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Pseudo generalized quasi-Einstein manifolds with applications to general relativity

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Abstract. Quasi-Einstein manifold and generalized quasi-Einstein manifold are the generalization of Einstein manifold. In the present paper we discuss about a set of some geometric properties of pseudo generalized quasi-Einstein manifold and we give three and four examples (both Riemannian and Lorentzian) of pseudo generalized quasi-Einstein manifold to show the existence of such manifold. We also discuss $PG(QE)_4$ spacetime with space-matter tensor and some properties related to it. Lastly we prove the existence of a pseudo generalized quasi-Einstein spacetime by constructing a non-trivial example.

1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the condition

$$Ric(X, Y) = \frac{r}{n} g(X, Y) \tag{1}$$

where Ric and r denote the Ricci tensor and scalar curvature tensor respectively. Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [3]. Also the Einstein manifold plays a very important role in Riemannian geometry as well as in general theory of relativity.

M. C. Chaki and R. K. Maity had given the notion of quasi Einstein manifold [7] in 2000. A non-flat n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ is said to be a quasi-Einstein manifold if its non-zero Ricci tensor Ric of type $(0, 2)$ satisfies the following condition

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{2}$$

for all $X, Y \in \chi(M)$ and a, b are scalars, A is a non-zero 1-form such that

$$g(X, \rho) = A(X), \tag{3}$$

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for all vector field X. ρ being a unit vector field, called the generator of the manifold. Also the 1-form A is called the associated 1-form. From the above definition it follows that every Einstein manifold is a subclass of quasi-Einstein manifold. This manifold is denoted by $(QE)_n$. We can note that Robertson-Walker spacetimes are quasi-Einstein spacetimes. In the recent papers ([17],[26]), the application of quasi-Einstein spacetime and generalized quasi-Einstein spacetime in general relativity have been studied. Many more works have been done in the spacetime of general relativity ([2],[5],[6],[18]).

Then M.C. Chaki initiated the notion of generalized quasi-Einstein manifold [8] in 2001. A Riemannian manifold of dimension $(n > 2)$ is said to be generalized quasi Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the following condition

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{4}$$

where a, b, c are scalars and A, B are two non-zero 1-forms. The unit vector fields ρ and σ corresponding to the 1-forms A and B respectively defined by

$$g(X, \rho) = A(X), \quad g(X, \sigma) = B(X) \tag{5}$$

also,

$$g(\rho, \rho) = 1, \quad g(\sigma, \sigma) = 1, \quad g(\rho, \sigma) = 0. \tag{6}$$

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds ([8],[10],[27]), pseudo quasi-Einstein manifolds [24] and many others.

In 2009, A. A. Shaikh [24] introduced the notion of pseudo quasi-Einstein manifold. A semi Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a pseudo quasi-Einstein manifold if the non-zero Ricci tensor Ric of type $(0, 2)$ satisfies the condition

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cE(X, Y) \tag{7}$$

where a, b, c are non-zero scalars, A is non-zero 1-form, ρ is unit vector fields corresponding to the 1-form A and E is symmetric tensor of type $(0, 2)$ with zero trace which satisfies the condition

$$E(X, \rho) = 0, \quad \forall \quad X. \tag{8}$$

An n-dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$.
From (5), (6), (7) and (8), we obtain

$$Ric(\sigma, \sigma) = a, \quad Ric(\rho, \rho) = a + b, \quad Ric(Y, \rho) = (a + b)A(Y), \quad Ric(Y, \sigma) = aC(Y). \tag{9}$$

In 2008, A. A. Shaikh and S. K. Jana [25] introduced the concept of pseudo generalized quasi-Einstein manifold and also verified it by a suitable example. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a pseudo generalized quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the condition

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + dE(X, Y) \tag{10}$$

where a, b, c, d are non-zero scalars A, B are two non-zero 1-forms, ρ and σ are unit vector fields corresponding to the 1-forms A and B respectively and E is symmetric tensor of type $(0, 2)$ with zero trace which satisfies $E(X, \rho) = 0$. An n-dimensional pseudo generalized quasi-Einstein manifold is denoted by $PG(QE)_n$.
Now, Contracting of (10) over X and Y, we get

$$r = na + b + c. \tag{11}$$

Since $a, b, c \in \mathbb{R}$ we obtain

$$dr(X) = 0, \quad \forall \quad X. \tag{12}$$

From (5), (6), (8) and (10), we have

$$Ric(\sigma,\sigma) = a + c + dE(\sigma,\sigma), \quad Ric(\rho,\rho) = a + b, \quad Ric(X,\rho) = (a + b)A(X), \quad Ric(X,\sigma) = (a + c)B(X). \tag{13}$$

