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Conformal (σ, τ)-derivations on Lie conformal algebras

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Abstract. In this paper, we focus on the conformal (σ, τ)-derivation theory of Lie conformal algebras. Firstly, we study the fundamental properties of conformal (σ, τ)-derivations. Secondly, we mainly research the interiors of conformal G-derivations. Finally, we discuss the relationships between the conformal (σ, τ)-derivations and some generalized conformal derivations of Lie conformal algebras.

1. Introduction

Lie conformal algebras, introduced by Kac in [4, 5], encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory. Furthermore, the category of Lie conformal algebras is equivalent to the category of formal distribution Lie algebras, which are essentially infinite-dimensional Lie algebras. Namely, they are closely connected to the notion of a formal distribution Lie algebra $(\mathfrak{g}, \mathcal{F})$, which is a Lie algebra $\mathfrak{g}$ spanned by the coefficients of a family $\mathcal{F}$ of mutually local formal distributions. See [2] for details.

The derivation theory of Lie conformal algebras was introduced in [2]. The generalized derivation theory of Lie conformal (super)algebras was developed in [3, 6, 7]. [1] studied a kind of new generalized derivations of Lie algebras, that is the (σ, τ)-derivation theory of Lie algebras. In the present paper, we aim to do the same as in [1] for Lie conformal algebras, extending the (σ, τ)-derivations of Lie algebras to that of the Lie conformal algebras.

This paper is organized as follows. In Section 1, we recall several basic definitions of Lie conformal algebras and introduce the concept of conformal (σ, τ)-derivations of Lie conformal algebras. In Section 2, we obtain some fundamental properties of conformal (σ, τ)-derivations. In Section 3, we describe the interiors of conformal G-derivations and compute the corresponding Hilbert series to show its complexity. In Section 4, we devote ourselves to studying the connections between the conformal (σ, τ)-derivations and some generalized conformal derivations, such as centroids and conformal (α, β, γ)-derivations.

Throughout this paper, we denote by $\mathbb{C}$ the field of complex numbers. Denote by $\mathbb{Z}$ the ring of integers and $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers. The set of strictly positive integers will be denoted by $\mathbb{N}^{+}$.

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2. Preliminaries

In this section, for the reader's convenience, we shall summarize some basic facts about Lie conformal algebras used in this paper, see [2, 4]. At the end of this section, we introduce the notion of a conformal (σ, τ)-derivation for arbitrary Lie conformal algebras.

Definition 2.1. [4] *A Lie conformal algebra $\mathcal{R}$ is a left $\mathbb{C}[\partial]$-module, and for any $n \in \mathbb{Z}_{\geq 0}$ there is a family of $\mathbb{C}$-linear n-products from $\mathcal{R} \otimes \mathcal{R}$ to $\mathcal{R}$ satisfying*

(C0) *For any $a, b \in \mathcal{R}$, $a_{(n)}b = 0$ for $n \gg 0$,*
(C1) *For any $a, b \in \mathcal{R}$ and $n \in \mathbb{Z}_{\geq 0}$, $(\partial a)_{(n)}b = -na_{(n-1)}b$,*
(C2) *For any $a, b \in \mathcal{R}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$a_{(n)}b = -\sum_{j=0}^{\infty}(-1)^{j+n}\frac{1}{j!}\partial^{j}(b_{(n+j)}a),$$

(C3) *For any $a, b, c \in \mathcal{R}$ and $m, n \in \mathbb{Z}_{\geq 0}$,*

$$a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + b_{(n)}(a_{(m)}c).$$

(Convention: $a_{(n)}b = 0$ if $n < 0$).

Define the λ-bracket $[-_\lambda -]$ by

$$[a_\lambda b] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}a_{(n)}b, \quad \forall a, b \in \mathcal{R}. \tag{1}$$

Then $\mathcal{R}$ is a Lie conformal algebra if and only if $[-_\lambda -]$ satisfies

$(C1)_\lambda$ Conformal sesquilinearity : $[(\partial a)_\lambda b] = -\lambda[a_\lambda b]$;
$(C2)_\lambda$ Skew − symmetry : $[a_\lambda b] = -[b_{-\partial-\lambda} a]$;
$(C3)_\lambda$ Jacobi identity : $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]$.

A Lie conformal algebras is called *finite* if $\mathcal{R}$ is a finitely generated $\mathbb{C}[\partial]$-module. The *rank* of a conformal algebra $\mathcal{R}$ is its rank as a $\mathbb{C}[\partial]$-module (recall that this is the dimension over $\mathbb{C}(\partial)$, the field of fractions of $\mathbb{C}[\partial]$, of $\mathbb{C}(\partial) \otimes_{\mathbb{C}[\partial]} \mathcal{R}$).

Throughout this paper, we assume that $\mathcal{R}$ is finite.

Definition 2.2. [4] *An associative conformal algebra $\mathcal{R}$ is a left $\mathbb{C}[\partial]$-module endowed with a λ-product from $\mathcal{R} \otimes \mathcal{R}$ to $\mathbb{C}[\lambda] \otimes \mathcal{R}$, for any $a, b, c \in \mathcal{R}$, satisfying*

(1) *$(\partial a)_\lambda b = -\lambda a_\lambda b$, $a_\lambda(\partial b) = (\partial + \lambda)(a_\lambda b)$,*
(2) *$a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$.*

Definition 2.3. [2] *Let M and N be $\mathbb{C}[\partial]$-modules. A conformal linear map from M to N is a sequence $f = \{f_{(n)}\}_{n\in\mathbb{Z}_{\geq 0}}$ of $f_{(n)} \in \mathrm{Hom}_{\mathbb{C}}(M, N)$ satisfying that*

$$\partial_N f_{(n)} - f_{(n)}\partial_M = -nf_{(n-1)}, \ n \in \mathbb{Z}_{\geq 0}.$$

Set $f_\lambda = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f_{(n)}$. Then $f = \{f_{(n)}\}_{n\in\mathbb{Z}_{\geq 0}}$ is a conformal linear map if and only if

$$f_\lambda \partial_M = (\partial_N + \lambda)f_\lambda.$$

Let $\mathrm{Chom}(M,N)$ denote the set of conformal linear maps from M to N. Then $\mathrm{Chom}(M,N)$ is a $\mathbb{C}[\partial]$-module via:

$$\partial f_{(n)} = -nf_{(n-1)}, \text{ equivalently, } \partial f_\lambda = -\lambda f_\lambda.$$

The composition $f_\lambda g : L \to N \otimes \mathbb{C}[\lambda]$ of conformal linear maps $f : M \to N$ and $g : L \to M$ is given by

$$(f_\lambda g)_{\lambda+\mu} = f_\lambda g_\mu, \quad \forall f, g \in \mathrm{Chom}(M,N).$$

If M is a finitely generated $\mathbb{C}[\partial]$-module, then $\mathrm{Cend}(M) := \mathrm{Chom}(M,M)$ is an associative conformal algebra with respect to the above composition. Thus, $\mathrm{Cend}(M)$ becomes a Lie conformal algebra, called the general linear Lie conformal algebra, denoted as $gc(M)$, with respect to the λ-bracket(see [2, Example3.5]):

$$[f_\lambda g]_\mu = f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda. \tag{2}$$

Throughout this paper, we mainly deal with $\mathbb{C}[\partial]$-modules which are finitely generated.

Definition 2.4. [2] *Let $\mathcal{R}$ be a Lie conformal algebra. $d \in \mathrm{Cend}(\mathcal{R})$ is a conformal derivation if for any $a, b \in \mathcal{R}$ it holds that*

$$d_{(m)}(a_{(n)}b) = \sum_{j=0}^{m} \binom{m}{j} (d_{(j)}a)_{(m+n-j)}b + a_{(n)}(d_{(m)}(b));$$

equivalently,

$$d_\lambda([a_\mu b]) = [(d_\lambda(a))_{\lambda+\mu} b] + [a_\mu(d_\lambda(b))].$$

For any $r \in \mathcal{R}$, d^r_λ is called an *inner* conformal derivation of $\mathcal{R}$ if $d^r_\lambda(r') = [r_\lambda r']$, $\forall r' \in \mathcal{R}$.

