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Uniformly continuous extension in L-uniform convergence tower spaces

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Abstract. Using an extension theorem for continuous mappings between quantale-valued convergence tower spaces that we obtained in a previous paper, we prove an extension theorem for uniformly continuous mappings between quantale-valued uniform convergence spaces. To this end, we define and study suitable uniform diagonal axioms and uniform regularity for quantale-valued uniform convergence tower spaces.

1. Introduction

Extending a continuous mapping $f : A \longrightarrow Y$ from a dense subset $A \subseteq X$ to the whole of X, such that the extension $F : X \longrightarrow Y$ is again continuous, is in general not possible. To see this, we may consider the following simple example with $X = Y = [0, 1]$ with the standard topologies and $A = [0, 1/2) \cup (1/2, 1]$ and the mapping $f : A \longrightarrow [0, 1]$ defined by $f(x) = 0$ for $x \in [0, 1/2)$ and $f(x) = 1$ for $x \in (1/2, 1]$. So we need to impose further conditions on the mapping f in order to avoid the "gap" at $x_0 = 1/2$. One way out in the category of convergence spaces was given by Cook [4]. For the setting of our simple example, we would demand that images under f of sequences in A that converge to x_0 in X have a common convergence point $y_0 \in Y$. This rules out our simple example, however this condition guarantees that an extension $F(x) = f(x)$ for $x \in A$ and $F(x_0) = y_0$ is continuous.

Cook's Theorem on continuous extension relies on a diagonal axiom for the convergence space X and regularity – as a "dual" diagonal axiom – for the convergence space Y. In [14] this approach was generalized to L-convergence tower spaces, where L is a quantale. This class of spaces encompasses important examples, like L-metric spaces (also called L-categories) and, for certain choices of the quantale, also classical convergence spaces, approach convergence spaces and probabilistic convergence spaces.

We note that in the setting of our simple example, a continuous mapping F from $X = [0, 1]$ to $Y = [0, 1]$ is uniformly continuous and hence, also the restriction to A can be assumed uniformly continuous. The setting of the general problem in the category of L-uniform convergence tower spaces, where uniform continuity of mappings can be defined, therefore seems natural. L-uniform tower spaces and L-metric spaces are natural examples for L-uniform convergence tower spaces and for special choices of the quantale, also classical uniform convergence spaces, approach uniform convergence spaces and probabilistic uniform convergence spaces are covered. In this paper, we obtain an extension theorem for a uniformly continuous mapping

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between L-uniform convergence tower spaces, defined on a dense subset. Following a classical approach for uniform convergence spaces given by Gähler [8], our approach will be, firstly, to apply the extension theorem previously obtained and show that a continuous extension exists. Secondly, we show that this extension is even uniformly continuous. In order to achieve our goal, we define and study uniform diagonal axioms and uniform regularity for L-uniform convergence tower spaces.

The paper is organized as follows. In the second section, we collect basic notions and concepts that are needed later on and fix the notation. Section 3 studies quantale-valued uniform convergence tower spaces and its relation to quantale-valued convergence tower spaces as well as the subclasses of quantale-valued uniform tower spaces and quantale-valued metric spaces. The fourth section is devoted to the study of several uniform diagonal axioms for quantale-valued uniform convergence tower spaces that are in a natural way related to diagonal axioms for quantale-valued convergence tower spaces. We show that in particular quantale-valued uniform tower spaces and quantale-valued metric spaces satisfy these uniform diagonal axioms. Section 5 treats regularity for quantale-valued uniform convergence tower spaces and again quantale-valued uniform tower spaces and quantale-valued metric spaces are always uniformly regular. The following section 6 ties the uniform diagonal axioms and uniform regularity together in the desired extension theorem.

2. Preliminaries

In this paper, we will consider *commutative and integral quantales* $\mathsf{L} = (L, \leq, *)$, that is $(L, \leq)$ is a complete lattice with distinct top and bottom elements $\top, \bot$, and $(L, *)$ is a commutative semigroup with the top element of L as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$ for all $\alpha_i, \beta \in L$, $i \in J$, see e.g. [9].

Typical examples for such quantales are $\mathsf{L} = ([0, 1], \leq, *)$ with a left-continuous t-norm on $[0, 1]$ or *Lawvere's quantale* $\mathsf{L} = ([0, \infty], \geq, +)$, the extended half line, ordered opposite to the natural order and addition as quantale operation [6]. Another important example is given by the *quantale of distance distribution functions* $\mathsf{L} = (\Delta^+, \leq, *)$, where Δ^+ is the set of all distance distribution functions $\varphi : [0, \infty] \longrightarrow [0, 1]$ which are left-continuous in the sense that $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0, \infty]$ and $*$ is a *sup-continuous triangle function*, see [6, 23]. It is shown in [6] that $(\Delta^+, \leq, *)$ is a commutative and integral quantale.

The *totally below relation* $\lhd$ in a complete lattice $(L, \leq)$ is defined by $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$, and a complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$, see e.g. [9].

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on X by $\mathsf{F}(X)$. We consider only *proper* filters, i.e. we do not allow the empty set to belong to a filter. On $\mathsf{F}(X)$ we use the subsethood order. In particular, for each $x \in X$, the point filter is defined by $[x] = \{A \subseteq X : x \in A\} \in \mathsf{F}(X)$. If $\mathbb{F} \in \mathsf{F}(X)$, $\mathbb{G} \in \mathsf{F}(Y)$ and $f : X \longrightarrow Y$ is a mapping, then we define *the image of* $\mathbb{F}$ *under* f, $f(\mathbb{F}) \in \mathsf{F}(Y)$, by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G$ for some $F \in \mathbb{F}\}$. The *inverse image of* $\mathbb{G}$ *under* f is defined by $f^{\leftarrow}(\mathbb{G}) = \{F \subseteq X : f^{\leftarrow}(G) \subseteq F$ for some $G \in \mathbb{G}\}$ with the inverse image of a set $G \subseteq Y$, $f^{\leftarrow}(G) = \{x \in X : f(x) \in G\}$. $f^{\leftarrow}(\mathbb{G})$ is not always a filter on X, only in case $f^{\leftarrow}(G) \neq \emptyset$ for all $G \in \mathbb{G}$. In this case, we also say that $f^{\leftarrow}(\mathbb{G})$ *exists*. A special case occurs, if $A \subseteq X$, for the embedding $\iota_A : A \longrightarrow X$, $\iota_A(x) = x$ for all $x \in A$. We denote, for $\mathbb{F} \in \mathsf{F}(A)$, $\iota_A(\mathbb{F}) = [\mathbb{F}]$ and for $\mathbb{G} \in \mathsf{F}(X)$, in case of existence, $\iota_A^{\leftarrow}(\mathbb{G}) = \mathbb{G}_A$ and we call $\mathbb{G}_A$ *the trace of* $\mathbb{G}$ *on* A. The trace $\mathbb{G}_A$ exists if, and only if, $G \cap A \neq \emptyset$ for all $G \in \mathbb{G}$.

For a set J, a filter $\mathbb{G} \in \mathsf{F}(J)$ and a mapping $\sigma : J \longrightarrow \mathsf{F}(X)$ we define the *diagonal filter* [17] $\kappa\sigma(\mathbb{G}) = \bigcup_{G\in\mathbb{G}} \bigwedge_{j\in G} \sigma(j)$. It is not difficult to see that $\kappa\sigma(\mathbb{G}) \in \mathsf{F}(X)$ and that $F \in \kappa\sigma(\mathbb{G})$ if, and only if, $F^{\sigma} = \{j \in J : F \in \sigma(j)\} \in \mathbb{G}$.

For filters $\mathbb{F} \in \mathsf{F}(X)$ and $\mathbb{G} \in \mathsf{F}(Y)$ we denote their Cartesian product $\mathbb{F} \times \mathbb{G} = \{H \subseteq X \times Y : F \times G \subseteq H$ for some $F \in \mathbb{F}, G \in \mathbb{G}\} \in \mathsf{F}(X \times Y)$.

For filters $\Phi, \Psi \in \mathsf{F}(X \times X)$ we define $\Phi^{-1} = \{H \subseteq X \times X : F^{-1} \subseteq H$ for some $F \in \Phi\} \in \mathsf{F}(X \times X)$, where $F^{-1} = \{(x, y) \in X \times X : (y, x) \in F\}$ and $\Phi \circ \Psi = \{H \subseteq X \times X : F \circ G \subseteq H$ for some $F \in \Phi, G \in \Psi\} \in \mathsf{F}(X \times X)$, whenever $F \circ G \neq \emptyset$ for all $F \in \Phi, G \in \Psi$, where $F \circ G = \{(x, y) \in X \times X : (x, s) \in F, (s, y) \in G$ for some $s \in X\}$.

3. L-uniform convergence tower spaces, L-limit tower spaces and L-metric spaces

For a set X we call a family $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha \in L}$, with $\Lambda_\alpha \subseteq \mathsf{F}(X \times X)$, which satisfies the axioms, for all $\alpha, \beta \in L$,

(LUC1) $[(x,x)] \in \Lambda_\alpha$ for all $x \in X$;

(LUC2) $\Psi \in \Lambda_\alpha$ whenever $\Phi \leq \Psi$ and $\Phi \in \Lambda_\alpha$;

(LUC3) $\Phi \wedge \Psi \in \Lambda_\alpha$ whenever $\Phi, \Psi \in \Lambda_\alpha$;

(LUC4) $\Lambda_\beta \subseteq \Lambda_\alpha$ whenever $\alpha \leq \beta$;

(LUC5) $\Phi^{-1} \in \Lambda_\alpha$ whenever $\Phi \in \Lambda_\alpha$;

(LUC6) $\Phi \circ \Psi \in \Lambda_{\alpha * \beta}$ whenever $\Phi \in \Lambda_\alpha$, $\Psi \in \Lambda_\beta$ and $\Phi \circ \Psi$ exists;

(LUC7) $\Lambda_\perp = \mathsf{F}(X \times X)$

an L-*uniform convergence tower on* X and we call the pair $(X, \overline{\Lambda})$ an L-*uniform convergence tower space* [13]. A mapping $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ between L-uniform convergence tower spaces is called *uniformly continuous* if $(f \times f)(\Phi) \in \Lambda'_\alpha$ whenever $\Phi \in \Lambda_\alpha$. The category of L-uniform convergence tower spaces with uniformly continuous mappings as morphisms is denoted by L-UCTS.