Further, we know that if the Riemannian curvature tensor $\overline{K}$ of type (0,4) has the form

$$\overline{K} = k[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{14}$$

then manifold is said to be of constant curvature k. The generalization of this manifold is the manifold of quasi-constant curvature and in this case the curvature tensor has the following form

$$\begin{aligned}\overline{K}(X,Y,Z,W) = {} & f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(Y,Z)A(X)A(W) \\ & - g(Y,W)A(X)A(Z) + g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W)],\end{aligned} \tag{15}$$

where $g(K(X,Y)Z,W) = \overline{K}(X,Y,Z,W)$, K is the curvature tensor of type $(1,3)$ and a,b are scalar function of which $b \neq 0$ and A is non-zero 1-form defined by

$$g(X,\rho) = A(X) \quad \forall \quad X. \tag{16}$$

It can be easily seen that if the curvature tensor $\overline{K}$ of the form (15), then the manifold is conformally flat. Hence a Riemannian or semi-Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $\overline{K}$ satisfied the relation (15). Such a manifold is denoted by $(QC)_n$. Also according to De and Ghosh [10], a Riemannian manifold $(M^n,g)(n \geq 3)$ is said to be of generalized quasi-constant curvature if it is conformally flat and its curvature tensor $\overline{K}$ of type $(0,4)$ has the form

$$\begin{aligned}\overline{K}(X,Y,Z,W) = {} & f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ & + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] + f_3[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ & + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)],\end{aligned} \tag{17}$$

where A and B are 1-forms and f_1, f_2, f_3 are non-zero scalars. Generalizing this notion we define the manifold of pseudo generalized quasi-constant curvature as follows:

A Riemannian manifold $(M^n,g)(n \geq 3)$ is said to be of pseudo generalized quasi-constant curvature if it is conformally flat and its curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\overline{K}(X,Y,Z,W) = {} & f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ & + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] + f_3[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ & + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] + f_4[D(Y,Z)g(X,W) - D(X,Z)g(Y,W) \\ & + D(X,W)g(Y,Z) - D(Y,W)g(X,Z)].\end{aligned} \tag{18}$$

where f_1, f_2, f_3, f_4 are scalars of which $f_1 \neq 0, f_2 \neq 0, f_3 \neq 0, f_4 \neq 0$, A, B are two non-zero 1-forms defined earlier, ρ, σ being two unit vector fields such that $g(\rho,\sigma) = 0$ and E is a symmetric tensor of type $(0,2)$ defined earlier. Such an n-dimensional manifold shall be denoted by $PG(QC)_n$. If in (17), $f_3 = f_4 = 0$, then the manifold reduces to a manifold of quasi-constant curvature. The notion of quasi-conformal curvature tensor was given by Yano and Sawaki [28] and defined as follows

$$\begin{aligned}C(X,Y,Z,W) = {} & f_1\overline{K}(X,Y,Z,W) + f_2[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) + g(Y,Z)Ric(X,W) - \\ & g(X,Z)Ric(Y,W)] - \frac{r}{n}\Big[\frac{f_1}{n-1} + 2f_2\Big][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{19}$$

where f_1 and f_2 are constants and r is the scalar curvature of the manifold.

If $f_1 = 1$ and $f_2 = -\frac{1}{n-2}$, then (19) reduces to the conformal curvature tensor. Thus the conformal curvature tensor is a particular case of the quasi conformal curvature tensor. The quasi-conformal curvature tensor have been studied by various authors in various ways such as Amur and Maralabhavi [1], De and Sarkar [12], De and Matsuyama [13], De, Jun and Gazi [14] and many others.

The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it ([4],[9]) and many others. Also several authors studied spacetimes in different way such as ([15],[20]) and many others.

2. Sufficient condition for a pseudo generalized quasi-Einstein manifold to be a pseudo generalized quasi constant curvature

From (19) it follows that in a quasi-conformally flat Riemannian manifold $(M^n, g)(n > 3)$, the curvature tensor $\overline{K}$ of type (0, 4) has the following form:

$$\begin{aligned} f_1\overline{K}(X,Y,Z,W) &= -f_2[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) + g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)] \\ &\quad + \frac{r}{n}[\frac{f_1}{n-1} + 2f_2][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{20}$$

Using (10) in (20), we have

$$\begin{aligned} \overline{K}(X,Y,Z,W) &= a_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + a_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &\quad + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] + a_3[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &\quad + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] + a_4[D(Y,Z)g(X,W) - D(X,Z)g(Y,W) \\ &\quad + D(X,W)g(Y,Z) - D(Y,W)g(X,Z)]. \end{aligned} \tag{21}$$

where $a_1 = \frac{2f_2(-an+1)(n-1)+rf_1}{f_1 n(n-1)}$, $a_2 = \frac{-bf_2}{f_1}$, $a_3 = \frac{-cf_2}{f_1}$, $a_4 = \frac{-df_2}{f_1}$
it follows from (21) that the manifold under consideration is a $PG(QC)_n$. This leads us to the following theorem.

