we can use the method in Theorem 3.1 to obtain a 2-uninorm via a conjunctive uninorm on $[0_L,k]$ and a disjunctive uninorm on $[k,1_L]$.

Example 1. Let $(L^\star = \{0_{L^\star}, e, a, k, b, f, c, m, n, 1_{L^\star}\}, \leq, 0_{L^\star}, 1_{L^\star})$ be a bounded lattice, and Figure 2 be its Hasse diagram.

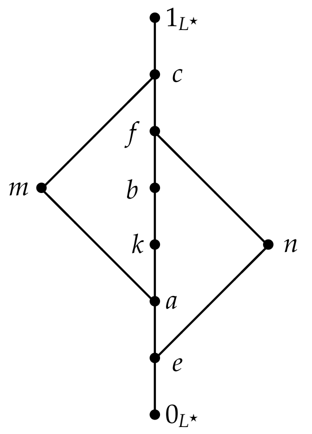

Figure 2: Hasse diagram of the lattice $L^\star$

Firstly, we define the uni-nullnorm F on $[0_{L^\star}, f]$ by the method in Theorem 2.9 (shown as Table 1) and the t-conorm S on $[f, 1_{L^\star}]$ (shown as Table 2).

Table 1 F on $[0_{L^\star}, f]$

F	$0_{L^\star}$	e	a	k	b	f	n
$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$
e	$0_{L^\star}$	e	a	k	k	k	e
a	$0_{L^\star}$	a	k	k	k	k	a
k	$0_{L^\star}$	k	k	k	k	k	k
b	$0_{L^\star}$	k	k	k	k	b	k
f	$0_{L^\star}$	k	k	k	b	f	k
n	$0_{L^\star}$	e	a	k	k	k	e

Table 2 S on $[f, 1_{L^\star}]$

S	f	c	$1_{L^\star}$
f	f	c	$1_{L^\star}$
c	c	c	$1_{L^\star}$
$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$

If the order-preserving mapping $h : L \to [f, 1_L]$ is defined as $h(x) = x \vee f$ for any $x \in L$, then we can obtain the structure of H_{S_f} shown as Table 3 from Theorem 3.1. It is obvious from Table 3 that $H_{S_f}|_{[0_{L^\star},k]^2}$ is a conjunctive uninorm and $H_{S_f}|_{[k,1_{L^\star}]^2}$ is a disjunctive uninorm. This means that one can use a conjunctive uninorm on $[0_{L^\star}, k]$ and a disjunctive uninorm on $[k, 1_{L^\star}]$ to construct a 2-uninorm on L.

Table 3 2-uninorm H_{S_f} on $L^\star$

H_{S_f}	$0_{L^\star}$	e	a	k	b	f	c	$1_{L^\star}$	m	n
$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	$0_{L^\star}$	c	$1_{L^\star}$	c	$0_{L^\star}$
e	$0_{L^\star}$	e	a	k	k	k	c	$1_{L^\star}$	c	e
a	$0_{L^\star}$	a	k	k	k	k	c	$1_{L^\star}$	c	a
k	$0_{L^\star}$	k	k	k	k	k	c	$1_{L^\star}$	c	k
b	$0_{L^\star}$	k	k	k	k	b	c	$1_{L^\star}$	c	k
f	$0_{L^\star}$	k	k	k	b	f	c	$1_{L^\star}$	c	k
c	c	c	c	c	c	c	c	$1_{L^\star}$	c	c
$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$	$1_{L^\star}$
m	c	c	c	c	c	c	c	$1_{L^\star}$	c	c
n	$0_{L^\star}$	e	a	k	k	k	c	$1_{L^\star}$	c	e

Next we will present another theorem for obtaining 2-uninorms on bounded lattices by a order-preserving mapping, a uni-nullnorm and a t-conorm.

Theorem 3.2. *Let $f \in L\backslash\{0_L, 1_L\}$, F be a uni-nullnorm on $[0_L, f]$ with a neutral element e and an absorbing element k, S be a t-conorm on $[f, 1_L]$, $h : L \to [0_L, f]$ be an order-preserving mapping such that $h(x) = x$ for any $x \in [0_L, f]$. Then $H_{F_f} : L^2 \to L$ shown by Eq. (9) is a 2-uninorm,*

$$H_{F_f}(x, y) = \begin{cases} F(x, y) & \text{when } x, y \text{ in } [0_L, f], \\ S(x, y) & \text{when } x, y \text{ in } [f, 1_L], \\ F(h(x), h(y)) & \text{otherwise.} \end{cases} \tag{9}$$

if and only if one of the following conditions is satisfied.