For $\mathsf{L} = (\{0,1\}, \leq, \wedge)$ we obtain uniform convergence spaces [21], for $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm we obtain probabilistic uniform convergence spaces in the definition of Nusser [20], for $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic uniform convergence spaces in [1] and for $\mathsf{L} = ([0,\infty], \geq, +)$ we obtain the approach uniform convergence spaces of Lee and Windels [19].

It can be shown in the standard way that the category L-UCTS is topological. In particular, initial constructions are done as follows. For a source $(f_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k \in K}$ we define the initial L-uniform convergence tower $\overline{\Lambda} = init(\overline{\Lambda^k})$ on X by $\Phi \in \Lambda_\alpha$ if $(f_k \times f_k)(\Phi) \in \Lambda^k_\alpha$ for all $k \in K$.

Important examples of initial constructions are product spaces and subspaces. For product spaces, we put $X = \prod_{k \in K} X_k$ and consider the source $(pr_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k \in K}$ with the projection mappings pr_k. The initial L-uniform convergence tower space is called the *product space* $(\prod_{k \in K} X_k, \overline{\pi\Lambda})$ and we have $\Phi \in \pi\Lambda_\alpha$ if $(pr_k \times pr_k)(\Phi) \in \Lambda^k_\alpha$ for all $k \in K$. For the product of two L-uniform convergence tower spaces $(X, \overline{\Lambda^X}), (Y, \overline{\Lambda^Y})$ we also write $(X \times Y, \overline{\Lambda^X \times \Lambda^Y})$. For a subset $A \subseteq X$, $(X, \overline{\Lambda})$ an L-uniform convergence tower space, we call the inital construction for the source $\iota_A : A \longrightarrow (X, \overline{\Lambda})$ a *subspace* and denote it by $(A, \overline{\Lambda}|_A)$. We have, for $\Phi \in \mathsf{F}(A \times A)$, that $\Phi \in \Lambda_\alpha|_A$ if $[\Phi] \in \Lambda_\alpha$.

An L-uniform convergence tower space $(X, \overline{\Lambda})$ is called *left-continuous* if for all subsets $M \subseteq L$ we have $\Phi \in \Lambda_{\bigvee M}$ whenever $\Phi \in \Lambda_\alpha$ for all $\alpha \in M$.

For an L-uniform convergence tower space $(X, \overline{\Lambda})$, $\mathbb{F} \in \mathsf{F}(X)$, $x \in X$ and $\alpha \in L$ we define

$$x \in q_\alpha^{\overline{\Lambda}}(\mathbb{F}) \iff [x] \times \mathbb{F} \in \Lambda_\alpha.$$

It is not difficult to show that $(X, \overline{q^{\overline{\Lambda}}} = (q_\alpha^{\overline{\Lambda}})_{\alpha \in L})$ is an L-*limit tower space*, i.e. satisfies the axioms (see [12]), for all $\alpha, \beta \in L$, $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$,

(LC1) $x \in q_\alpha^{\overline{\Lambda}}([x])$ for all $x \in X$;

(LC2) $q_\alpha^{\overline{\Lambda}}(\mathbb{F}) \subseteq q_\alpha^{\overline{\Lambda}}(\mathbb{G})$ whenever $\mathbb{F} \leq \mathbb{G}$;

(LC3) $q_\alpha^{\overline{\Lambda}}(\mathbb{F} \wedge \mathbb{G}) = q_\alpha^{\overline{\Lambda}}(\mathbb{F}) \cap q_\alpha^{\overline{\Lambda}}(\mathbb{G})$;

(LC4) $q_\beta^{\overline{\Lambda}}(\mathbb{F}) \subseteq q_\alpha^{\overline{\Lambda}}(\mathbb{F})$ whenever $\alpha \leq \beta$;

(LC5) $q_{\perp}^{\overline{\Lambda}}(\mathbb{F}) = X$.

Moreover, a uniformly continuous mapping $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ is continuous as a mapping $f : (X, \overline{q^{\overline{\Lambda}}}) \longrightarrow (X', \overline{q^{\overline{\Lambda'}}})$ in the sense that $x \in q_\alpha^{\overline{\Lambda}}(\mathbb{F})$ implies $f(x) \in q_\alpha^{\overline{\Lambda'}}(f(\mathbb{F}))$. A limit tower space $(X, \overline{q})$ is called *left-continuous* if $x \in q_\alpha(\mathbb{F})$ for all $\alpha \in M \subseteq L$ implies $x \in q_{\bigvee M}(\mathbb{F})$. If the L-uniform convergence tower space $(X, \overline{\Lambda})$ is left-continuous, then $(X, \overline{q^{\overline{\Lambda}}})$ is left-continuous, too.

For $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm, we obtain the probabilistic limit spaces of [20, 22]. For Lawvere's quantale, an L-limit tower space is a limit tower space in the definition of [2] and for $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic convergence spaces in [11].

For L-limit tower spaces, initial constructions are done as follows, see [15]. For a source $(f_k : X \longrightarrow (X_k, \overline{q^k}))_{k \in K}$, we define the initial L-convergence tower on X by $x \in init(\overline{q^k})_\alpha(\mathbb{F})$ if $f_k(x) \in q_\alpha^k(f_k(\mathbb{F}))$ for all $k \in K$. Later, the following result will be important for us.

Proposition 3.1. *Let $(f_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k \in K}$ be a source and let $(X, init(\overline{\Lambda^k}))$ be the initial construction. Furthermore let $(X, init(\overline{q^{\overline{\Lambda_k}}}))$ be the initial construction for the source $(f_k : X \longrightarrow (X_k, \overline{q^{\overline{\Lambda^k}}}))_{k \in K}$. Then we have $\overline{q^{init(\overline{\Lambda^k})}} = init(\overline{q^{\overline{\Lambda_k}}})$.*

Proof. We have $x \in init(\overline{q^{\overline{\Lambda_k}}})_\alpha(\mathbb{F})$ if, and only if, $f_k(x) \in q_\alpha^{\overline{\Lambda^k}}(f_k(\mathbb{F}))$ for all $k \in K$. This is equivalent to $(f_k \times f_k)([x] \times \mathbb{F}) = [f_k(x)] \times f_k(\mathbb{F}) \in \Lambda_\alpha^k$ for all $k \in K$, i.e. to $[x] \times \mathbb{F} \in init(\overline{\Lambda^k})_\alpha$, which is equivalent to $x \in q_\alpha^{init(\overline{\Lambda^k})}(\mathbb{F})$. □

A system of filters $\overline{\mathcal{U}} = (\mathcal{U}_\alpha)_{\alpha \in L}$ with $\mathcal{U}_\alpha \in \mathsf{F}(X \times X)$ for all $\alpha \in L$, with the properties, for all $\alpha, \beta \in L$,

(LU1) $\mathcal{U}_\alpha \leq [\Delta]$ with $[\Delta] = \bigwedge_{x \in X}[(x,x)]$;

(LU2) $\mathcal{U}_\alpha \leq (\mathcal{U}_\alpha)^{-1}$;

(LU3) $\mathcal{U}_{\alpha * \beta} \leq \mathcal{U}_\alpha \circ \mathcal{U}_\beta$;

(LU4) $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$ whenever $\alpha \leq \beta$;

(LU5) $\mathcal{U}_\perp = \bigwedge \mathsf{F}(X \times X)$

is called an L-*uniform tower on* X and the pair $(X, \overline{\mathcal{U}})$ is called an L-*uniform tower space* [13]. $(X, \overline{\mathcal{U}})$ is called *left-continuous* if $\mathcal{U}_{\bigvee M} \leq \bigvee_{\alpha \in M} \mathcal{U}_\alpha$ whenever $\emptyset \neq M \subseteq L$

For $\mathsf{L} = (\{0,1\}, \leq, \wedge)$ we obtain classical uniformities [3], for $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm, we obtain probabilistic uniformities in the definition of Florescu [7] and for Lawvere's quantale, $\mathsf{L} = ([0,\infty], \geq, +)$, a left-continuous L-uniform tower is an approach uniformity [19]. For $\mathsf{L} = (\Delta^+, \leq, *)$, an L-uniform tower is a probabilistic uniformity in [1].

For an L-uniform tower space $(X, \overline{\mathcal{U}})$ we define $\Phi \in \Lambda_\alpha^{\mathcal{U}}$ if, and only if, $\Phi \geq \mathcal{U}_\alpha$. It is then not difficult to see that $(X, \overline{\Lambda^{\overline{\mathcal{U}}}})$ is an L-uniform convergence tower space.

An L-*metric space* [6] is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ which satisfies the axioms

(LM1) $d(x,x) = \top$ for all $x \in X$;

(LM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(LM3) $d(x,y) * d(y,z) \leq d(x,z)$ for all $x, y, z \in X$.