Theorem 2.1. *In a Riemannian manifold, every quasi-conformally flat pseudo generalized quasi-Einstein manifold is a pseudo generalized quasi constant curvature.*

3. $PG(QE)_n$ with the Killing vector field generators

let us consider the generators ρ and σ of the manifold are Killing vector fields. Then we have

$$(L_\rho g)(X,Y) = 0, \tag{22}$$

$$(L_\sigma g)(X,Y) = 0 \tag{23}$$

where L is the Lie derivative.
From (22) and (23), we get

$$g(D_X\rho, Y) + g(X, D_Y\rho) = 0, \tag{24}$$

$$g(D_X\sigma, Y) + g(X, D_Y\sigma) = 0 \tag{25}$$

since $g(D_X\rho, Y) = (D_XA)(Y)$ *and* $g(D_X\sigma, Y) = (D_XB)(Y)$, therefore from (24) and (25), we have

$$(D_XA)(Y) + (D_YA)(X) = 0, \tag{26}$$

$$(D_X B)(Y) + (D_Y B)(X) = 0, \tag{27}$$

for all X, Y. Similarly we have

$$(D_X A)(Z) + (D_Z A)(X) = 0, \tag{28}$$

$$(D_Z A)(Y) + (D_Y A)(Z) = 0, \tag{29}$$

$$(D_X B)(Z) + (D_Z B)(X) = 0, \tag{30}$$

$$(D_Z B)(Y) + (D_Y B)(Z) = 0, \tag{31}$$

for all X, Y, Z.
We also assume that the associated scalars are constants. Then form (10), we have

$$\begin{aligned}(D_Z Ric)(X, Y) &= b[(D_Z A)(X)A(Y) + A(X)(D_Z A)(Y)] + c[(D_Z B)(X)B(Y) + B(X)(D_Z B)(Y)] \\ &\quad + d(D_Z E)(X, Y).\end{aligned} \tag{32}$$

Permuting equation (32) with respect to X, Y, Z, we get

$$\begin{aligned}&(D_X Ric)(Y, Z) + (D_Y Ric)(Z, X) + (D_Z Ric)(X, Y) \\ &= b[\{(D_X A)(Y) + (D_Y A)(X)\}A(Z) + \{(D_X A)(Z) + (D_Z A)(X)\}A(Y) + \{(D_Y A)(Z) + (D_Z A)(Y)\}A(X)] \\ &+ c[\{(D_X B)(Y) + (D_Y B)(X)\}B(Z) + \{(D_X B)(Z) + (D_Z B)(X)\}B(Y) + \{(D_Y B)(Z) + (D_Z B)(Y)\}B(X)] \\ &+ d[(D_X E)(Y, Z) + (D_Y E)(Z, X) + (D_Z E)(X, Y)].\end{aligned} \tag{33}$$

Which in view of (26) to (31), the relation (33) reduces to

$$\begin{aligned}&(D_X Ric)(Y, Z) + (D_Y Ric)(Z, X) + (D_Z Ric)(X, Y) \\ &= e[(D_X E)(Y, Z) + (D_Y E)(Z, X) + (D_Z E)(X, Y)].\end{aligned} \tag{34}$$

This leads us to the following theorem.

Theorem 3.1. *If the generators of a $PG(QE)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor if the symmetric tensor E is cyclic parallel.*

4. $PG(QE)_n$ with the recurrent vector fields generators

Let us consider the generators ρ and σ corresponding to the associated 1-forms A and B are recurrent. Then we have

$$(D_X A)(Y) = \eta(X)A(Y), \tag{35}$$

$$(D_X B)(Y) = \varphi(X)B(Y) \tag{36}$$

where η and φ are non-zero 1-forms.

A non-flat Riemannian or semi-Riemannian manifold $(M^n, g)(n > 2)$ is said to be a generalized Ricci recurrent manifold [11] if its Ricci tensor Ric of type (0,2) is non-zero and satisfies the condition

$$(D_X Ric)(Y, Z) = \alpha(X)Ric(Y, Z) + \beta(X)g(Y, Z) \tag{37}$$

where α and β are non-zero 1-forms. If $\beta = 0$, then the manifold reduces to a Ricci recurrent manifold [23]. Using (35) and (36) in (32), we have

$$(D_Z Ric)(X, Y) = 2b\eta(Z)A(X)A(Y) + 2c\varphi(Z)B(X)B(Y) + e(D_Z E)(X, Y). \tag{38}$$

Taking $\eta(Z) = \varphi(Z)$ in (38), we get

$$(D_Z Ric)(X, Y) = 2b\eta(Z)A(X)A(Y) + 2c\eta(Z)B(X)B(Y) + e(D_Z E)(X, Y). \tag{39}$$

Using (10) in (39), we have

$$(D_X Ric)(X, Y) = \pi_1(Z)Ric(X, Y) + \pi_2 g(X, Y) + e(D_Z E)(X, Y). \tag{40}$$

where $\pi_1(Z) = 2\eta(z)$ and $\pi_2(Z) = -2a\eta(Z)$. This leads us to the following theorem.