(i) $I_f = \emptyset$;

(ii) $I_f \neq \emptyset$ and $x \parallel k$ for any $x \in I_f$.

Proof. It is easy for us to get that $h(x) = f$ for any $x \in [f, 1_L]$ from the definition of the order-preserving mapping and the fact $h(f) = f$.

Necessity. Assume that $I_f \neq \emptyset$ and there exists some $x_0 \in I_f$ such that $x_0 \nparallel k$. There are two cases: $x_0 \leq k$ and $x_0 > k$. If $x_0 \leq k$, then $x_0 \leq k < f$, which contradicts with $x_0 \in I_f$. If $x_0 > k$, then from $k = h(k) \leq h(x_0) \leq f$ it follows that $x_0 = H_{F_f}(x_0, f) = F(h(x_0), f) = h(x_0) \in [0_L, f]$, which contradicts with $x_0 \in I_f$.

Sufficiency. It is clear that the commutativity of H_{F_f} holds. We can obtain that $H_{F_f}(x, e) = x$ for all $x \in [0_L, k]$ and $H_{F_f}(x, f) = x$ for all $[k, 1_L]$ from the condition (i) or (ii). The monotonicity of H_{F_f} can be easily verified from the inequality $F(x, y) \leq F(x, f) < F(f, f) = f = S(f, f)$ for any $x, y \in [0_L, f[$. Now let us prove that H_{F_f} satisfies the associativity, that is, $H_{F_f}(x, H_{F_f}(y, z)) = H_{F_f}(H_{F_f}(x, y), z)$ for all $x, y, z \in L$.

Case 1. If $x, y, z \in [f, 1_L]$, then $H_{F_f}(x, H_{F_f}(y, z)) = H_{F_f}(x, S(y, z)) = S(x, S(y, z)) = S(S(x, y), z) = H_{F_f}(S(x, y), z) = H_{F_f}(H_{F_f}(x, y), z)$.

Case 2. If only one of x, y, z belongs to $L \backslash [f, 1_L]$, and assume that $z \in L \backslash [f, 1_L]$ without loss of generality, then $H_{F_f}(x, H_{F_f}(y, z)) = H_{F_f}(x, F(f, h(z))) = F(f, F(f, h(z))) = F(F(f, f), h(z)) = F(f, h(z)) = H_{F_f}(S(x, y), z) = H_{F_f}(H_{F_f}(x, y), z)$.

Case 3. If only one of x, y, z belongs to $[f, 1_L]$, and assume that $x \in [f, 1_L]$ without loss of generality, then $H_{F_f}(x, H_{F_f}(y, z)) = H_{F_f}(x, F(h(y), h(z))) = F(f, F(h(y), h(z))) = F(F(f, h(y)), h(z)) = H_{F_f}(F(f, h(y)), z) = H_{F_f}(H_{F_f}(x, y), z)$.

Case 4. If $x, y, z \in L \backslash [f, 1_L]$, then $H_{F_f}(x, H_{F_f}(y, z)) = H_{F_f}(x, F(h(y), h(z))) = F(h(x), F(h(y), h(z))) = F(F(h(x), h(y)), h(z)) = H_{F_f}(F(h(x), h(y)), z) = H_{F_f}(H_{F_f}(x, y), z)$.

Therefore, H_{F_f} is a 2-uninorm on L. □

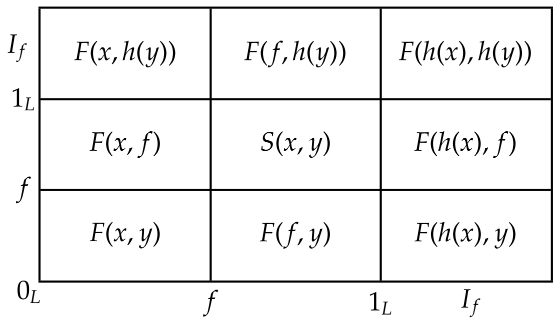

Figure 3: H_{F_f} on L

It is obvious that $H_{F_f}|_{[k,1_L]^2}$ is a conjunctive uninorm from the fact $H_{F_f}(k, 1_L) = F(k, f) = k$. But this construction methods of 2-uninorms on L is different from those proposed by Ertuğrul and Xie et al. The following example showing the difference between 2-uninorms constructed by the methods in Theorems 5, 6 and 9.