If we leave away the *symmetry axiom* (LM2), then we shall speak of an L-*quasimetric space*. In case $L = (\{0,1\}, \leq, \wedge)$, an L-quasimetric space is a preordered set. If $\mathsf{L} = ([0,\infty], \geq, +)$ is Lawvere's quantale, an L-metric space is a pseudometric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic pseudometric space, see [6].

If the lattice $(L, \leq)$ is completely distributive, what we tacitly assume if we consider L-metric spaces in this paper, for an L-metric space (X, d) we define an L-convergence tower $\overline{q^d}$ and an L-uniform convergence tower $\overline{\Lambda^d}$ as follows ([12],[13]). For $\alpha \in L$, $\mathbb{F} \in \mathsf{F}(X)$ and $\Phi \in \mathsf{F}(X \times X)$ we have

$$x \in q^d_\alpha(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \alpha;$$

$$\Phi \in \Lambda^d_\alpha \iff \bigvee_{M \in \Phi} \bigwedge_{(x,y) \in M} d(x, y) \geq \alpha.$$

We showed in [12, 13] that L-metric spaces can be characterized by their L-(uniform) convergence towers. We also showed in [13] that we can define an L-uniform tower $\overline{\mathcal{U}^d}$ by $\mathcal{U}^d_\alpha = \bigwedge_{\Phi \in \Lambda^d_\alpha} \Phi$ for $\alpha \in L$.

Proposition 3.2. *Let L be completely distributive. Then we have $\overline{\Lambda^{\overline{\mathcal{U}^d}}} = \overline{\Lambda^d}$.*

Proof. If $\Psi \in \Lambda^d_\alpha$, then clearly $\Psi \geq \mathcal{U}^d_\alpha$, i.e. $\Psi \in \Lambda^{\overline{\mathcal{U}^d}}_\alpha$. For the converse, let $\Psi \in \Lambda^{\overline{\mathcal{U}^d}}_\alpha$ and let $\epsilon \lhd \alpha$. For each $\Phi \in \Lambda^d_\alpha$, there is $M_\Phi \in \Phi$ such that for all $(x, y) \in M_\Phi$ we have $d(x, y) \geq \epsilon$. Then $M = \bigcup_{\Phi \in \Lambda^d_\alpha} M_\Phi \in \mathcal{U}^d_\alpha \leq \Psi$ and hence, $\bigvee_{N \in \Psi} \bigwedge_{(x,y) \in N} d(x, y) \geq \bigwedge_{(x,y) \in M} d(x, y) \geq \epsilon$. The complete distributivity yields $\bigvee_{N \in \Psi} \bigwedge_{(x,y) \in N} d(x, y) \geq \alpha$, i.e. $\Psi \in \Lambda^d_\alpha$. $\square$

We have shown in [13] that $\overline{q^d} = \overline{q^{\overline{\Lambda^d}}}$. If we define, for an L-uniform tower space $(X, \overline{\mathcal{U}})$, $\overline{q^{\overline{\mathcal{U}}}} = \overline{q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}}$, i.e. we define $x \in q^{\overline{\mathcal{U}}}_\alpha(\mathbb{F})$ if $[x] \times \mathbb{F} \geq \mathcal{U}_\alpha$, then we see that for an L-metric space the L-convergence towers resulting from (X, d), $(X, \overline{\Lambda^d})$ and $(X, \overline{\mathcal{U}^d})$ coincide.

4. Diagonal axioms for L-uniform convergence tower spaces

In the sequel, we fix a mapping $\gamma : L \times L \longrightarrow L$. We say that an L-uniform convergence tower space $(X, \overline{\Lambda})$ satisfies the *uniform Fischer diagonal axiom (LUF-γ)* if for all sets J and mappings $\psi : J \longrightarrow X \times X$, for all *selection functions* $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ and $\mathbb{G} \in \mathsf{F}(J)$ with $\psi(\mathbb{G}) \in \Lambda_\alpha$ and $\psi(j) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\sigma(j))$ for all $j \in J$, we have $\kappa\sigma(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}$.

A similar axiom for L-convergence tower spaces was studied in [14]. We say that an L-convergence tower space $(X, \overline{q})$ satisfies the *Fischer diagonal axiom* (LF-γ) if for all sets J and all mappings $\varphi : J \longrightarrow X$, and for all selection functions $\sigma : J \longrightarrow \mathsf{F}(X)$ and $\mathbb{G} \in \mathsf{F}(J)$ with $x \in q_\alpha(\varphi(\mathbb{G}))$ and $\varphi(j) \in q_\beta(\sigma(j))$ for all $j \in J$, we have $x \in q_{\gamma(\alpha,\beta)}(\kappa\sigma(\mathbb{G}))$. For classical convergence spaces this axiom is attributed to Fischer [5].

Lemma 4.1. *We have $(z, y) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta([z] \times \mathbb{F})$ if, and only if, $y \in q^{\overline{\Lambda}}_\beta(\mathbb{F})$.*

Proposition 4.2. *Let $(X, \overline{\Lambda})$ be an L-uniform convergence tower space and let $\gamma : L \times L \longrightarrow L$ be a mapping. If $(X, \overline{\Lambda})$ satisfies (LUF-γ), then $(X, \overline{q^{\overline{\Lambda}}})$ satisfies (LF-γ).*

Proof. Let J be a set, $\varphi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathsf{F}(X)$, $\mathbb{G} \in \mathsf{F}(J)$ and $x \in X$ such that $x \in q^{\overline{\Lambda}}_\alpha(\varphi(\mathbb{G}))$ and $\varphi(j) \in q^{\overline{\Lambda}}_\beta(\sigma(j))$ for all $j \in J$.

We define $\widetilde{J} = X \times J$, $\psi = id_X \times \varphi$, $\mathbb{G}_x = [x] \times \mathbb{G}$ and $\widetilde{\sigma} : \widetilde{J} \longrightarrow \mathsf{F}(X \times X)$ by $\widetilde{\sigma}(z, j) = [z] \times \sigma(j)$. Then $\psi(\mathbb{G}_x) = [x] \times \varphi(\mathbb{G}) \in \Lambda_\alpha$, because $x \in q^{\overline{\Lambda}}_\alpha(\varphi(\mathbb{G}))$, and $\psi(z, j) = (z, \varphi(j)) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\widetilde{\sigma}(z, j))$ because $\varphi(j) \in q^{\overline{\Lambda}}_\beta(\sigma(j))$ for all $j \in J$. The axiom (LUF-γ) implies $\kappa\widetilde{\sigma}(\mathbb{G}_x) \in \Lambda_{\gamma(\alpha,\beta)}$. Now we have $F \in \kappa\widetilde{\sigma}(\mathbb{G}_x)$ if, and only if, $\{(z, j) \in X \times J : F \in [z] \times \sigma(j)\} = F^{\widetilde{\sigma}} \in [x] \times \mathbb{G}$ if, and only if, $x \in pr_X(F)$ and $pr_J(F)^\sigma = \{j \in J : pr_J(F) \in \sigma(j)\} \in \mathbb{G}$. This is equivalent to $x \in pr_X(F)$ and $pr_J(F) \in \kappa\sigma(\mathbb{G})$, that is, to $F \in [x] \times \kappa\sigma(\mathbb{G})$. Hence we have $[x] \times \kappa\sigma(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}$, which means $x \in q^{\overline{\Lambda}}_{\gamma(\alpha,\beta)}(\kappa\sigma(\mathbb{G}))$ and the axiom (LF-γ) is shown. $\square$

The axiom (LUF-γ) is preserved by initial constructions. First we need a lemma and we omit its straightforward proof.

Lemma 4.3. *Let* $(f_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k\in K}$ *be a source and let* $(u,v) \in q_\beta^{init(\overline{\Lambda^k})} \times q_\beta^{init(\overline{\Lambda^k})}(\Phi)$. *Then* $(f_k(u), f_k(v)) \in q_\beta^{\overline{\Lambda^k}} \times q_\beta^{\overline{\Lambda^k}}((f_k \times f_k)(\Phi))$ *for all* $k \in K$.

Proposition 4.4. *Let* $(X_k, \overline{\Lambda^k})$ *be* L*-uniform convergence tower spaces that satisfy the axiom (LUF-γ) for all* $k \in K$, *let* $(f_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k\in K}$ *be a source and let* $(X, \overline{\Lambda})$ *be the initial construction. Then* $(X, \overline{\Lambda})$ *satisfies (LUF-γ).*

Proof. Let J be a set, $\psi : J \longrightarrow X \times X$ be a mapping, $\mathbb{G} \in \mathsf{F}(J)$ and $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ such that $\psi(\mathbb{G}) \in \Lambda_\alpha$ and $\psi(j) \in q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}(\sigma(j))$ for all $j \in J$. We define $\psi_k = (f_k \times f_k) \circ \psi$ and $\sigma_k = (f_k \times f_k) \circ \sigma$. By definition of the initial construction, then $\psi_k(\mathbb{G}) \in \Lambda^k_\alpha$ for all $k \in K$ and by Lemma 4.3 we conclude $\psi_k(j) \in q_\beta^{\overline{\Lambda^k}} \times q_\beta^{\overline{\Lambda^k}}(\sigma_k(j))$ for all $j \in J$. We conclude with (LUF-γ) that $\kappa\sigma_k(\mathbb{G}) \in \Lambda^k_{\gamma(\alpha,\beta)}$ for all $k \in K$.

We note that $F^{\sigma_k} = ((f_k \times f_k)^{\leftarrow}(F))^\sigma$ and hence, $(f_k \times f_k)(\kappa\sigma(\mathbb{G})) = \kappa\sigma_k(\mathbb{G}) \in \Lambda^k_{\gamma(\alpha,\beta)}$ for all $k \in K$, which entails $\kappa\sigma(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}$. $\square$

Proposition 4.5. *Let* $(X, \overline{\mathcal{U}})$ *be an* L*-uniform tower space. Then* $(X, \overline{\Lambda^{\mathcal{U}}})$ *satisfied (LUF-γ) for* $\gamma(\alpha,\beta) = \alpha * \beta * \beta$.