Theorem 4.1. *If the generators of a $PG(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants with the additional condition E is covariant constant, then the manifold is a generalized Ricci recurrent manifold.*

5. $PG(QE)_n$ with the concurrent vector fields generators

Let us consider the generators ρ and σ corresponding to the associated 1-forms A and B are concurrent. Then we have

$$(D_X A)(Y) = \lambda g(X, Y), \tag{41}$$

$$(D_X B)(Y) = \mu g(X, Y) \tag{42}$$

where λ and μ are non-zero constants.
Using (41) and (42) in (32), we have

$$(D_Z Ric)(X, Y) = b[\lambda g(X, Z)A(Y) + \lambda g(Y, Z)A(X)] + c[\mu g(X, Z)B(Y) + \mu g(Y, Z)B(X)] + e(D_Z E)(X, Y). \tag{43}$$

Now, contraction over X and Y, we have

$$dr(Z) = A(Z)[2b\lambda] + B(Z)[2c\mu]. \tag{44}$$

Using (12) in (44) yields

$$A(Z)[2b\lambda] + B(Z)[2c\mu] = 0. \tag{45}$$

By virtue of (45) in (10), we have

$$Ric(X, Y) = ag(X, Y) + \left[b + \frac{b^2\lambda^2}{c\mu^2}\right]A(X)A(Y) + dE(X, Y) \tag{46}$$

which is a pseudo quasi-Einstein manifold. This leads us to the following theorem.

Theorem 5.1. *If the associated vector fields of a $PG(QE)_n$ are concurrent vector fields and the associated scalars are constants, then manifold reduces to a pseudo quasi-Einstein manifold.*

6. $PG(QE)_n$ with the parallel vector fields generators

Let us consider the generators ρ and σ corresponding to the associated 1-forms A and B are parallel. Then we have

$$(D_X A)(Y) = 0, \tag{47}$$

$$(D_X B)(Y) = 0. \tag{48}$$

Using (12) in (19), we get

$$\begin{aligned}(D_W C)(X, Y, Z) = {} & f_1(D_W K)(X, Y, Z)] + f_2[(D_W Ric)(Y, Z)X \\ & - (D_W Ric)(X, Z)Y - g(Y, Z)(D_W R)(X) - g(X, Z)(D_W R)(Y)].\end{aligned} \tag{49}$$

From (10), we have

$$\begin{aligned}(D_X Ric)(Y, Z) = {} & b[(D_X A)(Y)A(Z) + A(Y)(D_X A)(Z)] + c[(D_X B)(Y)B(Z) + B(Y)(D_X B)(Z)] \\ & + e(D_X E)(Y, Z)\end{aligned} \tag{50}$$

Contraction (49) and using (12), we obtain

$$(divC)(X,Y,Z) = f_1[(divK)(X,Y,Z)] + f_2[(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z)]. \tag{51}$$

In a Riemannian manifold we know that

$$(divK)(X,Y,Z) = (D_X Ric)(Y,Z) - (D_Y Ric)(X,Z). \tag{52}$$

From (50), (51) and (52), we have

$$\begin{aligned}(divC)(X,Y,Z) = (f_1+f_2)[&b\{(D_XA)(Y)A(Z) + A(Y)(D_XA)(Z) - (D_YA)(X)A(Z) - A(X)(D_YA)(Z)\}\\ &+ c\{(D_XB)(Y)B(Z) + B(Y)(D_X)B(Z) - (D_YB)(X)B(Z) - B(X)(D_YB)(Z)\}\\ &+ e\{(D_XE)(Y,Z) - (D_YE)(X,Z)\}].\end{aligned} \tag{53}$$

Now, using (47) and (48) in (53), we have

$$(divC)(X,Y,Z) = (f_1+f_2)d[(D_XE)(Y,Z) - (D_YE)(X,Z)]. \tag{54}$$

This leads us to the following theorem.