Example 2. Let $(L^* = \{0_{L^*}, e, k, f, m, n, 1_{L^*}\}, \leq, 0_{L^*}, 1_{L^*})$ be a bounded lattice, and Figure 4 be its Hasse diagram.

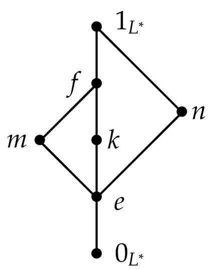

Figure 4: Hasse diagram of the lattice L^*

We define the disjunctive uninorm U_1 on $[0_{L^*}, k]$ (shown as Table 4), the conjunctive uninorm U_2 on $[k, 1_{L^*}]$ (shown as Table 5), and the uni-nullnorm F on $[0_{L^*}, f]$ by the method in Theorem 2.10 (shown as Table 6).

Table 4 U_1 on $[0_{L^*}, k]$

U_1	0_{L^*}	e	k
0_{L^*}	0_{L^*}	0_{L^*}	k
e	0_{L^*}	e	k
k	k	k	k

Table 5 U_2 on $[k, 1_{L^*}]$

U_2	k	f	1_{L^*}
k	k	k	k
f	k	f	1_{L^*}
1_{L^*}	k	1_{L^*}	1_{L^*}

If the order-preserving mapping $h : L \to [0_{L^*}]$ is defined as $h(x) = x \wedge f$ for any $x \in L$, then we can obtain the structures of H, H_W and H_{F_f} which are respectively shown as Table 7, Table 8 and Table 9 from Theorem 2.12, Theorem 2.13 and Theorem 3.2.

Table 6 F on $[0_{L^*}, f]$

F	0_{L^*}	e	k	f	m
0_{L^*}	0_{L^*}	0_{L^*}	k	k	k
e	0_{L^*}	e	k	k	k
k	k	k	k	k	k
f	k	k	k	f	f
m	k	k	k	f	f

Table 7 2-uninorm H on L^*

H	0_{L^*}	e	k	f	1_{L^*}	m	n
0_{L^*}	0_{L^*}	0_{L^*}	k	k	k	k	k
e	0_{L^*}	e	k	k	k	k	k
k	k	k	k	k	k	k	k
f	k	k	k	f	1_{L^*}	k	k
1_{L^*}	k	k	k	1_{L^*}	1_{L^*}	k	k
m	k	k	k	k	k	k	k
n	k	k	k	k	k	k	k

Table 8 2-uninorm H_W on L^*

H_W	0_{L^*}	e	k	f	1_{L^*}	m	n
0_{L^*}	0_{L^*}	0_{L^*}	k	k	k	0_{L^*}	0_{L^*}
e	0_{L^*}	e	k	k	k	e	e
k	k	k	k	k	k	k	k
f	k	k	k	f	1_{L^*}	k	k
1_{L^*}	k	k	k	1_{L^*}	1_{L^*}	k	k
m	0_{L^*}	e	k	k	k	e	e
n	0_{L^*}	e	k	k	k	e	e

Table 9 2-uninorm H_{S_f} on L^*

H_{S_f}	0_{L^*}	e	k	f	1_{L^*}	m	n
0_{L^*}	0_{L^*}	0_{L^*}	k	k	k	k	0_{L^*}
e	0_{L^*}	e	k	k	k	k	e
k	k	k	k	k	k	k	k
f	k	k	k	f	1_{L^*}	f	k
1_{L^*}	k	k	k	1_{L^*}	1_{L^*}	f	k
m	k	k	k	f	f	f	k
n	0_{L^*}	e	k	k	k	k	e

Obviously, the 2-uninorms in Tables 7, 8 and 9 are different.

Although Theorem 2.13 and Theorem 3.2 show two different ways to construct a 2-uninorm on L, we can still get the same 2-uninorm from both ways under some constraints.