Proof. Let J be a set, $\psi : J \longrightarrow X \times X$ be a mapping, $\mathbb{G} \in \mathsf{F}(J)$ and $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ such that $\psi(\mathbb{G}) \in \Lambda^{\overline{\mathcal{U}}}_\alpha$ and $\psi(j) \in q_\beta^{\overline{\Lambda^{\mathcal{U}}}} \times q_\beta^{\overline{\Lambda^{\mathcal{U}}}}(\sigma(j))$ for all $j \in J$. Then $\psi(\mathbb{G}) \geq \mathcal{U}_\alpha$ and for all $j \in J$ we have $[pr_1(\psi(j))] \times pr_1(\sigma(j)) \geq \mathcal{U}_\beta$ and $[pr_2(\psi(j))] \times pr_2(\sigma(j)) \geq \mathcal{U}_\beta$.

Let $G \in \mathbb{G}$. For $j \in G$ and $M_j \in \sigma(j)$ we have

$$M_j \subseteq (pr_1(M_j) \times \{pr_1(\psi(j))\}) \circ \psi(G) \circ (\{pr_2(\psi(j))\} \times pr_2(M_j)\}).$$

This follows as for $(s,t) \in M_j$ we have $s \in pr_1(M_j)$, $(pr_1(\psi(j)), pr_2(\psi(j))) = \psi(j) \in \psi(G)$ and $t \in pr_2(M_j)$ and hence $(s, pr_1(\psi(j))) \in pr_1(M_j) \times \{pr_1(\psi(j))\}$, $(pr_1(\psi(j)), pr_2(\psi(j))) \in \psi(G)$ and $(pr_2(\psi(j)), t) \in \{pr_2(\psi(j))\} \times pr_2(M_j)$.

Let now $H \in \left(\bigwedge_{j\in J}(pr_1(\sigma(j)) \times [pr_1(\psi(j))])\right) \circ \psi(\mathbb{G}) \circ \left(\bigwedge_{j\in J}([pr_2(\psi(j))] \times pr_2(\sigma(j)))\right)$. Then there are sets $P_1 \in \bigwedge_{j\in J}(pr_1(\sigma(j)) \times [pr_1(\psi(j))])$, $G \in \mathbb{G}$ and $P_2 \in \bigwedge_{j\in J}([pr_2(\psi(j))] \times pr_2(\sigma(j)))$ such that $P_1 \circ \psi(G) \circ P_2 \subseteq H$. For $j \in G$ we have $P_1 \in pr_1(\sigma(j)) \times [pr_1(\psi(j))]$ and $P_2 \in [pr_2(\psi(j))] \times pr_2(\sigma(j))$, i.e. there is $M_j \in \sigma(j)$ such that $pr_1(M_j) \times \{pr_1(\psi(j))\} \subseteq P_1$ and $\{pr_2(\psi(j))\} \times pr_2(M_j) \subseteq P_2$. We conclude, for $j \in G$, that $M_j \subseteq H$ and hence, $H \in \sigma(j)$. So we have $G \subseteq H^\sigma$, which implies $H^\sigma \in \mathbb{G}$, i.e. $H \in \kappa\sigma(\mathbb{G})$.

=Therefore, we conclude

$$\begin{aligned}
\mathcal{U}_{\alpha*\beta*\beta} &\leq \mathcal{U}_\beta \circ \mathcal{U}_\alpha \circ \mathcal{U}_\beta \\
&\leq \left(\bigwedge_{j\in J}(pr_1(\sigma(j)) \times [pr_1(\psi(j))])\right) \circ \psi(\mathbb{G}) \circ \left(\bigwedge_{j\in J}([pr_2(\psi(j))] \times pr_2(\sigma(j)))\right) \\
&\leq \kappa\sigma(\mathbb{G}),
\end{aligned}$$

and we have $\kappa\sigma(\mathbb{G}) \in \Lambda^{\mathcal{U}}_{\alpha*\beta*\beta}$ and the proof is complete. $\square$

We conclude that also for an L-metric space, $(X, \overline{\Lambda^d})$ satisfies (LUF-γ) for $\gamma(\alpha,\beta) = \alpha * \beta * \beta$. Proposition 4.2 entails that $(X, \overline{q^d})$ satisfies (LF-γ) for $\gamma(\alpha,\beta) = \alpha * \beta * \beta$. However, in [14] we have shown the stronger result that $(X, \overline{q^d})$ satisfies (LF-γ) for the mapping γ defined by $\gamma(\alpha,\beta) = \alpha * \beta$.

A special case of the axiom (LUF-γ) arises if we restrict to $J = X \times X$ and $\psi = id_{X\times X}$. We say that $(X, \overline{\Lambda})$ satisfies the *uniform Kowalsky diagonal axiom* (LUK-γ) if for all $\Psi \in \mathsf{F}(X \times X)$, $\sigma : X \times X \longrightarrow \mathsf{F}(X \times X)$ such that

$\Psi \in \Lambda_\alpha$ and $(u,v) \in q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}(\sigma(u,v))$ for all $(u,v) \in X \times X$ we have $\kappa\sigma(\Psi) \in \Lambda_{\gamma(\alpha,\beta)}$. Clearly, (LUF-$\gamma$) implies (LUK-$\gamma$).

A similar axiom for L-convergence tower spaces was introduced in [14] and goes back, for classical convergence spaces, to Kowalsky [18]. We say that $(X,\overline{q})$ satisfies the axiom (LK-γ) if for all $\mathbb{G} \in \mathsf{F}(X)$ and $\sigma : X \longrightarrow \mathsf{F}(X)$ such that $x \in q_\alpha(\mathbb{G})$ and $y \in q_\beta(\sigma(y))$ for all $y \in X$ we have $x \in q_{\gamma(\alpha,\beta)}(\kappa\sigma(\mathbb{G}))$.

Proposition 4.6. *If the* L*-uniform convergence tower space* $(X,\overline{\Lambda})$ *satisfies (LUK-γ), then* $(X,\overline{q^{\overline{\Lambda}}})$ *satisfies (LK-γ).*

Proof. This proof is similar to the proof of Proposition 4.2 and it is therefore not presented. □

Proposition 4.7. *Let* $(X_k,\overline{\Lambda^k})$ *be* L*-uniform convergence tower spaces that satisfy the axiom (LUK-γ) for all* $k \in K$ *and let* $(f_k : X \longrightarrow (X_k,\overline{\Lambda^k}))_{k\in K}$ *be a source with all* f_k *injective and let* $(X,\overline{\Lambda})$ *be the initial construction. Then* $(X,\overline{\Lambda})$ *satisfies (LUK-γ).*

Proof. Let $\Psi \in \mathsf{F}(X \times X)$, $\sigma : X \times X \longrightarrow \mathsf{F}(X \times X)$ such that $\Psi \in \Lambda_\alpha$ and $(u,v) \in q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}(\sigma(u,v))$ for all $(u,v) \in X \times X$. Then $(f_k \times f_k)(\Psi) \in \Lambda^k_\alpha$ and $(f_k(u), f_k(v)) \in q_\beta^{\overline{\Lambda^k}} \times q_\beta^{\overline{\Lambda^k}}((f_k \times f_k) \circ \sigma(u,v))$ for all $k \in K$. We define, for $k \in K$, the selection function $\sigma^k : X_k \times X_k \longrightarrow \mathsf{F}(X_k \times X_k)$ by $\sigma^k(u_k,v_k) = (f_k \times f_k) \circ \sigma(u,v)$ if $f_k(u) = u_k$ and $f_k(v) = v_k$, and $\sigma^k(u_k,v_k) = [(u_k,v_k)]$ otherwise. Note that σ^k is well-defined by the injectivity of the f_k. Then, for all $k \in K$, $(u_k,v_k) \in q_\beta^{\overline{\Lambda^k}} \times q_\beta^{\overline{\Lambda^k}}(\sigma^k(u_k,v_k))$ and hence, by (LUK-γ), we have $\kappa\sigma^k((f_k \times f_k)(\Psi)) \in \Lambda^k_{\gamma(\alpha,\beta)}$ for all $k \in K$. Noting that $F \in \kappa\sigma^k((f_k \times f_k)(\Psi))$ if, and only if, $((f_k \times f_k)^{\leftarrow}(F))^\sigma = (f_k \times f_k)^{\leftarrow}(F^{\sigma^k}) \in \Psi$, i.e. if $F \in (f_k \times f_k)(\kappa\sigma(\Psi))$ we have $(f_k \times f_k)(\kappa\sigma(\Psi)) \in \Lambda^k_{\gamma(\alpha,\beta)}$ for all $k \in K$, and therefore $\kappa\sigma(\Psi) \in \Lambda_{\gamma(\alpha,\beta)}$, as desired. □

We consider now an L-uniform tower space $(X,\overline{\mathcal{U}})$ and define, for $(x,y) \in X \times X$ and $\beta \in L$,

$$\mathbb{U}_\beta^{(x,y)} = \bigwedge_{(x,y)\in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\Phi)} \Phi.$$

Lemma 4.8. *Let* $(X,\overline{\mathcal{U}})$ *be an* L*-uniform tower space. Then* $(x,y) \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\mathbb{U}_\beta^{(x,y)})$.