Theorem 6.1. *If in a $PG(QE)_n$ the associated scalars are constants and the generators ρ and σ of the manifold are parallel vector fields, then (i) div C = 0 and (ii) the tensor E is of Codazzi type are equivalent provided $f_1 + f_2 \neq 0$.*

7. Examples of 3 and 4-dimensional $PG(QE)_n$

Example 7.1. *We define a Riemannian metric g in 3-dimensional space $\mathbb{R}^3$ by the relation*

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 \tag{55}$$

where x^1, x^2, x^3 are non-zero finite. The covariant and contravariant components of the metric tensor are

$$g_{11} = g_{22} = (x^3)^{4/3}, \quad g_{33} = 1, \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{56}$$

and

$$g^{11} = g^{22} = \frac{1}{(x^3)^{4/3}}, \quad g^{33} = 1, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{57}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix}1\\13\end{Bmatrix} = \begin{Bmatrix}2\\23\end{Bmatrix} = \frac{2}{3x^3}, \quad \begin{Bmatrix}3\\11\end{Bmatrix} = \begin{Bmatrix}3\\22\end{Bmatrix} = \frac{-2}{3}(x^3)^{\frac{1}{3}}. \tag{58}$$

The non-zero derivatives of (58), we have

$$\frac{\partial}{\partial x^3}\begin{Bmatrix}1\\13\end{Bmatrix} = \frac{\partial}{\partial x^3}\begin{Bmatrix}2\\23\end{Bmatrix} = \frac{-2}{3(x^3)^2}, \quad \frac{\partial}{\partial x^3}\begin{Bmatrix}3\\11\end{Bmatrix} = \frac{\partial}{\partial x^3}\begin{Bmatrix}3\\22\end{Bmatrix} = \frac{-2}{9(x^3)^{\frac{2}{3}}}. \tag{59}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix}\frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k}\\ \begin{Bmatrix}l\\ij\end{Bmatrix} & \begin{Bmatrix}l\\ik\end{Bmatrix}\end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix}\begin{Bmatrix}m\\ik\end{Bmatrix} & \begin{Bmatrix}m\\ij\end{Bmatrix}\\ \begin{Bmatrix}l\\mk\end{Bmatrix} & \begin{Bmatrix}l\\mj\end{Bmatrix}\end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$R^1_{331} = \frac{\partial}{\partial x^3}\begin{Bmatrix}1\\31\end{Bmatrix} = \frac{-2}{3(x^3)^2},$$
$$K^2_{332} = \frac{\partial}{\partial x^3}\begin{Bmatrix}2\\32\end{Bmatrix} = \frac{-2}{3(x^3)^2},$$

and the non-zero components of (II) are:

$$K^1_{331} = \begin{Bmatrix}m\\31\end{Bmatrix}\begin{Bmatrix}1\\m3\end{Bmatrix} - \begin{Bmatrix}m\\33\end{Bmatrix}\begin{Bmatrix}1\\m1\end{Bmatrix} = \begin{Bmatrix}1\\31\end{Bmatrix}\begin{Bmatrix}1\\13\end{Bmatrix} = \frac{4}{9(x^3)^2},$$

$$K^2_{332} = \begin{Bmatrix}m\\32\end{Bmatrix}\begin{Bmatrix}2\\m3\end{Bmatrix} - \begin{Bmatrix}m\\33\end{Bmatrix}\begin{Bmatrix}2\\m2\end{Bmatrix} = \begin{Bmatrix}2\\32\end{Bmatrix}\begin{Bmatrix}2\\23\end{Bmatrix} = \frac{4}{9(x^3)^2},$$

$$K^1_{221} = \begin{Bmatrix}m\\21\end{Bmatrix}\begin{Bmatrix}1\\m2\end{Bmatrix} - \begin{Bmatrix}m\\22\end{Bmatrix}\begin{Bmatrix}1\\m1\end{Bmatrix} = -\begin{Bmatrix}3\\22\end{Bmatrix}\begin{Bmatrix}1\\31\end{Bmatrix} = \frac{4}{9(x^3)^{\frac{2}{3}}},$$

Adding components corresponding (I) and (II), we have

$$K^1_{221} = \frac{4}{9(x^3)^{\frac{2}{3}}},\ K^1_{331} = \frac{-2}{9(x^3)^2} = K^2_{332}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are,

$$\overline{K}_{1331} = \overline{K}_{2332} = \frac{-2}{9(x^3)^{\frac{2}{3}}},\ \overline{K}_{1221} = \frac{4}{9}(x^3)^{\frac{2}{3}},$$

and the Ricci tensor

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} = \frac{2}{9(x^3)^{\frac{2}{3}}},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} = \frac{2}{9(x^3)^{\frac{2}{3}}},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} = \frac{-4}{9(x^3)^2},$$