Define $\mathrm{CDer}(\mathcal{R})$ as the set of conformal derivations of $\mathcal{R}$, then it is obvious that $\mathrm{CDer}(\mathcal{R})$ is a subalgebra of $\mathrm{Cend}(\mathcal{R})$.

Definition 2.5. [2] *Let $\mathcal{R}$ and $\mathcal{R}'$ be two Lie conformal algebras. A homomorphism ϕ from $\mathcal{R}$ to $\mathcal{R}'$ of Lie conformal algebras is a $\mathbb{C}[\partial]$-linear homomorphism if for any $a, b \in \mathcal{R}$ it holds that*

$$\phi(a_{(n)}b) = (\phi(a)_{(n)}\phi(b));$$

equivalently,

$$\phi([a_\lambda b]) = [\phi(a)_\lambda \phi(b)].$$

We call ϕ an isomorphism if it is bijective. We call ϕ an endomorphism if $\mathcal{R} = \mathcal{R}'$. We call ϕ an automorphism if it is bijective and if $\mathcal{R} = \mathcal{R}'$.

In the following, we denote $\mathrm{Aut}(\mathcal{R})$ the automorphism group of $\mathcal{R}$.

Definition 2.6. [2] *Let $\mathcal{R}$ be a Lie conformal algebra and G a subgroup of $\mathrm{Aut}(\mathcal{R})$. Then $d \in \mathrm{Cend}(\mathcal{R})$ is a conformal G-derivation of $\mathcal{R}$ if there exist two elements σ, τ in G such that*

$$d_\lambda([a_\mu b]) = [(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] + [(\tau(a))_\mu(d_\lambda(b))], \quad \forall a, b \in \mathcal{R}.$$

In this case, σ and τ are called the associated automorphisms of d.

Denote by $\mathrm{CDer}_G(\mathcal{R})$ the set of all conformal G-derivations of $\mathcal{R}$. It is clear that $\mathrm{CDer}_G(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$ if G is a trivial group, that is $G = \{\mathrm{id}_\mathcal{R}\}$. Thus, conformal G-derivations can be viewed as a generalization of conformal derivations. What's more, if $G \leq H$ are two subgroups of $\mathrm{Aut}(\mathcal{R})$, then $\mathrm{CDer}_G(\mathcal{R}) \subseteq \mathrm{CDer}_H(\mathcal{R})$ and $\mathrm{CDer}(\mathcal{R})$ is contained in $\mathrm{CDer}_G(\mathcal{R})$ for any subgroup G of $\mathrm{Aut}(\mathcal{R})$.

Fix two automorphisms $\sigma, \tau \in G$, we denote by $\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})$ the set of all conformal G-derivations associated to σ and τ, called the *conformal (σ, τ)-derivation*. It is clear that $\mathrm{CDer}_{\sigma,\tau}(\mathcal{R}) \subseteq \mathrm{CDer}_G(\mathcal{R})$ is a $\mathbb{C}[\partial]$-module and $\mathrm{CDer}_{\mathrm{id}_\mathcal{R},\mathrm{id}_\mathcal{R}}(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$. For convenience, we denote $\mathrm{CDer}_{\sigma,\mathrm{id}_\mathcal{R}}(\mathcal{R})$ by $\mathrm{CDer}_\sigma(\mathcal{R})$.

Hereafter, G always denotes a subgroup of $\mathrm{Aut}(\mathcal{R})$.

3. Fundamental properties

In this section, we aim to show several fundamental properties of conformal (σ, τ)-derivations.

Proposition 3.1. *Let $\mathcal{R}$ be a Lie conformal algebra. If $\sigma, \tau \in G$, then* $\mathrm{rank}(\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})) = \mathrm{rank}(\mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R}))$.

Proof. Define a map $\varphi_\tau : \mathrm{CDer}_{\sigma,\tau}(\mathcal{R}) \to \mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R})$ by

$$\varphi_\tau(d) = \tau^{-1}d, \quad \forall d \in \mathrm{CDer}_{\sigma,\tau}(\mathcal{R}).$$

Note that

$$\begin{aligned} &\tau^{-1}(d_\lambda([a_\mu b])) = \tau^{-1}([(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] + [(\tau(a))_\mu(d_\lambda(b))]) \\ &= [(\tau^{-1}(d_\lambda(a)))_{\lambda+\mu}(\tau^{-1}(\sigma(b)))] + [a_\mu(\tau^{-1}(d_\lambda(b)))], \end{aligned}$$

for any $a, b \in \mathcal{R}$. Hence $\tau^{-1}d \in \mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R})$, and thus the map φ_τ is well-defined. What's more

$$\varphi_\tau(d_1 + d_2) = \tau^{-1}(d_1 + d_2) = \tau^{-1}d_1 + \tau^{-1}d_2 = \varphi_\tau d_1 + \varphi_\tau d_2,$$

$$\varphi_\tau(\partial d) = \varphi_\tau(-\lambda d_\lambda) = -\lambda\varphi_\tau(d_\lambda) = -\lambda\tau^{-1}(d_\lambda) = \partial\varphi_\tau(d_\lambda).$$

That is to say, φ_τ is a $\mathbb{C}[\partial]$-module homomorphism.

In addition, it still needs to show that φ_τ is an isomorphism. So we try to see its inverse. We can define a map $\psi_\tau : \mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R}) \to \mathrm{CDer}_{\sigma,\tau}(\mathcal{R})$ by $\psi_\tau(d) = \tau d$ for any d in $\mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R})$. Similarly, we can verify that ψ_τ is a well-defined $\mathbb{C}[\partial]$-module homomorphism. What's more, $\psi_\tau\varphi_\tau = \mathrm{id}_{\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})}$ and $\varphi_\tau\psi_\tau = \mathrm{id}_{\mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R})}$, which means ψ_τ is the inverse of φ_τ. Therefore, $\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})$ and $\mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R})$ are isomorphic as $\mathbb{C}[\partial]$-modules, and thus $\mathrm{rank}(\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})) = \mathrm{rank}(\mathrm{CDer}_{\tau^{-1}\sigma}(\mathcal{R}))$. □

Proposition 3.1 means that the study of $\mathrm{CDer}_{\sigma,\tau}(\mathcal{R})$ with two parameters σ, τ can be reduced to the study of $\mathrm{CDer}_{\sigma'}(\mathcal{R})$ with one parameter $\sigma' = \tau^{-1}\sigma$. Particularly, if we take $\tau = \sigma$, then $\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R})$ and $\mathrm{CDer}(\mathcal{R})$ are isomorphic as $\mathbb{C}[\partial]$-modules. Moreover, we may extend this isomorphic relation to the level of Lie conformal algebras.

Proposition 3.2. *Let $\mathcal{R}$ be a Lie conformal algebra. If $\sigma \in G$, then there exists a Lie conformal algebra structure $[-_\lambda-]^\sigma$ on $\mathrm{Der}_{\sigma,\sigma}(\mathcal{R})$ such that $\mathrm{Der}_{\sigma,\sigma}(\mathcal{R}) \cong \mathrm{CDer}(\mathcal{R})$ as Lie conformal algebras.*

Proof. Define a λ-bracket $[-_\lambda-]^\sigma$ on $\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R}) \times \mathrm{CDer}_{\sigma,\sigma}(\mathcal{R})$ as follow:

$$[f_\lambda g]^\sigma = \varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]), \quad \forall f, g \in \mathrm{CDer}_{\sigma,\sigma}(\mathcal{R}),$$

where φ_σ is defined as Proposition 3.1. It's obvious that $[-_\lambda-]^\sigma$ is well-defined since φ_σ is a bijective map. And it is bilinear because both φ_σ and φ_σ^{-1} are $\mathbb{C}[\partial]$-module homomorphisms.