Remark 3.3. *If $I_f = \emptyset$ or $x \parallel k$ for any $x \in I_f$, and requiring*

(i) $U_2 \in \mathcal{U}_{\min}$,
(ii) *the uni-nullnorm F be obtained by the method in Theorem 2.9,*
(iii) *the order-preserving mapping h be defined as $h(x) = x \wedge f$ for any $x \in L$,*

then the 2-uninorms on L constructed by Theorem 2.13 are same as the ones constructed by Theorem 3.2.

In fact, it is easy to know that $[k, 1_L] = [k, f] \cup [f, 1_L]$ when $I_f = \emptyset$. It follows that $I_f \subseteq I_k$ from $x \parallel k$ for any $x \in I_f$, then we have $[k, 1_L] = [k, f] \cup [f, 1_L]$. Further, if $U_2 \in \mathcal{U}_{\min}$ in Theorem 2.13, then the function H_W becomes the following Eq. (10).

$$H_W(x,y) = \begin{cases} U_1(x,y) & \text{when } x, y \textit{ in } [0_L, k]; \\ T(x,y) & \text{when } x, y \textit{ in } [k, f]; \\ S(x,y) & \text{when } x, y \textit{ in } [f, 1_L]; \\ x & \text{when } x \textit{ in } [k, f] \textit{ and } y \textit{ in } [f, 1_L]; \\ y & \text{when } x \textit{ in } [f, 1_L] \textit{ and } y \textit{ in } [k, f]; \\ U_1(x \wedge k, y \wedge k) & \text{otherwise.} \end{cases} \tag{10}$$

If $I_f = \emptyset$, $h(x) = x \wedge f$ for any $x \in L$ and the uni-nullnorm F is obtained by the method from Theorem 2.9, then $H_{F_f}(x, y) = F(x \wedge f, y \wedge f) = F(x, f) = x$ for $x \in [k, f]$ and $y \in [f, 1_L]$, $H_{F_f}(x, y) = F(x \wedge f, y \wedge f) = F(x, f) = U(x \wedge k, k) = U(x \wedge k, y \wedge k)$ for $x \in [0_L, k] \cup I_k$ and $y \in [f, 1_L]$. If $x \parallel k$ for any $x \in I_f$, $h(x) = x \wedge f$ for any $x \in L$ and the uni-nullnorm F is obtained by the method from Theorem 2.9, then $x \wedge f = x \wedge k$ for any $x \in I_f$ from the fact $I_f \subseteq I_k$. Thus, $H_{F_f}(x, y) = F(x \wedge f, y \wedge f) = F(x \wedge k, y \wedge k) = U(x \wedge k, y \wedge k)$ for $x \in I_f$ or $y \in I_f$. Therefore, the function H_{F_f} becomes the following Eq. (11).

$$H_{F_f}(x,y) = \begin{cases} U(x,y) & \text{when } x, y \textit{ in } [0_L, k]; \\ T(x,y) & \text{when } x, y \textit{ in } [k, f]; \\ S(x,y) & \text{when } x, y \textit{ in } [f, 1_L]; \\ x & \text{when } x \textit{ in } [k, f] \textit{ and } y \textit{ in } [f, 1_L]; \\ y & \text{when } x \textit{ in } [f, 1_L] \textit{ and } y \textit{ in } [k, f]; \\ U(x \wedge k, y \wedge k) & \text{otherwise.} \end{cases} \tag{11}$$

Obviously, the 2-uninorms given by Eq. (10) and Eq. (11) are same when the underlying uninorms U_1 and U are same.

3.2. The methods to obtain a 2-uninorm via a t-norm and a null-uninorm on L

In this subsection, we propose two construction methods for a 2-uninorm on L via a order-preserving mapping, a t-norm and a null-uninorm. We omit the proofs of the following two theorems since their proofs are similar to those of theorems in the previous subsection.