Let us consider the associated scalars a, b, c, d and associated tensor E are defined by

$$a = \frac{-4}{9(x^3)^2},\ b = \frac{5(x^3)^{\frac{4}{3}}}{9},\ c = \frac{6}{9(x^3)^2},\ d = \frac{-1}{(x^3)^{\frac{5}{3}}}$$

and

$$E_{ij} = \begin{cases} \frac{5x^3}{9}, & \textit{if i=j=1} \\ \frac{-5x^3}{3}, & \textit{if i=j=2} \\ \frac{15}{(x^3)^{\frac{1}{3}}}, & \textit{if i=j=1,2} \\ 0, & \textit{otherwise} \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} \frac{1}{x^3}, & \textit{if i=1} \\ 0, & \textit{otherwise} \end{cases} \quad \textit{and} \quad B_i(x) = \begin{cases} (x^3)^{\frac{2}{3}}, & \textit{if i=2} \\ 0, & \textit{otherwise} \end{cases}$$

where generators are unit vector fields, then from (10), we have

$$Ric_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + dE_{11}, \tag{60}$$

$$Ric_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + dE_{22}, \tag{61}$$

$$Ric_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + dE_{33}, \tag{62}$$

$$\begin{aligned} R.H.S. \quad of \quad (60) &= ag_{11} + bA_1A_1 + cB_1B_1 + dE_{11} \\ &= \frac{-4}{9(x^3)^{\frac{2}{3}}} + \frac{5}{9(x^3)^{\frac{2}{3}}} + \frac{6}{9(x^3)^{\frac{2}{3}}} - \frac{5}{9(x^3)^{\frac{2}{3}}} \\ &= \frac{2}{9(x^3)^{\frac{2}{3}}} \\ &= L.H.S. \quad of \quad (60) \end{aligned}$$

By similar argument it can be shown that (61) *and* (62) *are also true.*
Hence $(\mathbb{R}^3, g)$ *is a* $PG(QE)_3$.

Example 7.2. *Lorentzian manifold* $(\mathbb{R}^3, g)$ *endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = -(x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2, \tag{63}$$

where x^1, x^2, x^3 *are non-zero finite, then* $(\mathbb{R}^3, g)$ *is a* $PG(QE)_3$.

Example 7.3. *We define a Riemannian metric* g *in* 4*-dimensional space* $\mathbb{R}^4$ *by the relation [16]*

$$ds^2 = g_{ij}dx^idx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \tag{64}$$

where x^1, x^2, x^3, x^4 *are non-zero finite and* $p = e^{x^1}k^{-2}$*. Then the covariant and contravariant components of the metric tensor are*

$$g_{11} = g_{22} = g_{33} = g_{44} = (1 + 2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{65}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1 + 2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{66}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned} &\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1 + 2p}, \\ &\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1 + 2p}. \end{aligned} \tag{67}$$

The non-zero derivatives of (67), we have

$$\begin{aligned} &\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p}{(1 + 2p)^2}, \\ &\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1 + 2p)^2}. \end{aligned} \tag{68}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$K^1_{221} = -\frac{\partial}{\partial x^1}\begin{Bmatrix}1\\22\end{Bmatrix} = \frac{p}{(1+2p)^2},$$
$$K^1_{331} = -\frac{\partial}{\partial x^1}\begin{Bmatrix}1\\33\end{Bmatrix} = \frac{p}{(1+2p)^2},$$
$$K^1_{441} = -\frac{\partial}{\partial x^1}\begin{Bmatrix}1\\44\end{Bmatrix} = \frac{p}{(1+2p)^2}$$

and the non-zero components of (II) are:

$$K^2_{332} = \begin{Bmatrix}m\\32\end{Bmatrix}\begin{Bmatrix}2\\m3\end{Bmatrix} - \begin{Bmatrix}m\\33\end{Bmatrix}\begin{Bmatrix}2\\m2\end{Bmatrix} = -\begin{Bmatrix}1\\33\end{Bmatrix}\begin{Bmatrix}2\\12\end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$K^2_{442} = \begin{Bmatrix}m\\42\end{Bmatrix}\begin{Bmatrix}2\\m4\end{Bmatrix} - \begin{Bmatrix}m\\44\end{Bmatrix}\begin{Bmatrix}2\\m2\end{Bmatrix} = -\begin{Bmatrix}1\\44\end{Bmatrix}\begin{Bmatrix}2\\12\end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$K^3_{443} = \begin{Bmatrix}m\\43\end{Bmatrix}\begin{Bmatrix}3\\m4\end{Bmatrix} - \begin{Bmatrix}m\\44\end{Bmatrix}\begin{Bmatrix}3\\m3\end{Bmatrix} = -\begin{Bmatrix}1\\44\end{Bmatrix}\begin{Bmatrix}3\\13\end{Bmatrix} = \frac{p^2}{(1+2p)^2}.$$

Adding components corresponding (I) and (II), we have

$$K^1_{221} = K^1_{331} = K^1_{441} = \frac{p}{(1+2p)^2},$$

$$K^2_{332} = K^2_{442} = K^3_{443} = \frac{p^2}{(1+2p)^2}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are,

$$\overline{K}_{1221} = \overline{K}_{1331} = \overline{K}_{1441} = \frac{p}{1+2p},$$

$$\overline{K}_{2332} = \overline{K}_{2442} = \overline{K}_{3443} = \frac{p^2}{1+2p}$$

and the Ricci tensor

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} + g^{44}\overline{K}_{1414} = \frac{3p}{(1+2p)^2},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} + g^{44}\overline{K}_{2424} = \frac{p}{(1+2p)},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} + g^{44}\overline{K}_{3434} = \frac{p}{(1+2p)},$$

$$Ric_{44} = g^{jh}\overline{K}_{4j4h} = g^{11}\overline{K}_{4141} + g^{22}\overline{K}_{4242} + g^{33}\overline{K}_{4343} = \frac{p}{(1+2p)}.$$