A direct computation shows that

$$\begin{aligned} &[\partial f_\lambda g]^\sigma = \varphi_\sigma^{-1}([(\varphi_\sigma(\partial f))_\lambda(\varphi_\sigma(g))]) \\ &= \varphi_\sigma^{-1}([(\varphi_\sigma(-\lambda f))_\lambda(\varphi_\sigma(g))]) = -\lambda\varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]) = -\lambda[f_\lambda g]^\sigma, \end{aligned}$$

and

$$[f_\lambda g]^\sigma = \varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]) = -\varphi_\sigma^{-1}([(\varphi_\sigma(g))_{-\partial-\lambda}(\varphi_\sigma(f))]) = -[g_{-\partial-\lambda} f]^\sigma,$$

for any f, g in $\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R})$.

To check the Jacobi identity, we compute

$$\begin{aligned} &[[f_\lambda g]^\sigma_{\lambda+\mu} h]^\sigma = [\varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))])_{\lambda+\mu} h]^\sigma \\ &= \varphi_\sigma^{-1}([\varphi_\sigma(\varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]))_{\lambda+\mu}(\varphi_\sigma(h))]) \\ &= \varphi_\sigma^{-1}([[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]_{\lambda+\mu}(\varphi_\sigma(h))]), \end{aligned}$$

for any f, g in $\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R})$. Similarly,

$$[f_\lambda[g_\mu h]^\sigma]^\sigma = \varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda[(\varphi_\sigma(g))_\mu(\varphi_\sigma(h))]]),$$

$$[g_\mu[f_\lambda h]^\sigma]^\sigma = \varphi_\sigma^{-1}([(\varphi_\sigma(g))_\mu[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(h))]]).$$

So

$$\begin{aligned}
&[[f_\lambda g]^\sigma_{\lambda+\mu}h]^\sigma - [f_\lambda[g_\mu h]^\sigma]^\sigma + [g_\mu[f_\lambda h]^\sigma]^\sigma \\
= &\varphi_\sigma^{-1}([[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]_{\lambda+\mu}(\varphi_\sigma(h))]) - \varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda[(\varphi_\sigma(g))_\mu(\varphi_\sigma(h))]]) \\
&+ \varphi_\sigma^{-1}([(\varphi_\sigma(g))_\mu[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(h))]]) \\
= &\varphi_\sigma^{-1}([[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]_{\lambda+\mu}(\varphi_\sigma(h))] - [\varphi_\sigma(f)_\lambda[(\varphi_\sigma(g))_\mu(\varphi_\sigma(h))]] \\
&+ [(\varphi_\sigma(g))_\mu[(\varphi_\sigma(f))_\lambda(\varphi_\sigma(h))]]) \\
= &\varphi_\sigma^{-1}(0) = 0,
\end{aligned}$$

which means that $[f_\lambda[g_\mu h]^\sigma]^\sigma = [[f_\lambda g]^\sigma_{\lambda+\mu}h]^\sigma + [g_\mu[f_\lambda h]^\sigma]^\sigma$. Hence, $(\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R}), [-_\lambda-]^\sigma)$ is a Lie conformal algebra.

To complete the proof, it still needs to show that φ_σ is a Lie conformal algebras homomorphism. Actually, $\varphi_\sigma([f_\lambda g]^\sigma) = \varphi_\sigma(\varphi_\sigma^{-1}([(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))])) = [(\varphi_\sigma(f))_\lambda(\varphi_\sigma(g))]$ and φ_σ is a Lie conformal algebras homomorphism between $\mathrm{CDer}_{\sigma,\sigma}(\mathcal{R})$ and $\mathrm{CDer}(\mathcal{R})$, as desired. □

Recall that the center of a Lie conformal algebra $\mathcal{R}$ is the set $\mathrm{Z}(\mathcal{R}) = \{a \in \mathcal{R} \mid [a_\lambda b] = 0, \forall b \in \mathcal{R}\}$, and the centralizer of a in $\mathcal{R}$ is the set $\mathrm{Z}_a(\mathcal{R}) = \{b \in \mathcal{R} \mid [a_\lambda b] = 0\}$.

Proposition 3.3. *Let σ and τ be two elements in G such that $(\sigma - \tau)(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$. Then $\mathrm{CDer}_\sigma(\mathcal{R}) = \mathrm{CDer}_\tau(\mathcal{R})$. In addition, if $(\sigma - \mathrm{id}_\mathcal{R})(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$, then $\mathrm{CDer}_\sigma(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$ is a Lie conformal subalgebra of $\mathrm{Cend}(\mathcal{R})$.*

Proof. For any d in $\mathrm{CDer}_\sigma(\mathcal{R})$ and since $(\sigma - \tau)(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$, we have

$$[(d_\lambda(a))_{\lambda+\mu}((\sigma - \tau)(b))] = 0, \quad \forall a, b \in \mathcal{R},$$

that is $[(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] = [(d_\lambda(a))_{\lambda+\mu}(\tau(b))]$. So we can get

$$d_\lambda([a_\mu b]) = [(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] + [a_\mu(d_\lambda(b))] = [(d_\lambda(a))_{\lambda+\mu}(\tau(b))] + [a_\mu(d_\lambda(b))],$$

which implies that $d \in \mathrm{CDer}_\tau(\mathcal{R})$ and $\mathrm{CDer}_\sigma(\mathcal{R}) \subseteq \mathrm{CDer}_\tau(\mathcal{R})$. By switching the roles of σ and τ, we can get $\mathrm{CDer}_\tau(\mathcal{R}) \subseteq \mathrm{CDer}_\sigma(\mathcal{R})$. Consequently, we obtain $\mathrm{CDer}_\sigma(\mathcal{R}) = \mathrm{CDer}_\tau(\mathcal{R})$.

In particular, if we take $\tau = \mathrm{id}_\mathcal{R}$, then $\mathrm{CDer}_\sigma(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$ is a Lie conformal subalgebra of $\mathrm{Cend}(\mathcal{R})$. □

Proposition 3.4. *Let G be an abelian group. If σ and σ' are two elements in G such that σ commutes with every element of $\mathrm{CDer}_{\sigma'}(\mathcal{R})$ and σ' commutes with every element of $\mathrm{CDer}_\sigma(\mathcal{R})$, then $[f_\lambda g]_\mu \in \mathrm{CDer}_{\sigma\sigma'}(\mathcal{R})$ for any $f \in \mathrm{CDer}_\sigma(\mathcal{R})$ and $g \in \mathrm{CDer}_{\sigma'}(\mathcal{R})$.*

Proof. For any $a, b \in \mathcal{R}$, we observe that

$$\begin{aligned}
&f_\lambda(g_{\mu-\lambda}([a_\gamma b])) = f_\lambda([(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\sigma'(b))] + [a_\gamma(g_{\mu-\lambda}(b))]) \\
= &[(f_\lambda(g_{\mu-\lambda}(a)))_{\mu+\gamma}(\sigma(\sigma'(b)))] + [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(f_\lambda(\sigma'(b)))] \\
&+ [(f_\lambda(a))_{\lambda+\gamma}(\sigma(g_{\mu-\lambda}(b)))] + [a_\gamma(f_\lambda(g_{\mu-\lambda}(b)))],
\end{aligned}$$

and

$$\begin{aligned}
&g_{\mu-\lambda}(f_\lambda([a_\gamma b])) = g_{\mu-\lambda}([(f_\lambda(a))_{\lambda+\gamma}(\sigma(b))] + [a_\gamma(f_\lambda(b))]) \\
= &[(g_{\mu-\lambda}(f_\lambda(a)))_{\mu+\gamma}(\sigma'(\sigma(b)))] + [(f_\lambda(a))_{\lambda+\gamma}(g_{\mu-\lambda}(\sigma(b)))] \\
&+ [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\sigma'(f_\lambda(b)))] + [a_\gamma(g_{\mu-\lambda}(f_\lambda(b)))].
\end{aligned}$$