Proof. We have for $\Phi \in \mathsf{F}(X\times X)$ that $(x,y) \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\Phi)$ is equivalent to $x \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(pr_1(\Phi))$ and $y \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(pr_2(\Phi))$. Hence $[x] \times pr_1(\Phi) \geq \mathcal{U}_\beta$ and therefore

$$[x] \times pr_1(\mathbb{U}_\beta^{(x,y)}) = \bigwedge_{(x,y)\in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\Phi)} ([x] \times pr_1(\Phi)) \geq \mathcal{U}_\beta.$$

In the same way we see that also $[y]\times pr_2(\mathbb{U}_\beta^{(x,y)}) \geq \mathcal{U}_\beta$. This means $x \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(pr_1(\mathbb{U}_\beta^{(x,y)}))$ and $y \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(pr_2(\mathbb{U}_\beta^{(x,y)}))$ which entails $(x,y) \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(pr_1(\mathbb{U}_\beta^{(x,y)}) \times pr_2(\mathbb{U}_\beta^{(x,y)})) \subseteq q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\mathbb{U}_\beta^{(x,y)})$. □

We define $\sigma_\beta : X \times X \longrightarrow \mathsf{F}(X \times X)$ by $\sigma_\beta(x,y) = \mathbb{U}_\beta^{(x,y)}$.

Proposition 4.9. *=Let* $(X,\overline{\mathcal{U}})$ *be an* L*-uniform tower space. Then* $(X,\overline{\Lambda^{\overline{\mathcal{U}}}})$ *satisfies (LUK-γ) if, and only if,* $\kappa\sigma_\beta(\mathcal{U}_\alpha) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$.

Proof. Let $(X, \overline{\Lambda^{\overline{\mathcal{U}}}})$ satisfy (LUK-γ). As $\mathcal{U}_\alpha \in \Lambda^{\overline{\mathcal{U}}}_\alpha$ and $(u,v) \in q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta \times q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta(\mathbb{U}^{(u,v)}_\beta)$ for all $(u,v) \in X \times X$ we conclude $\kappa\sigma_\beta(\mathcal{U}_\alpha) \in \Lambda^{\overline{\mathcal{U}}}_{\gamma(\alpha,\beta)}$, i.e. $\kappa\sigma_\beta(\mathcal{U}_\alpha) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$.

Let now $\kappa\sigma_\beta(\mathcal{U}_\alpha) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$. If $\Psi \in \Lambda^{\overline{\mathcal{U}}}_\alpha$ and $\sigma : X\times X \longrightarrow \mathsf{F}(X\times X)$ such that $(u,v) \in q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta \times q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta(\sigma(u,v))$ for all $(u,v) \in X\times X$, then $\Psi \geq \mathcal{U}_\alpha$ and $\sigma(u,v) \geq \mathbb{U}^{(u,v)}_\beta$ for all $(u,v) \in X\times X$. We conclude $\kappa\sigma(\Psi) \geq \kappa\sigma_\beta(\mathcal{U}_\alpha) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$, i.e. $\kappa\sigma(\Psi) \in \Lambda^{\overline{\mathcal{U}}}_{\gamma(\alpha,\beta)}$. $\square$

Proposition 4.10. *Let $(X, \overline{\mathcal{U}})$ be an L-uniform tower space. Then $(X, \overline{\Lambda^{\overline{\mathcal{U}}}})$ satisfies (LUK-γ) if, and only if, it satisfies (LUF-γ).*

Proof. It is sufficient to show that (LUK-γ) implies (LUF-γ). Let $\psi : J \longrightarrow X\times X$, $\sigma : J \longrightarrow \mathsf{F}(X\times X)$, $\mathbb{G} \in \mathsf{F}(J)$ such that $\psi(\mathbb{G}) \in \Lambda^{\overline{\mathcal{U}}}_\alpha$ and $\psi(j) \in q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta \times q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}_\beta(\sigma(j))$ for all $j \in J$. Then $\psi(\mathbb{G}) \geq \mathcal{U}_\alpha$ and $\sigma(j) \geq \mathbb{U}^{\psi(j)}_\beta$ for all $j \in J$. We define $\widetilde{\sigma} : X\times X \longrightarrow \mathsf{F}(X\times X)$ by $\widetilde{\sigma}(u,v) = \bigwedge_{\psi(j)=(u,v)} \sigma(j)$ if $(u,v) \in \psi(J)$ and $\widetilde{\sigma}(u,v) = [(u,v)]$ otherwise. Then $\widetilde{\sigma}(u,v) \geq \mathbb{U}^{(u,v)}_\beta$ for all $(u,v) \in X\times X$ and hence, $\kappa\widetilde{\sigma}(\psi(\mathbb{G})) \geq \kappa\sigma_\beta(\mathcal{U}_\alpha) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$. We finally show that $\kappa\sigma(\mathbb{G}) \geq \kappa\widetilde{\sigma}(\psi(\mathbb{G}))$. Let $H \in \kappa\widetilde{\sigma}(\psi(\mathbb{G}))$, then $\psi^{\leftarrow}(H^{\widetilde{\sigma}}) \in \mathbb{G}$. For $j \in \psi^{\leftarrow}(H^{\widetilde{\sigma}})$ we have $\psi(j) \in H^{\widetilde{\sigma}}$, i.e. $H \in \widetilde{\sigma} \circ \psi(j) \leq \sigma(j)$. Hence $H^\sigma \supseteq \psi^{\leftarrow}(H^{\widetilde{\sigma}})$ and therefore $H^\sigma \in \mathbb{G}$, which implies $H \in \kappa\sigma(\mathbb{G})$. Therefore $\kappa\sigma(\mathbb{G}) \geq \mathcal{U}_{\gamma(\alpha,\beta)}$, i.e. $\kappa\sigma(\mathbb{G}) \in \Lambda^{\overline{\mathcal{U}}}_{\gamma(\alpha,\beta)}$ and the proof is complete. $\square$

Finally, we shall consider a diagonal axiom that goes back to the work of W. Gähler [8]. For $\Phi \in \mathsf{F}(X\times X)$ we define

$$\mathbb{U}_\alpha(\Phi) = \kappa\sigma_\alpha(\Phi).$$

Lemma 4.11. *Let $(X, \overline{\Lambda})$ be an L-uniform convergence tower space and let $\Phi \in \mathsf{F}(X\times X)$, $\alpha \in L$ and $(x,y) \in X\times X$. Then $\mathbb{U}_\alpha(\Phi) \leq \Phi$ and $\mathbb{U}_\alpha([(x,y)]) = \mathbb{U}^{(x,y)}_\alpha$.*

Proof. The first assertion follows from $\mathbb{U}^{(x,y)}_\alpha \leq [(x,y)]$, which implies $H^{\sigma_\alpha} \subseteq H$.

For the second assertion, we have $H \in \mathbb{U}_\alpha([(x,y)])$ if, and only if, $H^{\sigma_\alpha} \in [(x,y)]$, if, and only if, $H \in \sigma_\alpha(x,y) = \mathbb{U}^{(x,y)}_\alpha$. $\square$

We say that the L-uniform convergence tower space $(X, \overline{\Lambda})$ satisfies the *uniform Gähler diagonal axiom* (LUG-γ) if for all $\alpha, \beta \in L$ and all $\Phi \in \mathsf{F}(X\times X)$ we have $\mathbb{U}_\beta(\Phi) \in \Lambda_{\gamma(\alpha,\beta)}$ whenever $\Phi \in \Lambda_\alpha$.

Proposition 4.12. *An L-uniform convergence tower space $(X, \overline{\Lambda})$ satisfies (LUG-γ) if, and only if, it satisfies (LUF-γ).*

Proof. Let first (LUG-γ) be satisfied and let J be a set, $\psi : J \longrightarrow X\times X$, $\sigma : J \longrightarrow \mathsf{F}(X\times X)$ and $\mathbb{G} \in \mathsf{F}(J)$ such that $\psi(\mathbb{G}) \in \Lambda_\alpha$ and $\psi(j) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\sigma(j))$ for all $j \in J$. Then $\sigma(j) \geq \sigma_\beta(\psi(j))$ for all $j \in J$ and, by (LUG-γ), $\mathbb{U}_\beta(\psi(\mathbb{G})) \in \Lambda_{\gamma(\alpha,\beta)}$. We show that $\mathbb{U}_\beta(\psi(\mathbb{G})) \leq \kappa\sigma(\mathbb{G})$, from which with (LUCT2) the axiom (LUF-γ) follows. We have $H \in \mathbb{U}_\beta(\psi(\mathbb{G}))$ if, and only if $H^{\sigma_\beta} \in \psi(\mathbb{G})$ if, and only if, $\psi^{\leftarrow}(H^{\sigma_\beta}) \in \mathbb{G}$. If $j \in \psi^{\leftarrow}(H^{\sigma_\beta})$, then $\psi(j) \in H^{\sigma_\beta}$, i.e. $H \in \sigma_\beta(\psi(j)) \leq \sigma(j)$ and we conclude $j \in H^\sigma$. Therefore, $\psi^{\leftarrow}(H^{\sigma_\beta}) \subseteq H^\sigma$ and we conclude $H \in \kappa\sigma(\mathbb{G})$.

