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Generalized solution of Burger equation

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Abstract. In this paper, The existence and uniqueness of the generalized solution of the Burger equation is studied with initial conditions are distributions (elements of Colombeau algebra). then we look at the association notion in conjunction with the classic solution.

1. Introduction

The Burgers equation is a partial differential equation (see e.g. [17–19]) derived from fluid mechanics. It appears in various fields of applied mathematics, such as the modeling of gas dynamics, acoustics or road traffic. It is named after Johannes Martinus Burgers who discussed it in 1948 [7]. One-dimensional nonlinear The Burger equation was first introduced by his Bateman H. [8] in 1915, who found it to be stable. A solution that describes a particular viscous flow. It was later proposed by Burger J.M. (1948). [9] belongs to the class of equations describing mathematical turbulence models. after that, The Burger equation was studied by Cole J.D. (1951) [10] and provided a theoretical solution. Moreover, Currò C., Donato A., Povzner A.Ya. they studied the Perturbation method for a generalized Burgers equation (1992)[20]. Fourier series analysis with appropriate initial and boundary conditions. Gorgis A. (2005) [11] presents a comparison of the Cole-Hopf transformation and decomposition. How to solve the Berger equation. Momani S., (2006) [12] published The unperturbed solution of the partial Burger equation in space and time from Atomic decomposition method. Inc (2008) [15] solved it using variational iteration. Space-time fractional equations. Wang Qi. (2008) extending the application of [16] Homotopy Perturbation and Adminian Decomposition Methods for Building Approximations Solution of the nonlinear fractional KdV-Burger equation. Biother J and Aminikhah H. (2009) [13] Solve the Burger equation using variational iteration (VIM). Which approximate solutions are found and which are better than his ADM. (2011), Pandey K. and Verma L. [14] gave a note on his Crank-Nicolson scheme for the Burger equation. The Hopf-Cole transform solution is obtained by ignoring nonlinear terms.
The space of distributions noted $\mathcal{D}^{\infty}(\mathbb{R})$ was formalized by the mathematician Laurent Schwartz, in 1940 whose the goal to extend the notion of the function to the Dirac mass ($\delta(0) = +\infty$ and $\delta(x) = 0\ \forall x \in \mathbb{R}^*$) and formalize solutions for partial differential equations .... In 1951, Laurent Schwartz evoked the problem of the non-linearity of distributions, namely δ^2 which has no meaning in the space of distributions. Moreover

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he asserted that there does not exist an algebra in which the distributions are injected and the product problem will be solved provided that the product of continuous functions coincides with the product in this algebra. In 1984, the mathematician Jean François Colombeau constructed an algebra noted, named algebra of generalized functions in which the space of distributions is linearly injected and the product remains well defined [1], [2],[6] .
The paper is organized as follows. After the introductory part,we give some basic preliminaries such as notations and definitions of the objects we shall work with, we also introduce different spaces of Colombeau algebra of generalized functions. In the third section we proved the existence and uniqueness of solution of Burger equation with variable speed and initial data in the Colombeau algebra $\mathcal{G}(\mathbb{R})$. Finally, in the fifth section we study the association (Application).

2. Preliminaries

In this section, we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in[0;1]}$ of elements of a set X will be denoted by $X^{[0;1]}$, such sequences will also be called nets and simply written as u_ε.
Let $\mathcal{D}(\mathbb{R})$ be the space of all smooth functions $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ with compact support. For $q \in \mathbb{N}$ we denote

$$\mathcal{A}_q(\mathbb{R}) = \left\{ \varphi \in \mathcal{D}(\mathbb{R}) \,/ \int \varphi(x)dx = 1 \;\text{ and }\; \int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \le \alpha \le q \right\}$$