According to the assumption, one can obtain that σ commutes with g and $\sigma^{'}$ commutes with f. By the fact that $\sigma\,\sigma^{'} = \sigma^{'}\sigma$, we can easily get

$$\begin{aligned}
&[f_\lambda g]_\mu([a_\gamma b]) = (f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda)([a_\gamma b]) = f_\lambda(g_{\mu-\lambda}([a_\gamma b])) - g_{\mu-\lambda}(f_\lambda([a_\gamma b]))\\
&= [(f_\lambda(g_{\mu-\lambda}(a)))_{\mu+\gamma}(\sigma(\sigma^{'}(b)))] + [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(f_\lambda(\sigma^{'}(b)))]\\
&\quad + [(f_\lambda(a))_{\lambda+\gamma}(\sigma(g_{\mu-\lambda}(b)))] + [a_\gamma(f_\lambda(g_{\mu-\lambda}(b)))]\\
&\quad - [(g_{\mu-\lambda}(f_\lambda(a)))_{\mu+\gamma}(\sigma^{'}(\sigma(b)))] - [(f_\lambda(a))_{\lambda+\gamma}(g_{\mu-\lambda}(\sigma(b)))]\\
&\quad - [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\sigma^{'}(f_\lambda(b)))] - [a_\gamma(g_{\mu-\lambda}(f_\lambda(b)))]\\
&= [([f_\lambda g]_\mu(a))_{\mu+\gamma}(\sigma(\sigma^{'}(b)))] + [a_\gamma([f_\lambda g]_\mu(b))].
\end{aligned}$$

Therefore, $[f_\lambda g]_\mu \in \mathrm{CDer}_{\sigma\sigma'}(\mathcal{R})$. □

Corollary 3.5. *Let σ be an idempotent automorphism of G. If σ commutes with every element of* $\mathrm{CDer}_\sigma(\mathcal{R})$*, then* $\mathrm{CDer}_\sigma(\mathcal{R})$ *is a Lie conformal algebra.*

Proof. Note that $\mathrm{CDer}_\sigma(\mathcal{R})$ is a $\mathbb{C}[\partial]$-module, thus it suffices to verify that $\mathrm{CDer}_\sigma(\mathcal{R})$ is closed under the λ-bracket, that is

$$[f_\lambda g]_\mu = f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda \in \mathrm{CDer}_\sigma(\mathcal{R}), \quad \forall f, g \in \mathrm{CDer}_\sigma(\mathcal{R}).$$

With a similar discussion as that in the proof of Proposition 3.4,

$$[f_\lambda g]_\mu([a_\gamma b])) = [([f_\lambda g]_\mu(a))_{\mu+\gamma}\sigma^2(b)] + [a_\gamma([f_\lambda g]_\mu(b))], \quad \forall a, b \in \mathcal{R}.$$

By the fact that $\sigma^2 = \sigma$, we can deduce that $[f_\lambda g]_\mu \in \mathrm{CDer}_\sigma(\mathcal{R})[\lambda]$. Therefore, $\mathrm{CDer}_\sigma(\mathcal{R})$ is a Lie conformal algebra. □

Proposition 3.6. *Let $\sigma, \tau \in G$ and $d \in \mathrm{CDer}_\sigma(\mathcal{R})$. Then $\tau d \in \mathrm{CDer}_{\tau\sigma,\tau}(\mathcal{R})$ and $d\tau \in \mathrm{CDer}_{\sigma\tau,\tau}(\mathcal{R})$.*

Proof. A direct computation shows that

$$\begin{aligned}
&\tau d_\lambda([a_\mu b]) = \tau([(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] + [a_\mu(d_\lambda(b))])\\
&= \tau([(d_\lambda(a))_{\lambda+\mu}(\sigma(b))]) + \tau([a_\mu(d_\lambda(b))])\\
&= [(\tau(d_\lambda(a)))_{\lambda+\mu}(\tau(\sigma(b)))] + [(\tau(a))_\mu(\tau(d_\lambda(b)))],
\end{aligned}$$

for any $a, b \in \mathcal{R}$. Thus, $\tau d \in \mathrm{CDer}_{\tau\sigma,\tau}(\mathcal{R})$.

Similarly, we can obtain that $d\tau \in \mathrm{CDer}_{\sigma\tau,\tau}(\mathcal{R})$. □

Proposition 3.7. *If σ, τ are two elements in G such that $c - \sigma^{-1}\tau(c) \notin Z_c(\mathcal{R})$ for any nonzero element c in $\mathcal{R}$, then* $\mathrm{CDer}_\sigma(\mathcal{R}) \cap \mathrm{CDer}_\tau(\mathcal{R}) = \{0\}$.

Proof. Assume that there exists a nonzero element $d \in \mathrm{CDer}_\sigma(\mathcal{R}) \cap \mathrm{CDer}_\tau(\mathcal{R})$. Then there exists an element $a_0 \in \mathcal{R}$ such that $d_\lambda(a_0) \neq 0$. So

$$[(d_\lambda(a))_{\lambda+\mu}(\sigma(b))] = [(d_\lambda(a))_{\lambda+\mu}(\tau(b))], \quad \forall a, b \in \mathcal{R},$$

which implies that

$$[(\sigma^{-1}(d_\lambda(a)))_{\lambda+\mu}(b - \sigma^{-1}\tau(b))] = 0.$$

If we take $a = a_0$ and $b = b_0 := \sigma^{-1}d_\lambda(a_0)$, then we can get

$$[b_{0_{\lambda+\mu}}(b_0 - \sigma^{-1}\tau(b_0))] = 0,$$

which means that $b_0 - \sigma^{-1}\tau(b_0) \in Z_{b_0}(\mathcal{R})$. Since $c - \sigma^{-1}\tau(c) \notin Z_c(\mathcal{R})$ with $c \in \mathcal{R}$ and $c \neq 0$, we can deduce that $b_0 = 0$. According to the assumption, $d_\lambda(a_0) \neq 0$ and σ^{-1} is an isomorphism, we observe that $b_0 = \sigma^{-1}d_\lambda(a_0) \neq 0$, which is a contradiction. □

Proposition 3.8. *If $d \in \mathrm{CDer}_\sigma(\mathcal{R})$ is an element such that $(d_\lambda\sigma - \sigma d_\lambda)(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$, then $[\mathcal{R}_\lambda\mathcal{R}]$ is contained in the kernel of $d_\lambda\sigma - \sigma d_\lambda$.*

Proof. We compute, respectively,

$$d_\lambda(\sigma([a_\mu b])) = d_\lambda([(\sigma(a))_\mu(\sigma(b))]) = [(d_\lambda(\sigma(a)))_{\lambda+\mu}(\sigma^2(b))] + [(\sigma(a))_\mu(d_\lambda(\sigma(b)))],$$

and

$$\begin{aligned} &\sigma(d_\lambda([a_\mu b])) = \sigma([(d_\lambda(a))_{\lambda+\mu}(\sigma(b))]) + [a_\mu(d_\lambda(b))] \\ &= [(\sigma(d_\lambda(a)))_{\lambda+\mu}(\sigma^2(b))] + [(\sigma(a))_\mu(\sigma(d_\lambda(b)))], \end{aligned}$$

for any $a, b \in \mathcal{R}$. Since $(d_\lambda\sigma - \sigma d_\lambda)(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$, we can get

$$\begin{aligned} &(d_\lambda\sigma - \sigma d_\lambda)([a_\mu b]) = d_\lambda\sigma([a_\mu b]) - \sigma d_\lambda([a_\mu b]) \\ &= [d_\lambda(\sigma(a))_{\lambda+\mu}(\sigma^2(b))] + [(\sigma(a))_\mu(d_\lambda(\sigma(b)))] \\ &\quad - [(\sigma(d_\lambda(a)))_{\lambda+\mu}(\sigma^2(b))] - [(\sigma(a))_\mu(\sigma(d_\lambda(b)))] \\ &= [((d_\lambda\sigma - \sigma d_\lambda)(a))_{\lambda+\mu}(\sigma^2(b))] + [(\sigma(a))_\mu((d_\lambda\sigma - \sigma d_\lambda)(b))] \\ &= 0. \end{aligned}$$

Consequently, $[\mathcal{R}_\lambda\mathcal{R}]$ is contained in the kernel of $d_\lambda\sigma - \sigma d_\lambda$. □

4. The interiors of conformal G-derivations

In this section, we will investigate the structures of $\mathrm{CDer}_\sigma(\mathcal{R})$ and $\mathrm{CDer}_G(\mathcal{R})$. To understand this, we focus on a special class of $\mathrm{CDer}_\sigma(\mathcal{R})$ called the *interiors* of G-derivations, $\mathrm{CDer}_G^\star(\mathcal{R})$. In addition, we study the rationality of the Hilbert series for the direct sum of these interiors of conformal G- derivations when G is a cyclic subgroup.