Let now (LUF-γ) be true and let $\Phi \in \Lambda_\alpha$. We define $J_\beta = \{(\Psi,(x,y)) : (x,y) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\Psi)\}$ and $\psi : J_\beta \longrightarrow X\times X$ by $\psi((\Psi,(x,y))) = (x,y)$, and $\sigma : X\times X \longrightarrow \mathsf{F}(X\times X)$ by $\sigma((\Psi,(x,y))) = \Psi$. As $(x,y) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta([(x,y)])$ we see that ψ is a surjection and hence $\mathbb{K} = \psi^{\leftarrow}(\Phi) \in \mathsf{F}(J_\beta)$ and $\psi(\mathbb{K}) = \Phi \in \Lambda_\alpha$. Furthermore, we have $\psi((\Psi,(x,y))) = (x,y) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\sigma(\Psi,(x,y)))$. The axiom (LUF-$\gamma$) yields $\kappa\sigma(\mathbb{K}) \in \Lambda_{\gamma(\alpha,\beta)}$ and we will show that $\kappa\sigma(\mathbb{K}) \leq \mathbb{U}_\beta(\Phi)$. Let $H \in \kappa\sigma(\mathbb{K})$. Then $H^\sigma \in \psi^{\leftarrow}(\Phi)$, i.e. there is $M \in \Phi$ such that $\psi^{\leftarrow}(M) \subseteq H^\sigma$. We note that $(\Psi,(x,y)) \in \psi^{\leftarrow}(M)$ is equivalent to $(x,y) \in M$ and implies $H \in \sigma((\Psi,(x,y))) = \Psi$. Therefore, noting that $(\Psi,(x,y)) \in J_\beta$ means $(x,y) \in q^{\overline{\Lambda}}_\beta \times q^{\overline{\Lambda}}_\beta(\Psi)$, we see that $(x,y) \in M$ implies $H \in \mathbb{U}^{(x,y)}_\beta = \sigma_\beta(x,y)$ and hence $M \subseteq H^{\sigma_\beta}$, i.e. $H^{\sigma_\beta} \in \Phi$. This entails $H \in \kappa\sigma_\beta(\Phi) = \mathbb{U}_\beta(\Phi)$ and the proof is complete. $\square$

Hence, for an L-uniform tower space, in particular for an L-metric space, all three diagonal axioms (LUF-γ), (LUK-γ) and (LUG-γ) are equivalent and they satisfy these axioms for $\gamma : L \times L \longrightarrow L$ defined by $\gamma(\alpha, \beta) = \alpha * \beta * \beta$.

5. Regularity

We call an L-uniform convergence tower space $(X, \overline{\Lambda})$ *uniformly γ-regular* if it satisfies the axiom (LUR-γ): for all sets J, mappings $\psi : J \longrightarrow X \times X$, selection mappings $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ and filters $\mathbb{G} \in \mathsf{F}(J)$ such that $\kappa\sigma(\mathbb{G}) \in \Lambda_\alpha$ and $\psi(j) \in q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}(\sigma(j))$ for all $j \in J$, we have $\psi(\mathbb{G}) \in \Lambda_{\gamma(\alpha,\beta)}$.

This axiom is in a sense "dual" to the axiom (LUF-γ). For L-convergence tower spaces, regularity was introduced in [14]. We call an L-convergence tower space $(X, \overline{q})$ *γ-regular* if it satisfies the axiom (LR-γ): for all sets J, mappings $\psi : J \longrightarrow X$, selection functions $\sigma : J \longrightarrow \mathsf{F}(X)$, filters $\mathbb{G} \in \mathsf{F}(J)$ such that $x \in q_\alpha(\kappa\sigma(\mathbb{G}))$ and $\psi(j) \in q_\beta(\sigma(j))$ for all $j \in J$ we have $x \in q_{\gamma(\alpha,\beta)}(\psi(\mathbb{G}))$.

The following two results can be proved in a similar way as Propositions 4.2 and 4.4.

Proposition 5.1. *Let $(X, \overline{\Lambda})$ be an L-uniform convergence tower space and let $\gamma : L \times L \longrightarrow L$ be a mapping. If $(X, \overline{\Lambda})$ is uniformly γ-regular, then $(X, \overline{q^{\overline{\Lambda}}})$ is γ-regular.*

Proposition 5.2. *Let $(X_k, \overline{\Lambda^k})$ be L-uniform convergence tower spaces that are uniformly γ-regular for all $k \in K$ and let $(f_k : X \longrightarrow (X_k, \overline{\Lambda^k}))_{k \in K}$ be a source and let $(X, \overline{\Lambda})$ be the initial construction. Then $(X, \overline{\Lambda})$ is uniformly γ-regular.*

An L-uniform tower space is always γ-regular for a certain function γ.

Proposition 5.3. *Let $(X, \overline{\mathcal{U}})$ be an L-uniform tower space. Then $(X, \overline{\Lambda^{\overline{\mathcal{U}}}})$ is uniformly γ-regular for $\gamma(\alpha, \beta) = \alpha * \beta * \beta$.*

Proof. Let J be a set, $\psi : J \longrightarrow X \times X$ be a mapping, $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ be a selection mapping and $\mathbb{G} \in \mathsf{F}(J)$ such that $\kappa\sigma(\mathbb{G}) \in \Lambda_\alpha^{\overline{\mathcal{U}}}$ and $\psi(j) \in q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}} \times q_\beta^{\overline{\Lambda^{\overline{\mathcal{U}}}}}(\sigma(j))$ for all $j \in J$. Then $\kappa\sigma(\mathbb{G}) \geq \mathcal{U}_\alpha$ and, for all $j \in J$, $[pr_1(\psi(j))] \times pr_1(\sigma(j)) \geq \mathcal{U}_j$ and $[pr_2(\psi(j))] \times pr_2(\sigma(j)) \geq \mathcal{U}_j$. We conclude

$$\mathcal{U}_{\alpha*\beta*\beta} \leq \left(\bigwedge_{j \in J} (pr_1(\sigma(j)) \times [pr_1(\psi(j))]) \right) \circ \kappa\sigma(\mathbb{G}) \circ \left(\bigwedge_{j \in J} ([pr_2(\psi(j))] \times pr_2(\sigma(j))) \right) =: \mathbb{K}$$

and we show that $\mathbb{K} \leq \psi(\mathbb{G})$. Let $K \in \mathbb{K}$. Then there are $P_1 \in \bigwedge_{j \in J}(pr_1(\sigma(j)) \times [pr_1(\psi(j))]), H \in \kappa\sigma(\mathbb{G}), P_2 \in \bigwedge_{j \in J}([pr_2(\psi(j))] \times pr_2(\sigma(j)))$ such that $P_1 \circ H \circ P_2 \subseteq K$. Then $H^\sigma \in \mathbb{G}$ and thus there is $G \in \mathbb{G}$ such that for all $j \in G$ we have $H \in \sigma(j)$. Let $(s, t) \in \psi(G)$. Then $\psi(j) = (s, t)$ for some $j \in G$ and there is $M_j \in \sigma(j)$ such that $\{s\} \times pr_1(M_j) \subseteq P_1$ and $pr_2(M_j) \times \{t\} \subseteq P_2$. We put $\widetilde{M}_j = M_j \cap H$ and have for $(u, v) \in \widetilde{M}_j$, $(s, u) \in P_1$, $(u, v) \in H$, $(v, t) \in P_2$, i.e. we have $(s, t) \in P_1 \circ H \circ P_2 \subseteq K$. Hence we have shown $\psi(G) \subseteq K$, i.e. $K \in \psi(\mathbb{G})$. Therefore, we have, as desired, $\mathcal{U}_{\alpha*\beta*\beta} \leq \psi(\mathbb{G})$ which means we have $\psi(\mathbb{G}) \in \Lambda_{\alpha*\beta*\beta}^{\mathcal{U}}$. $\square$

In particular, an L-metric space is γ-regular for $\gamma(\alpha, \beta) = \alpha * \beta * \beta$. Again, in [14] we showed that this is even true for the function $\gamma(\alpha, \beta) = \alpha * \beta$.

We can characterize uniform γ-regularity using certain closures. To this end, we define for $M \subseteq X \times X$, the *β-closure of* M, as the closure of M with respect to $q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}$, by $(x, y) \in \overline{M}^\beta$ if there is $\Psi \in \mathsf{F}(X \times X)$ such that $M \in \Psi$ and $(x, y) \in q_\beta^{\overline{\Lambda}} \times q_\beta^{\overline{\Lambda}}(\Psi)$. For a filter $\Phi \in \mathsf{F}(X \times X)$, the set $\{\overline{M}^\beta : M \in \Phi\}$ is then a filter basis and we denote the generated filter by $\overline{\Phi}^\beta \in \mathsf{F}(X \times X)$.

Proposition 5.4. *An L-uniform convergence tower space $(X, \overline{\Lambda})$ is γ-regular if, and only if, $\overline{\Phi}^\beta \in \Lambda_{\gamma(\alpha,\beta)}$ whenever $\Phi \in \Lambda_\alpha$.*

Proof. Let first $(X,\overline{\Lambda})$ be γ-regular and let $\Phi \in \Lambda_\alpha$. We define $J_\beta = \{((x,y),\Psi) \ : \ (x,y) \in q^{\Lambda}_\beta \times q^{\Lambda}_\beta(\Psi)\}$, $\psi : J_\beta \longrightarrow X\times X$ by $\psi((x,y),\Psi) = (x,y)$ and $\sigma : J_\beta \longrightarrow \mathsf{F}(X\times X)$ by $\sigma(((x,y),\Psi)) = \Psi$. Then for $j = ((x,y),\Psi) \in J_\beta$ we have $\psi(j) = (x,y) \in q^{\Lambda}_\beta \times q^{\Lambda}_\beta(\sigma(j))$ for all $j \in J_\beta$. For $M \in \Phi$ we define $S_M = \{((x,y),\Psi) \in J_\beta \ : \ M \in \Psi\}$. It is not difficult to see that the sets S_M with $M \in \Phi$ form a filter basis on J_β and we denote the generated filter by $\mathcal{S}_\Phi$. From $\psi(S_M) = \{(x,y) \ : \ (x,y) \in q^{\Lambda}_\beta \times q^{\Lambda}_\beta(\Psi), M \in \Psi\} = \overline{M}^\beta$ we see that $\psi(\mathcal{S}_\Phi) = \overline{\Phi}^\beta$. We show that $\Phi \leq \kappa\sigma(\mathcal{S}_\Phi)$. If $M \in \Phi$, then $S_M \in \mathcal{S}_\Phi$. For $j = ((x,y),\Psi) \in S_M$ we have $M \in \Psi = \sigma(j)$, i.e. $j \in M^\sigma$. Hence, we have $S_M \subseteq M^\sigma$ and therefore $M \in \kappa\sigma(\mathcal{S}_\Phi)$. $\Phi \leq \kappa\sigma(\mathcal{S}_\Phi)$ implies that $\kappa\sigma(\mathcal{S}_\Phi) \in \Lambda_\alpha$ and by (LUR-γ) then $\overline{\Phi}^\beta = \psi(\mathcal{S}_\Phi) \in \Lambda_{\gamma(\alpha,\beta)}$.