The elements of the set $\mathcal{A}_q$ are called test functions.
It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \ldots$. Colombeau in his books has proved that the sets $\mathcal{A}_k$ are non empty for all $k \in \mathbb{N}$.
For $\varphi \in \mathcal{A}_q(\mathbb{R})$ and $\epsilon > 0$ it is denoted as $\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\check{\varphi}(x) = \varphi(-x)$
We denote by

$$\mathcal{E}(\mathbb{R}) = \{u : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{C} / \quad \text{with} \quad u(\varphi, x) \text{is} \quad C^\infty \quad \text{to the second variable} \quad x\}$$

with $\forall \varphi \in \mathcal{A}_1$

$$u(\varphi_\varepsilon, x) = u_\varepsilon(x), \quad \forall x \in \mathbb{R}$$

$$\mathcal{E}_M(\mathbb{R}) = \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{N}_0, \exists N \in \mathbb{N} \text{ such that } \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \quad \text{as} \quad \varepsilon \to 0\}$$

this set named the set of moderate elements.

$$\mathcal{N}(\mathbb{R}) = \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \; \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{N}_0, \forall p \in \mathbb{N}; \quad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^p) \quad \text{as} \quad \varepsilon \to 0\}$$

this set named the set of negligible elements.
The generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G}(\mathbb{R}) = \mathcal{E}_M(\mathbb{R}) / \mathcal{N}(\mathbb{R})$$

Any element u of $\mathcal{G}(\mathbb{R})$ is called a generalized function, written as:

$$u = (u_\varepsilon) + \mathcal{N}(\mathbb{R}) \quad \text{with} \quad (u_\varepsilon) \in \mathcal{E}_M(\mathbb{R})$$

The meaning of the term association in Colombeau's algebra is given as follows:
Let $u, v \in \mathcal{G}(\mathbb{R})$, We say that u and v are associated and we note $u \approx v$, if

$$lim_{\varepsilon\to 0} \int_{\mathbb{R}} (u_\varepsilon - v_\varepsilon)(x)\psi(x)dx = 0$$

For all $\psi \in \mathcal{D}(\mathbb{R})$.

3. Existence and uniqueness of the generalized solution

. To show the existence and the uniqueness of the solution of this problem

$$(1)\begin{cases} \partial_t y(t,x) + y(t,x)\partial_x y(t,x) = F(t,x,y(t,x)), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \\ y(0,x) = y_0(x), \quad x \in \mathbb{R}. \end{cases}$$

where $y_0(x)$ is a discontinue function, we need the following definitions and propositions[5]:

Definition 3.1. *We say that f generalized function is of logarithmic type, if there exists a representative (f_ε) of f such that*

$$\sup_{x\in K}|f_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad \text{when} \quad \varepsilon \longrightarrow 0$$

Proposition 3.2. *If f is of logarithmic type, then for any representative of (f_ε)*

$$\sup_{x\in K}|f_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad \text{when} \quad \varepsilon \longrightarrow 0$$

Proof. f is of logarithmic type then there exists a representative of f, such that

$$\sup_{x\in K}|f_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad \text{when} \quad \varepsilon \longrightarrow 0$$

Let (g_ε) be another representative of f, Then

$$|f_\varepsilon(x) - g_\varepsilon(x)| \le c.\varepsilon^q \quad \forall q \quad \forall x \in K_T$$

We find

$$|g_\varepsilon(x)| \le |f_\varepsilon(x)| + c\cdot\varepsilon^q \quad \forall q \quad \forall x \in K_T, \quad \text{for} \quad q \longrightarrow +\infty$$

And then

$$|g_\varepsilon(x)| \le |f_\varepsilon(x)|$$

$$\sup_{x\in K}|g_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad \text{when} \quad \varepsilon \longrightarrow 0$$

□

Definition 3.3. *We say that a generalized function f is globally bounded, if there exists a representative (f_ε) of f and $M \ge 0$ such that :*

$$\sup_{x\in\mathbb{R}^n}|f_\varepsilon(x)| \le M$$

Example 3.4.