Set

$$\mathrm{CDer}_\sigma^+(\mathcal{R}) = \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid d_\lambda\sigma = \sigma d_\lambda\},\ \mathrm{CDer}_\sigma^-(\mathcal{R}) = \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid d_\lambda\tau = \tau d_\lambda,\ \forall\tau \in G\}.$$

It is obvious that $\mathrm{CDer}_\sigma^-(\mathcal{R}) \subseteq \mathrm{CDer}_\sigma^+(\mathcal{R}) \subseteq \mathrm{CDer}_\sigma(\mathcal{R})$ and they are all $\mathbb{C}[\partial]$-modules. We now consider some kind of "sum" of them respectively and observe how close these sums are to $\mathrm{CDer}_G(\mathcal{R})$.

Define

$$\mathrm{CDer}_G^+(\mathcal{R}) := \oplus_{\sigma\in G}\mathrm{CDer}_\sigma^+(\mathcal{R}),\ \mathrm{CDer}_G^-(\mathcal{R}) := \oplus_{\sigma\in G}\mathrm{CDer}_\sigma^-(\mathcal{R}),$$

called the *big interior* and the *small interior* of $\mathrm{CDer}_G(\mathcal{R})$ respectively. Besides, we may define

$$\mathrm{CDer}_G^\star(\mathcal{R}) := \oplus_{\sigma\in G}\mathrm{CDer}_\sigma(\mathcal{R}),$$

called the *interior* of $\mathrm{CDer}_G(\mathcal{R})$. Obviously, $\mathrm{CDer}_G^-(\mathcal{R}) \subseteq \mathrm{CDer}_G^+(\mathcal{R}) \subseteq \mathrm{CDer}_G^\star(\mathcal{R})$.

Example 4.1. *Let $G = \{\mathrm{id}_\mathcal{R}\}$, the trivial group. Since $\mathrm{CDer}_G^-(\mathcal{R}) = \mathrm{CDer}_G^+(\mathcal{R})$, we have $\mathrm{CDer}_G^-(\mathcal{R}) = \mathrm{CDer}_G^+(\mathcal{R}) = \mathrm{CDer}_G^\star(\mathcal{R}) = \mathrm{CDer}_G(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$.*

Example 4.2. *Let G be a cyclic group generator by σ. If $* \in \{-, +, \star\}$, then*

$$\mathrm{CDer}_G^*(\mathcal{R}) = \mathrm{CDer}_{\langle\sigma\rangle}^*(\mathcal{R}) = \oplus_{k\in\mathbb{Z}}\mathrm{CDer}_{\sigma^k}^*(\mathcal{R}),$$

where $\sigma^0 = \mathrm{id}_\mathcal{R}$, $\sigma^1 = \sigma$ and $\sigma^k = \sigma^{k-1}\sigma$. For convenience, we denote $\mathrm{CDer}_{\sigma^k}^\star(\mathcal{R})$ by $\mathrm{CDer}_{\sigma^k}(\mathcal{R})$. In this case, $\mathrm{CDer}_{\langle\sigma\rangle}^*(\mathcal{R})$ is a $\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module and recall that the Hilbert series of $\mathrm{CDer}_{\langle\sigma\rangle}^*(\mathcal{R})$ is defined by

$$H(\mathrm{CDer}_{\langle\sigma\rangle}^*(\mathcal{R}), t) := \sum_{k\in\mathbb{Z}} \mathrm{rank}(\mathrm{CDer}_{\sigma^k}^*(\mathcal{R}))t^k.$$

If σ is of finite order, then $H(\mathrm{CDer}_{\langle\sigma\rangle}^*(\mathcal{R}), t)$ is a polynomial function in $\mathbb{Z}[t]$.

Proposition 4.3. *If G is an abelian group, then $\mathrm{CDer}_G^-(\mathcal{R})$ is a Lie conformal algebra with the λ-bracket $[-_\lambda-]$.*

Proof. Since $\mathrm{CDer}_G^-(\mathcal{R})$ is a $\mathbb{C}[\partial]$-module, it is sufficient to show that $\mathrm{CDer}_G^-(\mathcal{R})$ is closed under the λ-bracket. For any $f \in \mathrm{CDer}_\sigma^-(\mathcal{R})$ and $g \in \mathrm{CDer}_\tau^-(\mathcal{R})$ with $\sigma, \tau \in G$ and $a, b \in \mathcal{R}$, we have

$$\begin{aligned} & f_\lambda(g_{\mu-\lambda}([a_\gamma b])) = f_\lambda([(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\tau(b))] + [a_\gamma(g_{\mu-\lambda}(b))]) \\ = & [(f_\lambda(g_{\mu-\lambda}(a)))_{\mu+\gamma}(\sigma(\tau(b)))] + [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(f_\lambda(\tau(b)))] \\ & + [(f_\lambda(a))_{\lambda+\gamma}(\sigma(g_{\mu-\lambda}(b)))] + [a_\gamma(f_\lambda(g_{\mu-\lambda}(b)))], \end{aligned}$$

and

$$\begin{aligned} & g_{\mu-\lambda}(f_\lambda([a_\gamma b])) = g_{\mu-\lambda}([(f_\lambda(a))_{\lambda+\gamma}(\sigma(b))] + [a_\gamma(f_\lambda(b))]) \\ = & [(g_{\mu-\lambda}(f_\lambda(a)))_{\mu+\gamma}(\tau(\sigma(b)))] + [(f_\lambda(a))_{\lambda+\gamma}(g_{\mu-\lambda}(\sigma(b)))] \\ & + [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\tau(f_\lambda(b)))] + [a_\gamma(g_{\mu-\lambda}(f_\lambda(b)))]. \end{aligned}$$

Since $f_\lambda\tau = \tau f_\lambda$, $g_{\mu-\lambda}\sigma = \sigma g_{\mu-\lambda}$ and G is abelian, we compute and get

$$\begin{aligned} & [f_\lambda g]_\mu([a_\gamma b]) = (f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda)([a_\gamma b]) \\ = & [(f_\lambda(g_{\mu-\lambda}(a)))_{\mu+\gamma}(\sigma(\tau(b)))] + [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(f_\lambda(\tau(b)))] \\ & + [(f_\lambda(a))_{\lambda+\gamma}(\sigma(g_{\mu-\lambda}(b)))] + [a_\gamma(f_\lambda(g_{\mu-\lambda}(b)))] \\ & - [(g_{\mu-\lambda}(f_\lambda(a)))_{\mu+\gamma}(\tau(\sigma(b)))] - [(f_\lambda(a))_{\lambda+\gamma}(g_{\mu-\lambda}(\sigma(b)))] \\ & - [(g_{\mu-\lambda}(a))_{\mu-\lambda+\gamma}(\tau(f_\lambda(b)))] - [a_\gamma(g_{\mu-\lambda}(f_\lambda(b)))]. \\ = & [(f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda)(a)_{\mu+\gamma}(\sigma(\tau(b)))] + [a_\gamma((f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda)(b))] \\ = & [([f_\lambda g]_\mu(a))_{\mu+\gamma}(\sigma(\tau(b)))] + [a_\gamma([f_\lambda g]_\mu(b))], \end{aligned}$$

which implies that $[f_\lambda g]_\mu \in \mathrm{CDer}_{\sigma\tau}(\mathcal{R})[\lambda]$. Obviously, $[f_\lambda g]_\mu$ commutes with every element in G, and so $[f_\lambda g]_\mu \in \mathrm{CDer}_{\sigma\tau}^-(\mathcal{R})[\lambda] \subseteq \mathrm{CDer}_G^-(\mathcal{R})[\lambda]$. Consequently, $\mathrm{CDer}_G^-(\mathcal{R})$ is a Lie conformal algebra. □

According to the above results, we can see that $\mathrm{CDer}_G(\mathcal{R})$ may be very large and complicated. From now on, we will focus on the interiors of $\mathrm{CDer}_G(\mathcal{R})$ where G is an infinite cyclic group. Particularly, we will investigate the important invariant, the Hilbert series, which encodes the ranks of submodules into an infinite series.