Theorem 3.4. *Let $e \in L \backslash \{0_L, 1_L\}$, T be a t-norm on $[0_L, e]$ and G be a null-uninorm on $[e, 1_L]$ with a neutral element f and an absorbing element k, $h : L \to [0_L, e]$ be an order-preserving mapping such that $h(x) = x$ for any $x \in [0_L, e]$. Then $H_{T_e} : L^2 \to L$ shown by Eq. (12) is a 2-uninorm,*

$$H_{T_e}(x,y) = \begin{cases} T(x,y) & \textit{when } x, y \textit{ in } [0_L, e], \\ G(x,y) & \textit{when } x, y \textit{ in } [e, 1_L], \\ T(h(x), h(y)) & \textit{otherwise.} \end{cases} \tag{12}$$

if and only if one of the following conditions is true.

(i) $I_e = \emptyset$;
(ii) $I_e \neq \emptyset$ *and* $x \parallel k$ *for any* $x \in I_e$.

Theorem 3.5. *Let $e \in L \backslash \{0_L, 1_L\}$, T be a t-norm on $[0_L, e]$ and G be a null-uninorm on $[e, 1_L]$ with a neutral element f and an absorbing element k, $h : L \to [e, 1_L]$ be an order-preserving mapping such that $h(x) = x$ for any $x \in [e, 1_L]$.*

Then $H_{G_e} : L^2 \to L$ *shown by Eq.* (13) *is a 2-uninorm,*

$$H_{G_e}(x,y) = \begin{cases} T(x,y) & \text{when } x,y \text{ in } [0_L, e], \\ G(x,y) & \text{when } x,y \text{ in } [e, 1_L], \\ G(h(x),h(y)) & \text{otherwise.} \end{cases} \tag{13}$$

if and only if one of the following conditions is true.

(i) $I_e = \emptyset$;

(ii) $I_e \neq \emptyset$ *and* $x \parallel k$ *for any* $x \in I_e$.

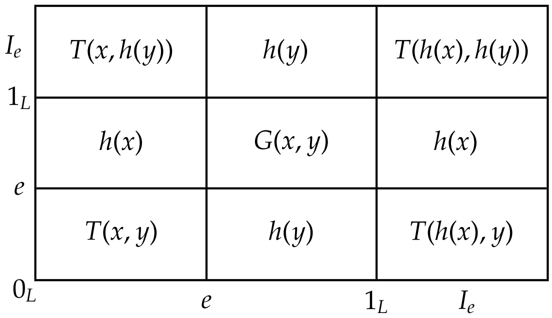

Figure 5: H_{T_e} on L

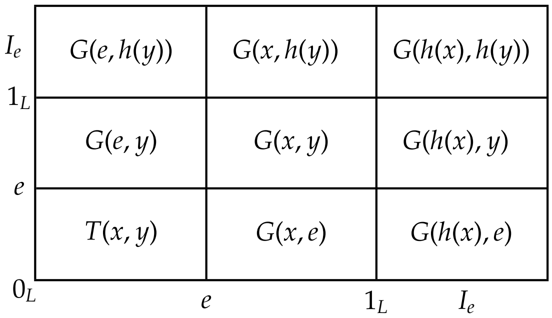

Figure 6: H_{G_e} on L

4. Conclusion

In this work, we first provide two ways to obtain a 2-uninorm on L by using a uni-nullnorm and a t-conorm. By the first method, we can obtain a 2-uninorm on L such that the uninorm on $[0_L, k]$ is not necessarily disjunctive and the uninorm on $[k, 1_L]$ is not necessarily conjunctive. In other words, we can use a conjunctive uninorm on $[0_L, k]$ and a disjunctive uninorm on $[k, 1_L]$ to construct a 2-uninorm on L (but it is not necessary). Furthermore, we present another new method for constructing 2-uninorm on L which differs from all existing ones. Finally, we present two approaches to construct a 2-uninorm on L via a t-norm and a null-uninorm.

In addition, we will consider the way to construct a 2-uninorm via a uni-nullnorm and a t-conorm (a t-norm and a null-uninorm) on a more general lattice without restrictions as our future work.