Let now $\overline{\Phi}^\beta \in \Lambda_{\gamma(\alpha,\beta)}$ whenever $\Phi \in \Lambda_\alpha$. Let J be a set, $\psi : J \longrightarrow X \times X$, $\sigma : J \longrightarrow \mathsf{F}(X \times X)$ and $\mathbb{G} \in \mathsf{F}(J)$ such that $\kappa\sigma(\mathbb{G}) \in \Lambda_\alpha$ and $\psi(j) \in q^{\Lambda}_\beta \times q^{\Lambda}_\beta(\sigma(j))$ for all $j \in J$. Then $\overline{\kappa\sigma(\mathbb{G})}^\beta \in \Lambda_{\gamma(\alpha,\beta)}$. To complete the proof, we show that $\psi(\mathbb{G}) \geq \overline{\kappa\sigma(\mathbb{G})}^\beta$. To this end, let $H \in \overline{\kappa\sigma(\mathbb{G})}^\beta$. Then there is $K \in \kappa\sigma(\mathbb{G})$ such that $\overline{K}^\beta \subseteq H$. As $K^\sigma \in \mathbb{G}$, there is $G \in \mathbb{G}$ such that $G \subseteq K^\sigma$, i.e. for $j \in G$ we have $K \in \sigma(j)$. Also, for $j \in G$ we have $\psi(j) \in q^{\Lambda}_\beta \times q^{\Lambda}_\beta(\sigma(j))$, which shows that $\psi(j) \in \overline{K}^\beta \subseteq H$. Hence, $\psi(G) \subseteq H$ and this shows $H \in \psi(\mathbb{G})$. □

Proposition 5.5. *Let $(X,\overline{\mathcal{U}})$ be an* L*-uniform tower space. Then $(X,\overline{\Lambda^{\mathcal{U}}})$ is γ-regular if, and only if, $\mathcal{U}_{\gamma(\alpha,\beta)} \leq \overline{\mathcal{U}_\alpha}^\beta$ for all $\alpha,\beta \in L$.*

Proof. If $(X,\overline{\Lambda^{\mathcal{U}}})$ is γ-regular, then, because $\mathcal{U}_\alpha \in \Lambda^{\mathcal{U}}_\alpha$ we conclude $\overline{\mathcal{U}_\alpha}^\beta \in \Lambda^{\mathcal{U}}_{\gamma(\alpha,\beta)}$, i.e. $\mathcal{U}_{\gamma(\alpha,\beta)} \leq \overline{\mathcal{U}_\alpha}^\beta$. For the converse, let $\Phi \in \Lambda^{\mathcal{U}}_\alpha$. Then $\Phi \geq \mathcal{U}_\alpha$ which implies $\overline{\Phi}^\beta \geq \overline{\mathcal{U}_\alpha}^\beta \geq \mathcal{U}_{\gamma(\alpha,\beta)}$, i.e. $\overline{\Phi}^\beta \in \Lambda^{\mathcal{U}}_{\gamma(\alpha,\beta)}$. □

We note that for $\gamma(\alpha,\beta) = \alpha * \beta * \beta$, $(X,\overline{\Lambda^{\mathcal{U}}})$ is γ-regular. Hence, we always have $\overline{\mathcal{U}_\alpha}^\beta \geq \mathcal{U}_{\alpha*\beta*\beta}$.

6. An extension theorem for uniformly continuous mappings

In [14], an extension theorem for continuous mappings between L-convergence tower spaces was given. The following notation and results are needed. Let $(X,\overline{q^X}), (Y,\overline{q^Y})$ be L-convergence tower spaces, let $A \subseteq X$ and let $f : A \longrightarrow Y$ be a mapping. For $x \in X$ and $\alpha \in L$ we denote

$$\begin{aligned} H^\alpha_A(x) &= \{\mathbb{F} \in \mathsf{F}(X) \ : \ \mathbb{F}_A \in \mathsf{F}(A), x \in q^X_\alpha(\mathbb{F})\} \\ F^\alpha_A(x) &= \begin{cases} \{y \in Y \ : \ y \in q^Y_\alpha(f(\mathbb{F}_A)) \forall \mathbb{F} \in H^\alpha_A(x)\} & \text{if} \quad H^\alpha_A(x) \neq \emptyset \\ Y & \text{if} \quad H^\alpha_A(x) = \emptyset \end{cases} \end{aligned}$$

We have $H^\beta_A(x) \subseteq H^\alpha_A(x)$ whenever $\alpha \leq \beta$. With the notation $\overline{A}^\alpha = \{x \in X \ : \ \exists \mathbb{F} \in \mathsf{F}(X), A \in \mathbb{F}, x \in q_\alpha(\mathbb{F})\}$ we have $x \in \overline{A}^\alpha$ if and only if $H^\alpha_A(x) \neq \emptyset$. We call $A \subseteq X$ *dense in* $(X,\overline{q^X})$ if $\overline{A}^\top = X$. For a dense subset $A \subseteq X$ all $H^\alpha_A(x)$ are non-empty. Furthermore, we call an L-convergence tower space $(X,\overline{q})$ a *T2-space* if $x,y \in q_\top(\mathbb{F})$ implies $x = y$.

Theorem 6.1. ([14]) *Let $\gamma : L \times L \longrightarrow L$ satisfy $\bigvee_{\beta \lhd \top} \gamma(\gamma(\alpha,\beta),\beta) \geq \alpha$ for all $\alpha,\beta \in L$.*

Let $(X,\overline{q^X}), (Y,\overline{q^Y})$ be L*-convergence tower spaces and let $(X,\overline{q^X})$ satisfy (LK-γ) and $(Y,\overline{q^Y})$ be a left-continuous T2-space and satisfy (LR-γ). Let further $A \subseteq X$ be dense in $(X,\overline{q^X})$ and let $f : (A,\overline{q^X|_A}) \longrightarrow (Y,\overline{q^Y})$ be continuous. The following are equivalent:*

(i) There is a unique continuous mapping $g : (X,\overline{q^X}) \longrightarrow (Y,\overline{q^Y})$ such that $g \circ \iota_A = f$.
(ii) for each $x \in X$, $\bigcap_{\alpha \in L} F^\alpha_A(x) \neq \emptyset$.

We now prove a related extension theorem for uniformly continuous mappings between L-uniform convergence tower spaces. For the classical result we refer to Gähler [8]. In the restricted framework of approach uniform convergence spaces a similar result was obtained in [10].

We have to restrict the lattice context to frames, i.e. to quantales with idempotent quantale operation, $\alpha * \alpha = \alpha$ for all $\alpha \in L$. It is not difficult to see that this implies that $\alpha * \beta = \alpha \wedge \beta$ for all $\alpha, \beta \in L$. Furthermore, we require a stronger separation axiom. We say that an L-convergence tower space $(X, \overline{q})$ is a *strong T2-space* if $x = y$ whenever $x \in q_\top(\mathbb{F})$ and $y \in q_\alpha(\mathbb{F})$ for some $\alpha > \bot$. Clearly, a strong T2-space is a T2-space. An L-uniform convergence tower space $(X, \overline{\Lambda})$ is a strong T2-space if $(X, \overline{q^{\overline{\Lambda}}})$ is a strong T2-space.

Example 6.2. Let $L = \{0, 0.5, 1\}$ with the usual order and consider the minimum as quantale operation. Let further (X, p) be a limit space. We define $q_0(\mathbb{F}) = X$ for all $\mathbb{F} \in \mathsf{F}(X)$, $q_{0.5} = p$ and q_1 by $x \in q_1(\mathbb{F})$ if and only if $\mathbb{F} = [x]$. Then $(X, \overline{q})$ is a left-continuous L-convergence tower space, which is a strong T2-space if (X, p) is a T1-space, i.e. if $y \in p([x])$ implies $x = y$.

We note that being a strong T2-space is a very strong requirement. For an L-metric space (X, d) the space $(X, \overline{q^d})$ is a strong T2-space only if (X, d) is discrete. In fact, if we assume that (X, d) is not discrete and $d(x, y) = \alpha \notin \{\bot, \top\}$ for $x \neq y$, then $y \in q^d_\alpha([x])$ by the definition of $\overline{q^d}$. As also $x \in q^d_\top([x])$ we conclude $x = y$, a contradiction.

An L-uniform convergence tower space $(X, \overline{\Lambda})$ is called *complete* if for all $\alpha \in L$, $\mathbb{F} \times \mathbb{F} \in \Lambda_\alpha$ implies that there is $x \in X$ such that $[x] \times \mathbb{F} \in \Lambda_\alpha$, see [16].