Proposition 4.4. *Let $G = \langle\sigma\rangle$ be an infinite cyclic group. If there exists $l_0 \in \mathbb{N}^+$ and $d \in \mathrm{CDer}_{\sigma^{l_0}}(\mathcal{R})$ such that ϕ_d is invertible restricted to $\mathrm{CDer}_{\sigma^i}(\mathcal{R})$ for all $i \in \mathbb{Z} \setminus \{l_0\}$, then $H(\mathrm{CDer}_G^-(\mathcal{R}), t)$ is a rational function.*

Proof. Since G is an infinite cyclic group generated by σ, $\phi_d : \mathrm{CDer}_{\sigma^k}^-(\mathcal{R}) \to \mathrm{CDer}_{\sigma^{k+l_0}}^-(\mathcal{R})$ is a $\mathbb{C}[\partial]$-module isomorphism for all $k \in \mathbb{Z} \setminus \{l_0\}$ by Proposition 4.3. Hence,

$$\mathrm{rank}(\mathrm{CDer}_{\sigma^k}^-(\mathcal{R})) = \mathrm{rank}(\mathrm{CDer}_{\sigma^{k+l_0}}^-(\mathcal{R})) = \mathrm{rank}(\mathrm{CDer}_{\sigma^{k-l_0}}^-(\mathcal{R}))$$

for each $k \in \mathbb{N} \setminus \{l_0\}$. Obviously,

$$H(\mathrm{CDer}_G^-(\mathcal{R}), t) = \sum_{k\in\mathbb{Z}} \mathrm{rank}(\mathrm{CDer}_{\sigma^k}^-(\mathcal{R}))\mathrm{t}^{\mathrm{k}} = \sum_{k=l_0+1}^{\infty} \mathrm{rank}(\mathrm{CDer}_{\sigma^k}^-(\mathcal{R}))t^k + \sum_{-\infty}^{k=l_0} \mathrm{rank}(\mathrm{CDer}_{\sigma^k}^-(\mathcal{R}))t^k.$$

In addition,

$$
\begin{aligned}
&\sum_{k=l_0+1}^{\infty} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^k}(\mathcal{R}))t^k \\
&= (m_0t^{l_0+1} + m_1t^{l_0+2} + \cdots + m_{l_0-1}t^{2l_0}) + (m_0t^{2l_0+1} + m_1t^{2l_0+2} + \cdots + m_{l_0-1}t^{3l_0}) + \cdots \\
&= (m_0 + m_1t + \cdots + m_{l_0-1}t^{l_0-1})t^{l_0+1} + (m_0 + m_1t + \cdots + m_{l_0-1}t^{l_0-1})t^{2l_0+1} + \cdots \\
&= t^{l_0+1}(m_0 + m_1t + \cdots + m_{l_0-1}t^{l_0-1})(1 + t^{l_0} + t^{2l_0} + \cdots) \\
&= t^{l_0+1}(m_0 + m_1t + \cdots + m_{l_0-1}t^{l_0-1})\frac{1}{1-t^{l_0}} \\
&= \frac{t^{l_0+1}\sum_{i=0}^{l_0-1} m_it^i}{1-t^{l_0}}
\end{aligned}
$$

where $m_i = \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^{l_0+1+i}}(\mathcal{R}))$ for $0 \le i \le l_0 - 1$. Similarly,

$$
\begin{aligned}
&\sum_{-\infty}^{k=l_0} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^k}(\mathcal{R}))t^k \\
&= \sum_{k=1}^{l_0} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^k}(\mathcal{R}))t^k + \sum_{-\infty}^{k=0} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^k}(\mathcal{R}))t^k \\
&= \sum_{k=1}^{l_0} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^k}(\mathcal{R}))t^k + \sum_{k=0}^{\infty} \operatorname{rank}(\mathrm{CDer}^{-}_{\sigma^{-k}}(\mathcal{R}))t^k \\
&= m_0t + \cdots + m_{l_0-1}t^{l_0} + (m_0t^{-l_0+1} + m_1t^{-l_0+2} + \cdots + m_{l_0-1}) \\
&\quad + (m_0t^{-2l_0+1} + m_1t^{-2l_0+2} + \cdots + m_{l_0-1}t^{-l_0}) + \cdots \\
&= m_0t + \cdots + m_{l_0-1}t^{l_0} + (m_0t^{-l_0+1} + m_1t^{-l_0+2} + \cdots + m_{l_0-1})(1 + t^{-l_0} + \cdots) \\
&= (m_0 + \cdots + m_{l_0-1}t^{l_0-1})t + \frac{\sum_{i=0}^{l_0-1} m_it^{-(l_0-1-i)}}{1-t^{-l_0}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&H(\mathrm{CDer}^{-}_{G}(\mathcal{R}), t) \\
&= \frac{t^{l_0+1}\sum_{i=0}^{l_0-1} m_it^i}{1-t^{l_0}} + (m_0 + \cdots + m_{l_0-1}t^{l_0-1})t + \frac{\sum_{i=0}^{l_0-1} m_it^{-(l_0-1-i)}}{1-t^{-l_0}} \\
&= \frac{t}{1-t^{l_0}}\sum_{i=0}^{l_0-1} m_it^i + \frac{1}{1-t^{-l_0}}\sum_{i=0}^{l_0-1} m_it^{-(l_0-1-i)}.
\end{aligned}
$$

Consequently, $H(\mathrm{CDer}^{-}_{G}(\mathcal{R}), t)$ is a rational function. □

5. Applications

In this section, we study the relation between conformal (σ, τ)-derivation and some well-known (generalized) conformal derivations of a Lie conformal algebra $\mathcal{R}$, such as centroids and conformal (α, β, γ)-derivations.

5.1. Relation with centroids

Recall that an element d in $\mathrm{Cend}(\mathcal{R})$ is called a *centroid* of $\mathcal{R}$, if it satisfies

$$[(d_\lambda(a))_{\lambda+\mu}b] = [a_\mu(d_\lambda(b))] = d_\lambda([a_\mu b]),\ \forall\, a, b \in \mathcal{R}.$$

Denote by $C(\mathcal{R})$ the sets of all centroids of $\mathcal{R}$.

We denote $\mathrm{ad} : \mathcal{R} \to \mathrm{Cend}(\mathcal{R})$ the adjoint map sending a to $\mathrm{ad}(a)$ with $\mathrm{ad}(a)_\lambda(b) = [a_\lambda b]$, where $a, b \in \mathcal{R}$. And we write $\mathrm{ad}(\mathcal{R})$ for the set $\{\mathrm{ad}(a) \mid a \in \mathcal{R}\}$.

Proposition 5.1. *Let $\sigma \in G$ and $d \in C(\mathcal{R}) \cap \mathrm{CDer}_\sigma(\mathcal{R})$. Then $\mathrm{ad}(d_\lambda(a))_\mu = 0$ for any $a \in \mathcal{R}$. In addition, if $Z(\mathcal{R}) = \{0\}$, then $C(\mathcal{R}) \cap \mathrm{CDer}_\sigma(\mathcal{R}) = \{0\}$.*

Proof. For any $d \in C(\mathcal{R}) \cap \mathrm{CDer}_\sigma(\mathcal{R})$ and $a, b \in \mathcal{R}$, we have

$$d_\lambda([a_\gamma b]) = [(d_\lambda(a))_{\lambda+\gamma}(\sigma(b))] + [a_\gamma(d_\lambda(b))],$$

and

$$[a_\gamma(d_\lambda(b))] = d_\lambda([a_\lambda b]),$$

which implies

$$[(d_\lambda(a))_{\lambda+\gamma}(\sigma(b))] = d_\lambda([a_\gamma b]) - [a_\gamma(d_\lambda(b))] = 0.$$

Since σ is a bijective map, we can obtain $d_\lambda(a) \in Z(\mathcal{R})[\lambda] = \mathrm{Ker}(\mathrm{ad})[\lambda]$. Hence, $\mathrm{ad}(d_\lambda(a))_\mu = 0$ for any $a \in \mathcal{R}$.