Let us consider the associated scalars a, b, c, d and the associated tensor E are defined by

$$a = \frac{3p}{(1+2p)^3}, \quad b = 2p, \quad c = \frac{-p}{(1+2p)^2}, \quad d = \frac{-2\sqrt{p}}{(1+2p)^2}$$

and

$$E_{ij} = \begin{cases} \sqrt{p}, & \text{if } i=j=1 \\ -\sqrt{p}, & \text{if } i=j=3 \\ 0, & \text{otherwise} \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} \frac{1}{1+2p}, & \text{if } i=1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \sqrt{p}, & \text{if } i=2 \\ -\sqrt{p}, & \text{if } i=2 \\ 0, & \text{otherwise} \end{cases}$$

where generators are unit vector fields, then from (10), we have

$$Ric_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + dE_{11}, \tag{69}$$

$$Ric_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + dE_{22}, \tag{70}$$

$$Ric_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + dE_{33}, \tag{71}$$

$$Ric_{44} = ag_{44} + bA_4A_4 + cB_4B_4 + dE_{44}, \tag{72}$$

$$\begin{aligned} R.H.S. \quad of \quad (69) &= ag_{11} + bA_1A_1 + cB_1B_1 + dE_{11} \\ &= \frac{3p}{(1+2p)^2} + \frac{2p}{(1+2p)^2} - \frac{2p}{(1+2p)^2} \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S. \quad of \quad (69) \end{aligned}$$

By similar argument it can be shown that (70) *and* (72) *are also true.*
Hence $(\mathbb{R}^4, g)$ *is a* $PG(QE)_4$.

Example 7.4. *Lorentzian manifold* $(\mathbb{R}^4, g)$ *endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = -(1+2p)(dx^1)^2 + (1+2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where x^1, x^2, x^3 *are non-zero finite, then* $(\mathbb{R}^4, g)$ *is a* $PG(QE)_4$.

8. Pseudo generalized quasi-Einstein spacetimes

Petrov [22] developed a tensor $\overline{P}$ of type $(0,4)$ in a smooth manifold (M^n, g) and defined it as follows:

$$\overline{P} = \overline{K} + \frac{\delta}{2} g \wedge T - \varrho H \tag{73}$$

where $\overline{P}$ is the space-matter tensor of type $(0,4)$, $\overline{K}$ is the curvature tensor of type $(0,4)$, T is the energy momentum tensor of type $(0,2)$, δ is the gravitational constant, ϱ is the energy density, H is a tensor of type $(0,4)$ given by

$$H(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \tag{74}$$

$\forall \quad X, Y, Z, W \in \chi(M)$ and Kulkarni-Nomizu product $U \wedge V$ of type $(0,2)$ tensors U and V is defined by

$$(U \wedge V)(X,Y,Z,W) = U(Y,Z)V(X,W) + U(X,W)V(Y,Z) - U(X,Z)V(Y,W) - U(Y,W)V(X,Z), \tag{75}$$

Using (74) and (75) in (73), we have

$$\begin{aligned} \overline{P}(X,Y,Z,W) = \overline{K}(X,Y,Z,W) &+ \frac{\delta}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z) - g(X,Z)T(Y,W) \\ &- g(Y,W)T(X,Z)] - \varrho[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{76}$$

If $\overline{P} = 0$, then (76) yields

$$\begin{aligned}\overline{K}(X,Y,Z,W) = &\frac{\delta}{2}[g(X,Z)T(Y,W) + g(Y,W)T(X,Z) - g(Y,Z)T(X,W) - g(X,W)T(Y,Z)] \\ &- \varrho[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)].\end{aligned} \tag{77}$$

The Einstein's field equation without cosmological constant is given by [21]

$$Ric(X,Y) - \frac{r}{2}g(X,Y) = \varrho T(X,Y). \tag{78}$$

Now using (10) and (78) in (77), we have

$$\begin{aligned}\overline{K}(X,Y,Z,W) = &\alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \beta[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] + \gamma[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] + \theta[D(Y,Z)g(X,W) - D(X,Z)g(Y,W) \\ &+ D(X,W)g(Y,Z) - D(Y,W)g(X,Z)].\end{aligned} \tag{79}$$

where $\alpha = (\varrho - a - \frac{r}{2}), \quad \beta = -\frac{b}{2}, \quad \gamma = -\frac{c}{2}, \quad \theta = -\frac{d}{2}.$
Which is the manifold under consideration is a manifold of pseudo generalized quasi-constant curvature. This leads us to the following theorem.