$$f_\varepsilon(x) = \mathcal{N}(0,1)$$

Proposition 3.5. *If f is globally bounded, then for any representative (f_ε) of f we have:*

$$\sup_{x\in\mathbb{R}^n}|f_\varepsilon(x)| < M \quad \text{when } \varepsilon \longrightarrow 0$$

Proof. f is globally bounded, then there exists a representative of (f_ε) of f such that :

$$\sup_{x\in K}|f_\varepsilon(x)| \le M$$

Let (g_ε) be another representative of f, then :

$$|f_\varepsilon(x) - g_\varepsilon(x)| \le c.\varepsilon^q \quad \forall q \in \mathbb{N}$$

As

$$|f_\varepsilon(x)| - |g_\varepsilon(x)| \leq c.\varepsilon^q \quad \forall q \in \mathbb{N}$$

So

$$|g_\varepsilon(x)| \leq |f_\varepsilon(x)| + c.\varepsilon^q \quad \forall q \in \mathbb{N}$$

So

$$|g_\varepsilon(x)| \leq M + c \cdot \varepsilon^q \quad \forall q \in \mathbb{N}$$

For $q \longrightarrow +\infty$ we find:

$$|g_\varepsilon(x)| \leq M \quad \forall x \in \mathbb{R}$$

□

Theorem 3.6. *If $\nabla_x f$ is of logarithmic type, then for the problem* (1) *has a unique solution in* $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R})$.

Proof. **Existence**
We have :

$$\begin{cases} \partial_t y(t,x) + y(t,x)\partial_x y(t,x) = f(t,x,y(t,x)), & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ y(0,x) = y_0(x), & t \geq 0 \end{cases}$$

with $y_0(x)$ are discontinuous functions, the solution according to the characteristic curves :

$$y_\varepsilon(t,x) = y_{0\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right) + \int_0^t f_\varepsilon\left(t, y_\varepsilon(t,x), \tau\right) d\tau$$

$$y_\varepsilon(t,x) = y_{0\varepsilon}\left(\lambda_\varepsilon(0,x,0)\right) + \int_0^t f_\varepsilon(t,0,\tau) d\tau + \int_0^t \int_0^1 \nabla_x f_\varepsilon\left(t, \sigma y_\varepsilon(t,x), \tau\right) y_\varepsilon(t,x) d\sigma d\tau$$

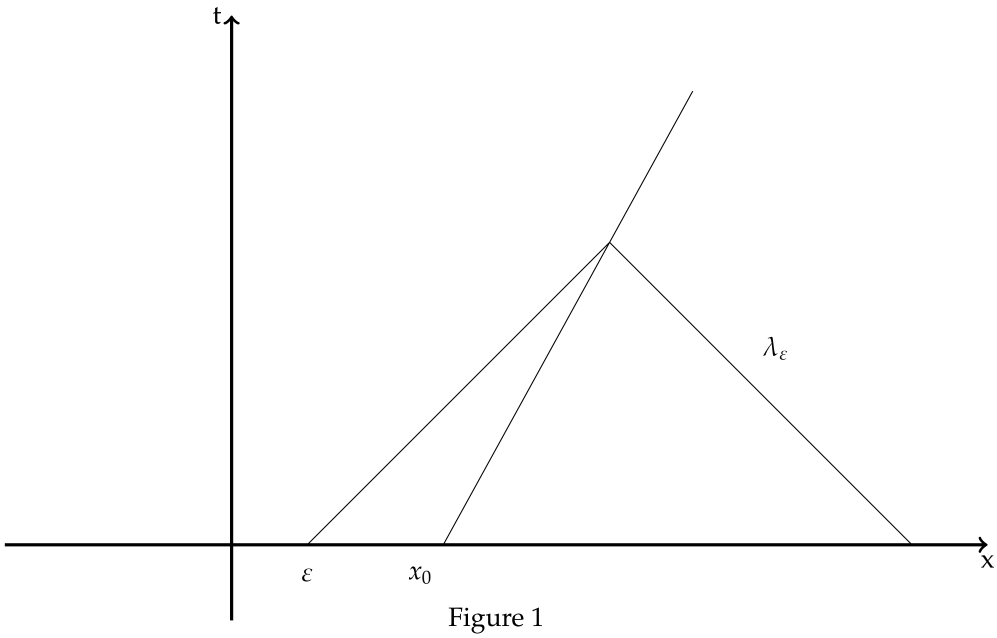

Figure 1

with λ_ε the characteristic curve corresponding to $y_{0,\varepsilon}$.
Let $T > t$, then we get :
By hypothesis, y_0 is globally bounded:

$$\exists M \geq 0, \quad \text{Such as} \quad \sup_{x\in\mathbb{R}} |y_{0\varepsilon}(x)| \leq M \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}$$

So

$$|\frac{d\lambda_\varepsilon(t,x,\tau)}{d\tau}| \leq M$$

With $\lambda_\varepsilon(t,x,\tau)$ the characteristic curve corresponding to $y_{0,\varepsilon}$ coming from the point $(0,x)$.

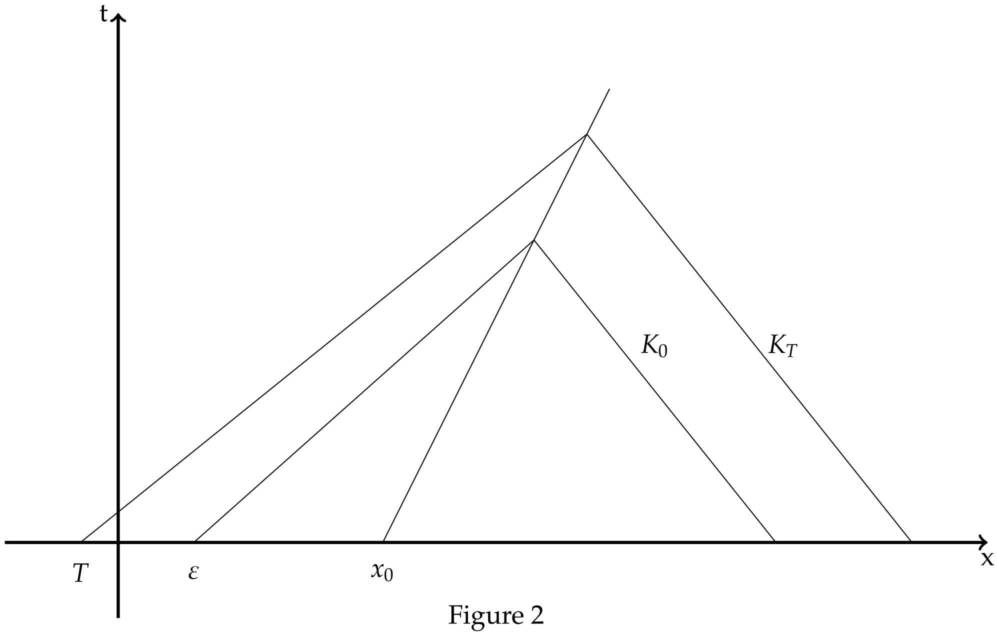

Figure 2

Let K_T be the triangle bounded by:

$$|y_\varepsilon(t,x)|_{K_T} \leq |y_{0i\varepsilon}(t)|_{K_0} + T|f_{i\varepsilon}(t,x)|_{K_T} + |\nabla_x f_\varepsilon|_{K_T} \int_0^t |y_\varepsilon(t,x)|_{K_\tau} d\tau$$

Apply Gronwall's inequality to the function

$$\tau \longmapsto |y_\varepsilon(t,x)|_{K_\tau}$$

$$|y_\varepsilon(t,x)|_{K_T} \leq [|y_{0\varepsilon}(t)|_{K_0} + T|f_\varepsilon(t,x)|_{K_T}] \times \exp\left(\int_0^T |\nabla f_\varepsilon|_{K_T} d\tau\right)$$

$$|y_\varepsilon(t,x)|_{K_T} \leq [|y_{0\varepsilon}(t)|_{K_0} + T|f_\varepsilon(t,x)|_{K_T}] \times \exp\left(T|\nabla_x f_\varepsilon|_{K_T}\right)$$

Such as $\nabla_x f_\varepsilon$ is of logarithmic type

$$\exists N \in \mathbb{N} \quad |y_{i\varepsilon}(t,x)|_{K_T} = O\left(\varepsilon^{-N}\right)$$

Uniqueness

Suppose problem (1) has two solutions $x, y \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R})$.