Theorem 6.3. *Let L be a frame and let $\gamma : L \times L \longrightarrow L$ satisfy $\bigvee_{\beta \lhd \top} \gamma(\gamma(\alpha, \beta), \beta) \geq \alpha$ for all $\alpha, \beta \in L$.*

Let $(X, \overline{\Lambda^X}), (Y, \overline{\Lambda^Y})$ be L-uniform convergence tower spaces and let $(X, \overline{\Lambda^X})$ satisfy the axiom (LUG-γ) and let $(Y, \overline{\Lambda^Y})$ be a complete, left-continuous, strong T2-space and satisfy (LUR-γ). Let further $A \subseteq X$ be dense in $(X, \overline{q^{\overline{\Lambda^X}}})$ and let $f : (A, \overline{\Lambda^X}|_A) \longrightarrow (Y, \overline{\Lambda^Y})$ be uniformly continuous. Then there is a uniformly continuous mapping $g : (X, \overline{\Lambda^X}) \longrightarrow (Y, \overline{\Lambda^Y})$ such that $g \circ \iota_A = f$.

Proof. We have seen in Proposition 4.12 that $(X, \overline{\Lambda^X})$ satisfies (LUF-γ) and hence also (LUK-γ). By Proposition 4.6 then $(X, \overline{q^{\overline{\Lambda}}})$ satisfies (LK-γ). Also, using Proposition 5.1, we know that $(X, \overline{q^{\overline{\Lambda}}})$ is a γ-regular, left-continuous T2-space. As $f : (A, \overline{\Lambda^X}|_A) \longrightarrow (Y, \overline{\Lambda^Y})$ is uniformly continuous, we have that $f : (A, \overline{q^{\overline{\Lambda^X}}}|_A) \longrightarrow (Y, \overline{q^{\overline{\Lambda^Y}}})$ is continuous. We want to show that for all $x \in X$, $\bigcap_{\alpha \in L} F^\alpha_A(x) \neq \emptyset$. Let $\mathbb{F} \in H^\top_A(x)$. Then $\mathbb{F}_A \in \mathsf{F}(A)$ and $[x] \times \mathbb{F} \in \Lambda^X_\top$ and hence, by (LUC5) and (LUC6), $\mathbb{F} \times \mathbb{F} = (\mathbb{F} \times [x]) \circ ([x] \times \mathbb{F}) \in \Lambda^X_\top$. From $\iota_A(\mathbb{F}_A) \geq \mathbb{F}$ we conclude $(\iota_A \times \iota_A)(\mathbb{F}_A \times \mathbb{F}_A) \geq \mathbb{F} \times \mathbb{F}$ and hence, $\mathbb{F}_A \times \mathbb{F}_A \in \Lambda^X_\top|_A$. From the uniform continuity of f we obtain $\varphi(\mathbb{F}_A) \times \varphi(\mathbb{F}_A) \in \Lambda^Y_\top$ and the completeness of $(Y, \overline{\Lambda^Y})$ ensures the existence of $y_0 \in Y$ such that $[y_0] \times \varphi(\mathbb{F}_A) \in \Lambda^Y_\top$. We claim that $y_0 \in F^\alpha_A(x)$ for all $\alpha \in L$. To this end, let $\mathbb{G} \in H^\alpha_A(x)$. Then $\mathbb{G}_A \in \mathsf{F}(A)$ and $x \in q^{\overline{\Lambda^X}}_\alpha(\mathbb{G})$. As also $x \in q^{\overline{\Lambda^X}}_\top(\mathbb{F}) \subseteq q^{\overline{\Lambda^X}}_\alpha(\mathbb{F})$, we conclude $x \in q^{\overline{\Lambda^X}}_\alpha(\mathbb{F} \wedge \mathbb{G})$. As before, we obtain $(\mathbb{F} \wedge \mathbb{G}) \times (\mathbb{F} \wedge \mathbb{G}) \in \Lambda^X_{\alpha * \alpha} = \Lambda^X_\alpha$ and also $(\mathbb{F} \wedge \mathbb{G})_A \times (\mathbb{F} \wedge \mathbb{G})_A \in \Lambda^X_\alpha|_A$. Uniform conituity of f and completeness of $(Y, \overline{\Lambda^Y})$ ensure that there exists $y_1 \in q^{\overline{\Lambda^Y}}_\alpha(f((\mathbb{F} \wedge \mathbb{G})_A)) \subseteq q^{\overline{\Lambda^Y}}_\alpha(f(\mathbb{F}_A)) \cap q^{\overline{\Lambda^Y}}_\alpha(f(\mathbb{G}_A))$. The strong T2-property implies $y_1 = y_0$ and as $\mathbb{G} \in H^\alpha_A(x)$ was arbitrary, we finally obtain $y_0 \in F^\alpha_A(x)$.

Theorem 6.1 yields the existence of a continuous extension $g : (X, \overline{q^{\overline{\Lambda^X}}}) \longrightarrow (Y, \overline{q^{\overline{\Lambda^Y}}})$ of f and we need to show that $g : (X, \overline{\Lambda^X}) \longrightarrow (Y, \overline{\Lambda^Y})$ is uniformly continuous. Let $\Phi \in \Lambda^X_\alpha$. For $\beta \lhd \top$ then $\mathbb{U}_\beta(\Phi) \in \Lambda^X_{\gamma(\alpha,\beta)}$.

We now show that $\mathbb{U}_\beta(\Phi)_{A \times A}$ exists. As $\overline{A}^\top = X$, for $(x, y) \in X \times X$ there are $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$ such that $A \in \mathbb{F}, \mathbb{G}$ and $x \in q^{\overline{\Lambda^X}}_\top(\mathbb{F}) \subseteq q^{\overline{\Lambda^X}}_\beta(\mathbb{F})$ and $y \in q^{\overline{\Lambda^X}}_\top(\mathbb{G}) \subseteq q^{\overline{\Lambda^X}}_\beta(\mathbb{G})$. For $\Theta := \mathbb{F} \times \mathbb{G}$ then $A \times A \in \Theta$ and $(x, y) \in q^{\overline{\Lambda^X}}_\beta \times q^{\overline{\Lambda^X}}_\beta(\Theta)$. Hence $\mathbb{U}^{(x,y)}_\beta \leq \Theta$ and the trace $(\mathbb{U}^{(x,y)}_\beta)_{A \times A}$ exists. For $H \in \mathbb{U}_\beta(\Phi)$ there is $M \in \Phi$ such that $H \in \mathbb{U}^{(x,y)}_\beta$ whenever $(x, y) \in M$. Therefore, $H \cap (A \times A) \neq \emptyset$ and $\mathbb{U}_\beta(\Phi)_{A \times A}$ exists.

We conclude that $(\iota_A \times \iota_A)(\mathbb{U}_\beta(\Phi)) \geq \mathbb{U}_\beta(\Phi) \in \Lambda^X_{\gamma(\alpha,\beta)}$ and therefore $\mathbb{U}_\beta(\Phi)_{A\times A} \in \Lambda^X_{\gamma(\alpha,\beta)}|_{A\times A}$. The uniform continuity of f then ensures that $(f\times f)(\mathbb{U}_\beta(\Phi)_{A\times A}) \in \Lambda^Y_{\gamma(\alpha,\beta)}$ and the regularity of $(Y, \overline{\Lambda^Y})$ yields $\overline{(f\times f)(\mathbb{U}_\beta(\Phi)_{A\times A})}^\beta \in \Lambda^Y_{\gamma(\gamma(\alpha,\beta),\beta)}$.

We next show that $(g\times g)(\Phi) \geq \overline{(f\times f)(\mathbb{U}_\beta(\Phi)_{A\times A})}^\beta$. Let $K \in \overline{(f\times f)(\mathbb{U}_\beta(\Phi)_{A\times A})}^\beta$. Then there is $H \in \mathbb{U}_\beta(\Phi)$ such that $\overline{(f\times f)(H\cap(A\times A))}^\beta \subseteq K$. There is $M \in \Phi$ such that $H \in \mathbb{U}^{(x,y)}_\beta$ for all $(x,y)\in M$. We show that $(g\times g)(M) \subseteq \overline{(f\times f)(H\cap(A\times A))}^\beta$. To this end, let $(s,t)\in(g\times g)(M)$. Then there are $(x,y)\in M$ with $g(x)=s$ and $g(y)=t$. As $\overline{A}^\top = X$ we can choose again $\Theta \in \mathsf{F}(X\times X)$ such that $A\times A \in \Theta$ and $(x,y) \in q^{\overline{\Lambda^X}}_\beta \times q^{\overline{\Lambda^X}}_\beta(\Theta)$. The continuity of g ensures $(s,t) = (g(x),g(y)) \in q^{\overline{\Lambda^Y}}_\beta \times q^{\overline{\Lambda^Y}}_\beta((g\times g)(\Theta))$. Moreover, from $H \in \mathbb{U}^{(x,y)}_\beta \leq \Theta$ we conclude $H\cap(A\times A)\in\Theta$ and hence, $(f\times f)(H\cap(A\times A)) = (g\times g)(H\cap(A\times A)) \in (g\times g)(\Theta)$. We conclude $(s,t) \in \overline{(f\times f)(\mathbb{U}_\beta(\Phi)_{A\times A})}^\beta$. This shows $(g\times g)(M)\subseteq K$, i.e. $K \in (g\times g)(\Phi)$.

We deduce finally that $(g\times g)(\Phi) \in \Lambda^Y_{\gamma(\gamma(\alpha,\beta),\beta)}$ for all $\beta \lhd \top$. The left-continuity of $(Y,\overline{\Lambda^Y})$ yields $(g\times g)(\Phi) \in \Lambda^Y_{\bigvee_{\beta\lhd\top}\gamma(\gamma(\alpha,\beta),\beta)} \subseteq \Lambda^Y_\alpha$ and the uniform continuity of g is established. $\square$

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