Theorem 8.1. *A pseudo generalized quasi-Einstein spacetime with vanishing space-matter tensor satisfying Einstein's field equation is a pseudo generalized quasi-constant curvature spacetime.*

Theorem 8.2. *If the associated vector fields of a viscous fluid pseudo generalized quasi-Einstein spacetime are concurrent vector fields and the associated scalars are constants and satisfying Einstein's field equation with a cosmological constant, then none of the energy density and isotropic pressure of the fluid can be a constant.*

The following is a discussion of the form of energy momentum tensor [19]

$$T_{ij} = (\phi + \psi)v_i v_j + q g_{ij} + d D_{ij}$$

where v_i is the timelike unit vector field and D_{ij} is the an isotropic stress tensor, symmetric, traceless and such that $D_{ij}v^j = 0$. We exclude heat transfer, which adds a term $p_i v_j + p_j v_i$ to the tensor T_{ij}, with $p^i vi = 0$. We prove in Theorem 5.1 that a pseudo generalized quasi-Einstein manifold reduces to a pseudo quasi-Einstein manifold if the associated vector fields are concurrent vector fields and the associated scalars are constants.

As a result, we argue that if a pseudo generalized quasi-Einstein spacetime's associated vector fields are concurrent vector fields and the associated scalars are constants, the spacetime is imperfect fluid spacetime.

9. Example of pseudo generalized quasi-Einstein spacetime

Example 9.1. *We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2 \tag{80}$$

where x^1, x^2, x^3, x^4 are non-zero finite. Then the covariant and contravariant components of the metric are

$$g_{11} = 1, \quad g_{22} = (x^1)^2, \quad g_{33} = (x^2)^2, \quad g_{44} = -1, \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{81}$$

and

$$g^{11} = 1, \quad g^{22} = \frac{1}{(x^1)^2}, \quad g^{33} = \frac{1}{(x^2)^2}, \quad g^{44} = -1, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{82}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -x^1, \quad \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\frac{x^2}{(x^1)^2}, \quad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{x^1}, \quad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \frac{1}{x^2}. \tag{83}$$

The non-zero derivatives of (83), we have

$$\begin{aligned} &\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -1, \quad \frac{\partial}{\partial x^2}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^2}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{-1}{(x^1)^2} \\ &\frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \frac{2x^2}{(x^1)^3}, \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \frac{-1}{(x^2)^2}. \end{aligned} \tag{84}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) and (II) are:

$$K^1_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 2 \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{-x^2}{x^1}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are,

$$\overline{K}_{1332} = \frac{-x^2}{x^1},$$

and the Ricci tensor

$$Ric_{12} = g^{jh}\overline{K}_{1j1h} = g^{33}\overline{K}_{1332} = \frac{-1}{x^1x^2}.$$

Let us consider the associated scalars a, b, c, d and the associated tensor E are defined by

$$a = \frac{1}{x^1}, \quad b = \frac{-4}{x^1}, \quad c = x^1x^2, \quad d = \frac{-1}{(x^1)^2}$$

and

$$E_{ij} = \begin{cases} 1, & \text{if i=j=1,3} \\ -2, & \text{if i=j=2} \\ \dfrac{x^1}{x^2}, & \text{if i=1,j=2} \\ 0, & \text{otherwise} \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} 1, & \text{if i=4} \\ 0, & \text{if 1=1,2,3} \end{cases} \quad and \quad B_i(x) = \begin{cases} \dfrac{x^1}{\sqrt{2}}, & \text{if i=2} \\ \dfrac{x^2}{\sqrt{2}}, & \text{if i=3} \\ 0, & \text{if i=4} \end{cases}$$

Then we have

$$R_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + dE_{12}, \tag{85}$$

$$\begin{aligned} R.H.S. \quad of \quad (85) &= ag_{12} + bA_1A_2 + cB_1B_2 + dE_{12} \\ &= \frac{-1}{(x^1)^2}\frac{x^1}{x^2} \\ &= \frac{-1}{x^1x^2} \\ &= L.H.S. \quad of \quad (85) \end{aligned}$$

It can be easily seen that (85) is true. Clearly, the trace of the $(0, 2)$ tensor E is zero.

The 1-forms will now be shown to be unit and orthogonal. Here

$$g^{ij}A_iA_j - 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

So, the manifold under consideration is a pseudo generalized quasi-Einstein spacetime. Thus, the following theorem holds.

Theorem 9.2. *We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation*

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2$$

where x^1, x^2, x^3, x^4 are non-zero finite. Then $(\mathbb{R}^4, g)$ is a pseudo generalized quasi-Einstein spacetime.

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