References

[1] P. Akella, *Structure of n-uninorms*, Fuzzy Sets Syst. **158** (2007), 1631–1651.
[2] G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publishers, Providence, RI, 1967.
[3] T. Calvo, B. De Baets, J.C. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, Fuzzy Sets Syst. **120** (2001), 385–394.
[4] B. De Baets, R. Mesiar, *Triangular norms on product lattices*, Fuzzy Sets Syst. **104** (1999), 61–75.
[5] P. Drygaś, E. Rak, *Distributivity equation in the class of 2-uninorms*, Fuzzy Sets Syst. **291** (2016), 82–97.
[6] D. Dubois, H. Prade, *Fundamentals of Fuzzy Sets*, Kluwer Academic Publisher, Boston, 2000.
[7] Ü. Ertuğrul, *A way to obtain 2-uninorm on bounded lattice from uninorms defined on subintervals of bounded lattice*, New Trends in Math. Sci. **5** (2017), 1–9.
[8] Ü. Ertuğrul, M.N. Kesicioğlu, F. Karaçal, *Construction methods for uni-nullnorms and null-uninorms on bounded lattice*, Kybernetika, **55(6)** (2019), 994–1015.
[9] J. Fodor, R.R. Yager, A. Rybalov, *Structure of uninorms*, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. **5(4)** (1997), 411–427.
[10] C.Y. Huang, F. Qin, *Migratiivty properties of uninorms over 2-uninorms*, Int. J. Approx. Reason. **139** (2021), 104–129.
[11] W. Ji, *Constructions of uninorms on bounded lattice by means of t-subnorms and t-subconorms*, Fuzzy Sets Syst. **403** (2021), 38–55.
[12] F. Karaçal, R. Mesiar, *Uninorms on bounded lattices*, Fuzzy Sets Syst. **261** (2015), 33–43.
[13] F. Sun, X.P. Wang, X.B. Qu, *Uni-nullnorms and null-uninorms*, J. Intell. Fuzzy Syst. **32** (2017), 1969–1981.

[14] A.K. Tsadiras, R.G. Margaritis, *The MYCIN certainty factor handling function as uninorm operator and its use as a threshold function in artifical neurons*, Fuzzy Sets Syst. **93** (1998), 263–274.
[15] Y.M. Wang, F. Qin, *Distributivity for 2-uninorms over semi-uninorms*, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. **25** (2017), 317–345.
[16] Y.M. Wang, W.W. Zong, H. Zhan, H.W. Liu, *On migrative 2-uninorms and nullnorms*, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. **27** (2019), 303–328.
[17] Y.M. Wang, H. Zhan, H.W. Liu, *Uni-nullnorms on bounded lattices*, Fuzzy Sets Syst. **386** (2020), 132–144.
[18] Y.M. Wang, *An addendum to "Construction of 2-uninorms on bounded lattices"*, Iran. J. Fuzzy Syst. **20(4)** (2023), 97–102.
[19] Z. Wang, W. Zong, Y. Su, *Discussing discrete 2-uninorms using lower and upper ordinal sums*, Inf. Sci. **542** (2021), 317–323.
[20] X. Wu, S. Liang, Y. Luo, G.D. Cayli, *Construction methods for the smallest and largest uni-nullnorms on bounded lattices*, Fuzzy Sets Syst. **427** (2022), 132–148.
[21] A. Xie, Z. Yi, *Construction of 2-uninorms on bounded lattices*, Iran. J. Fuzzy Syst. **19** (2022), 187–198.
[22] R.R. Yager, A. Rybalov, *Uninorms aggregation operators*, Fuzzy Sets Syst. **180** (1996), 111–120.
[23] R.R. Yager, *Uninorms in fuzzy system modeling*, Fuzzy Sets Syst. **122** (2001), 165–175.
[24] R.R. Yager, *Defending against strategic manipulation in uninorm-based multi-agent decision making*, Fuzzy Sets Syst. **140** (2003), 331–339.
[25] F.X. Zhang, E. Rak, J. Bazan, *On the distributivity of continuous triangular norms and triangular conorms with respect to 2-uninorms*, Fuzzy Sets Syst. **395** (2020), 168–177.
[26] H.P. Zhang, M. Wu, Z. Wang, Y. Ouyang, B. De Baets, *A characterization of the classes Umin and Umax of uninorms on a bounded lattice*, Fuzzy Sets Syst. **423** (2021), 107–121.
[27] T. Zhang, F. Qin, *On distributive laws between 2-uninorms and overlap (grouping) functions*, Int. J. Approx. Reason. **119** (2020), 353–372.
[28] W. Zong, Y. Su, H.W. Liu, B. De Baets, *On the structure of 2-uninorms*, Inf. Sci. **467** (2018), 506–527.