Particularly, if $Z(\mathcal{R}) = \{0\}$, then $d = 0$. Thus, $C(\mathcal{R}) \cap \mathrm{CDer}_\sigma(\mathcal{R}) = \{0\}$. □

Lemma 5.2. *If $\sigma \in G$ and $d \in \mathrm{CDer}_\sigma(\mathcal{R})$, then for any $a \in \mathcal{R}$,*

$$[d_\lambda(\mathrm{ad}(a))]_\mu = \sigma\mathrm{ad}(\sigma^{-1}d_\lambda(a))_\mu.$$

Proof. A direct computation shows that

$$\begin{aligned}
&[d_\lambda(\mathrm{ad}(a))]_\mu(b) = (d_\lambda(\mathrm{ad}(a))_{\mu-\lambda} - (\mathrm{ad}(a))_{\mu-\lambda}d_\lambda)(b)\\
&= d_\lambda((\mathrm{ad}(a))_{\mu-\lambda}(b)) - (\mathrm{ad}(a))_{\mu-\lambda}(d_\lambda(b)) = d_\lambda([a_{\mu-\lambda}b]) - [a_{\mu-\lambda}(d_\lambda(b))]\\
&= [(d_\lambda(a))_\mu(\sigma(b))] = \sigma([(\sigma^{-1}(d_\lambda(a)))_\mu b]) = \sigma(\mathrm{ad}(\sigma^{-1}d_\lambda(a))_\mu(b)),
\end{aligned}$$

for any $b \in \mathcal{R}$. Therefore, $[d_\lambda \mathrm{ad}(a)]_\mu = \sigma\mathrm{ad}(\sigma^{-1}d_\lambda(a))_\mu$. □

Lemma 5.3. *Let $a \in \mathcal{R}$ and $\sigma \in G$. Define a map $\phi_a^\sigma : \mathrm{CDer}_\sigma(\mathcal{R}) \to \mathrm{ad}(\mathcal{R})_\gamma$, given by $d \mapsto \mathrm{ad}(\sigma^{-1}d_\lambda(a))$. Then ϕ_a^σ is a $\mathbb{C}[\partial]$-module homomorphism.*

Proof. For any $f, g \in \mathrm{CDer}_\sigma(\mathcal{R})$ and $b \in \mathcal{R}$, we observe that

$$\begin{aligned}
&(\phi_a^\sigma(f_\lambda + g_\mu))(b) = \mathrm{ad}(\sigma^{-1}((f_\lambda + g_\mu)(a)))_\gamma(b) = [(\sigma^{-1}((f_\lambda + g_\mu)(a)))_\gamma b]\\
&= \sigma^{-1}([((f_\lambda + g_\mu)(a))_\gamma(\sigma(b))]) = \sigma^{-1}([(f_\lambda(a))_\gamma(\sigma(b))] + [(g_\mu(a))_\gamma(\sigma(b))])\\
&= \sigma^{-1}([(f_\lambda(a))_\gamma(\sigma(b))]) + \sigma^{-1}([(g_\mu(a))_\gamma(\sigma(b))]) = [(\sigma^{-1}(f_\lambda(a)))_\gamma b] + [(\sigma^{-1}(g_\mu(a)))_\gamma b]\\
&= \mathrm{ad}(\sigma^{-1}(f_\lambda(a)))_\gamma(b) + \mathrm{ad}(\sigma^{-1}(g_\mu(a)))_\gamma(b) = (\phi_a^\sigma(f_\lambda) + \phi_a^\sigma(g_\mu))_\gamma(b),
\end{aligned}$$

and

$$\begin{aligned}
&(\phi_a^\sigma(\partial f_\lambda))_\gamma(b) = \mathrm{ad}(\sigma^{-1}(\partial f_\lambda)(a))_\gamma(b) = \mathrm{ad}(\sigma^{-1}(-\lambda f_\lambda)(a))_\gamma(b)\\
&= \mathrm{ad}(-\lambda\sigma^{-1}f_\lambda(a))_\gamma(b) = [-\lambda\sigma^{-1}f_\lambda(a)_\gamma b] = -\lambda[\sigma^{-1}f_\lambda(a)_\gamma b]\\
&= -\lambda\mathrm{ad}(\sigma^{-1}(f_\lambda(a)))_\gamma(b) = \partial\mathrm{ad}(\sigma^{-1}(f_\lambda(a)))_\gamma(b) = (\partial(\phi_a^\sigma(f_\lambda)))_\gamma(b).
\end{aligned}$$

Therefore, ϕ_x^σ is a $\mathbb{C}[\partial]$-module homomorphism. □

Proposition 5.4. *If $a \in \mathcal{R}$ and $\sigma \in G$, then*

$$\mathrm{Ker}(\phi_a^\sigma) = \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid d_\lambda(a) \in Z(\mathcal{R})[\lambda]\}.$$

What's more, $\mathrm{Ker}(\phi_a^\sigma)$ is a subalgebra of $\mathrm{Cend}(\mathcal{R})$.

Proof. According to Lemma 5.2, we can get

$$\begin{aligned}\mathrm{Ker}(\phi_a^\sigma) &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid \mathrm{ad}(\sigma^{-1}(d_\lambda(a)))_\mu(b) = 0,\ \forall b \in \mathcal{R}\}\\ &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid \sigma(\mathrm{ad}(\sigma^{-1}d_\lambda(a))_\mu(b)) = 0,\ \forall b \in \mathcal{R}\}\\ &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid [d_\lambda(\mathrm{ad}(a))]_\mu(b) = 0,\ \forall b \in \mathcal{R}\}\\ &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid [(d_\lambda(a))_\mu(\sigma(b))] = 0,\ \forall b \in \mathcal{R}\}\\ &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid [(d_\lambda(a))_\mu b] = 0,\ \forall b \in \mathcal{R}\}\\ &= \{d \in \mathrm{CDer}_\sigma(\mathcal{R}) \mid d_\lambda(a) \in \mathrm{Z}(\mathcal{R})[\lambda]\}.\end{aligned}$$

Moreover, it is obvious that $\mathrm{Ker}(\phi_a^\sigma)$ is a $\mathbb{C}[\partial]$-module. We only need to show $\mathrm{Ker}(\phi_a^\sigma)$ is a Lie conformal algebra. For any $f, g \in \mathrm{Ker}(\phi_a^\sigma)$ and $b \in \mathcal{R}$, we have

$$\begin{aligned}&[([f_\lambda g]_\mu(a))_\gamma(\sigma(b))] = [(f_\lambda(g_{\mu-\lambda}(a)))_\gamma(\sigma(b))] - [(g_{\mu-\lambda}(f_\lambda(a)))_\gamma(\sigma(b))]\\ &= f_\lambda([(g_{\mu-\lambda}(a))_{\gamma-\lambda}b]) - [(g_{\mu-\lambda}(a))_{\gamma-\lambda}(f_\lambda(b))]\\ &\quad - g_{\mu-\lambda}([(f_\lambda(a))_{\gamma-\mu+\lambda}b]) + [(f_\lambda(a))_{\gamma-\mu+\lambda}(g_{\mu-\lambda}(b))]\\ &= 0.\end{aligned}$$

Since σ is an isomorphism, we can deduce that $[f_\lambda g]_\mu(a) \in \mathrm{Z}(\mathcal{R})[\lambda]$, which implies that $[f_\lambda g]_\mu(a) \in \mathrm{Ker}(\phi_a^\sigma)[\lambda]$. Therefore, $\mathrm{Ker}(\phi_x^\sigma)$ is a subalgebra of $\mathrm{Cend}(\mathcal{R})$. □

As a corollary, we can obtain a much deeper result.

Corollary 5.5. *Let $\mathcal{R}$ be a centerless Lie conformal algebra. If there exists an element $a_0 \in \mathcal{R}$ such that $d_\lambda(a_0) \neq 0$ for all $d \in \mathrm{CDer}_\sigma(\mathcal{R})$, then* $\mathrm{rank}(\mathrm{CDer}_\sigma(\mathcal{R})) \leq \mathrm{rank}(\mathcal{R})$.

Proof. Note that $\mathrm{ad} : \mathcal{R} \to \mathrm{ad}(\mathcal{R})$ is an isomorphism. Since $\mathrm{Z}(\mathcal{R}) = \{0\}$, $\phi_{a_0}^\sigma$ is injective by Proposition 5.4. Hence, as a $\mathbb{C}[\partial]$-module, $\mathrm{CDer}_\sigma(\mathcal{R})$ can be embedded into $\mathcal{R}$. □

5.2. *Relation with conformal (α, β, γ)-derivations*

Recall that a conformal linear map $d \in \mathrm{Cend}(\mathcal{R})$ is a *conformal (α, β, γ)-derivation* of $\mathcal{R}$ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying that for any $a, b \in \mathcal{R}$,

$$\alpha d_\lambda([a_\mu b]) = [(\beta d_\lambda(a))_{\lambda+\mu} b] + [a_\mu(\gamma d_\lambda(b))]. \tag{3}$$

For any given $\alpha, \beta, \gamma \in \mathbb{C}$, we denote the set of all conformal (α, β, γ)-derivations by $\mathrm{CDer}_{(\alpha,\beta,\gamma)}(\mathcal{R})$, i.e.,

$$\mathrm{CDer}_{(\alpha,\beta,\gamma)}(\mathcal{R}) = \{d \in \mathrm{Cend}(\mathcal{R}) \mid \alpha d_\lambda([a_\mu b]) = [(\beta d_\lambda(a))_{\lambda+\mu} b] + [a_\mu(\gamma d_\lambda(b))],\ \forall\, a, b \in \mathcal{R}\}.$$

We turn now to the problem of relation between conformal (σ, τ)-derivation and conformal (α, β, γ)-derivations.

Lemma 5.6. *Let $\mathcal{R}$ be a Lie conformal algebra. Then*

$$\mathrm{CDer}_{(\alpha,\beta,\gamma)}(\mathcal{R}) = \mathrm{CDer}_{(\frac{\alpha}{\beta+\lambda},1,0)}(\mathcal{R}),$$

for any $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. According to [3, Proposition 4.2], we observe that

$$\begin{aligned}&\mathrm{CDer}_{(\alpha,\beta,\gamma)}(\mathcal{R})\\ &= \mathrm{CDer}_{(0,\beta-\gamma,\gamma-\beta)}(\mathcal{R}) \cap \mathrm{CDer}_{(2\alpha,\beta+\gamma,\beta+\gamma)}(\mathcal{R})\\ &= \mathrm{CDer}_{(0,1,-1)}(\mathcal{R}) \cap \mathrm{CDer}_{(\frac{2\alpha}{\beta+\gamma},1,1)}(\mathcal{R})\\ &= \mathrm{CDer}_{(\frac{\alpha}{\beta+\lambda},1,0)}(\mathcal{R}).\end{aligned}$$

This lemma is proved. □

Proposition 5.7. *Suppose that σ is an element in G and there exists a scalar $\alpha \in \mathbb{C}$ such that $(\sigma - \alpha \mathrm{id}_{\mathcal{R}})(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$. If $\alpha \neq 1$, then $\mathrm{CDer}_{\sigma}(\mathcal{R}) = \mathrm{CDer}_{(\frac{1}{\alpha+1},1,0)}(\mathcal{R})$; if $\alpha = 1$, then $\mathrm{CDer}_{\sigma}(\mathcal{R}) = \mathrm{CDer}_{(1,1,1)}(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$.*

Proof. Assume that $d \in \mathrm{CDer}_{\sigma}(\mathcal{R})$. Since $(\sigma - \alpha \mathrm{id}_{\mathcal{R}})(\mathcal{R}) \subseteq \mathrm{Z}(\mathcal{R})$, we have

$$[(d_\lambda(a))_\mu(\sigma(b))] = [(d_\lambda(a))_\mu(\alpha b)], \quad \forall a, b \in \mathcal{R}.$$

Notice that

$$d \in \mathrm{CDer}_{\sigma}(\mathcal{R}) \Leftrightarrow d_\lambda([a_\mu b]) = [(d_\lambda(a))_\mu(\alpha b)] + [a_\mu(d_\lambda(b))] \Leftrightarrow d \in \mathrm{CDer}_{(1,\alpha,1)}(\mathcal{R}),$$

which implies that $\mathrm{CDer}_{\sigma}(\mathcal{R}) = \mathrm{CDer}_{(1,\alpha,1)}(\mathcal{R})$. If $\alpha \neq 1$, then $\mathrm{CDer}_{(1,\alpha,1)}(\mathcal{R}) = \mathrm{CDer}_{(\frac{1}{\alpha+1},1,0)}(\mathcal{R})$ by Lemma 5.6. Therefore, $\mathrm{CDer}_{\sigma}(\mathcal{R}) = \mathrm{CDer}_{(\frac{1}{\alpha+1},1,0)}(\mathcal{R})$. If $\alpha = 1$, then $\mathrm{CDer}_{\sigma}(\mathcal{R}) = \mathrm{CDer}_{(1,1,1)}(\mathcal{R}) = \mathrm{CDer}(\mathcal{R})$. □

Proposition 5.8. *Let $\mathcal{R}$ be a Lie conformal algebra. If $\delta \neq 0$, then $\mathrm{CDer}_{(\delta,1,-1)}(\mathcal{R}) = \mathrm{CDer}_{\mathrm{id}_{\mathcal{R}},-\mathrm{id}_{\mathcal{R}}}(\mathcal{R})$ and $\mathrm{CDer}_{(\delta,1,1)}(\mathcal{R}) = \mathrm{CDer}_{\frac{1}{\delta}\mathrm{id}_{\mathcal{R}},\frac{1}{\delta}\mathrm{id}_{\mathcal{R}}}(\mathcal{R})$.*

Proof. According to [3, Proposition 4.2], we can get

$$\begin{aligned}
&\mathrm{CDer}_{(\delta,1,-1)}(\mathcal{R})\\
&= \mathrm{CDer}_{(0,2,-2)}(\mathcal{R}) \cap \mathrm{CDer}_{(2\delta,0,0)}(\mathcal{R})\\
&= \mathrm{CDer}_{(0,2,-2)}(\mathcal{R}) \cap \mathrm{CDer}_{(2,0,0)}(\mathcal{R})\\
&= \mathrm{CDer}_{(1,1,-1)}(\mathcal{R}).
\end{aligned}$$

Note that

$$d_\lambda([a_\mu b]) = [(d_\lambda(a))_{\lambda+\mu} b] - [a_\mu(d_\lambda(b))],$$

for any $d \in \mathrm{CDer}_{(1,1,-1)}(\mathcal{R})$. Because of this, d is a (σ,τ)-derivation with $\sigma = \mathrm{id}_{\mathcal{R}}$ and $\tau = -\mathrm{id}_{\mathcal{R}}$. Therefore, $\mathrm{CDer}_{(\delta,1,-1)}(\mathcal{R}) = \mathrm{CDer}_{\mathrm{id}_{\mathcal{R}},-\mathrm{id}_{\mathcal{R}}}(\mathcal{R})$.

Similarly, $\mathrm{CDer}_{(\delta,1,1)}(\mathcal{R})$ is a (σ,τ)-derivation with $\sigma = \frac{1}{\delta}\mathrm{id}_{\mathcal{R}}$ and $\tau = \frac{1}{\delta}\mathrm{id}_{\mathcal{R}}$. □

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