Then,

$$\begin{aligned} x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x) &= x_{0i\varepsilon}(\lambda_{i\varepsilon}(t,x,0)) - y_{0i\varepsilon}(\lambda_{i\varepsilon}(t,x,0)) \\ &\quad + \int_0^t \left(f_{i\varepsilon}(t, x_{i\varepsilon}(t,x),\tau) - f_{i\varepsilon}(t, y_{i\varepsilon}(t,x),\tau)\right) d\tau \\ &= (x_{0i\varepsilon} - y_{0i\varepsilon})(\lambda_{i\varepsilon}(t,x,0)) + \int_0^t (f_{i\varepsilon}(t, x_{i\varepsilon}(t,x),\tau) - f_{i\varepsilon}(t, y_{i\varepsilon}(t,x),\tau))d\tau \end{aligned}$$

So

$$|x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x)|_{K_T} \leq |(x_{0i\varepsilon} - y_{0i\varepsilon})(\lambda_{i\varepsilon}(t,x,0))|_{K_0} + \int_0^T |\nabla f_{i\varepsilon}||x_{i\varepsilon}(t,x,\tau)) - y_{i\varepsilon}(t,x,\tau)|_{K_\tau} d\tau$$

Let us apply Granwall's inequality on the function,

$$T \longrightarrow |x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x)|_{K_T}$$

We obtain

$$|x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x)|_{K_T} \leq |x_{0i\varepsilon} - y_{0i\varepsilon}|_{K_0} \exp(\int_0^T |\nabla_x f_{i\varepsilon}| d\tau)$$

$$|x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x)|_{K_T} \leq |x_{0i\varepsilon} - y_{0i\varepsilon}|_{K_0} \exp(T|\nabla_x f_{i\varepsilon}|)$$

As $\nabla_x f_{i\varepsilon}$ is of logarithmic type and $\eta_{i\varepsilon} \in \mathcal{N}(\mathbb{R}^+)$
Then

$$|x_{i\varepsilon}(t,x) - y_{i\varepsilon}(t,x)|_{K_T} = O(\varepsilon^q) \qquad \forall q \in \mathbb{N}$$

□

4. Application

We consider the following problem which presents the propagation in a discontinuous medium.

$$(2) \begin{cases} \partial_t y(t,x) + y(t,x)\partial_x y(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ y(0,x) = y_0(x) \end{cases}$$

With

$$y_0(x) = \begin{cases} c_1, & x \leq x_0 \\ c_2, & x > x_0 \end{cases} \quad c_1 \neq c_2 > 0$$

$y_0(x)$ is discontinuous in $x_0 = 1$, The problem (2) admits a classical solution for the domain :

$$(I) = \{x \leq x_0, \quad t \geq 0\} \quad \text{and} \quad (II) = \{x > x_0, \quad t \geq 0\}$$

and while imposing a passing condition at x_0, then we have a solution on $\{x \geq x_0, \quad \text{and} \quad t \geq 0\}$.
For this, either $\phi \in \mathcal{D}(\mathbb{R}^+)$ $\operatorname{supp}(\phi) \in \left]1 - \eta_\varepsilon; 1 + \eta_\varepsilon\right[$.
We pose

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon = \frac{1}{|\ln(\varepsilon)|}$$
$$y_\varepsilon(x) = y * \phi_\varepsilon(x), \quad y_\varepsilon \in C^\infty(\mathbb{R}^+)$$

let λ_ε be the characteristic curve associated with y_ε

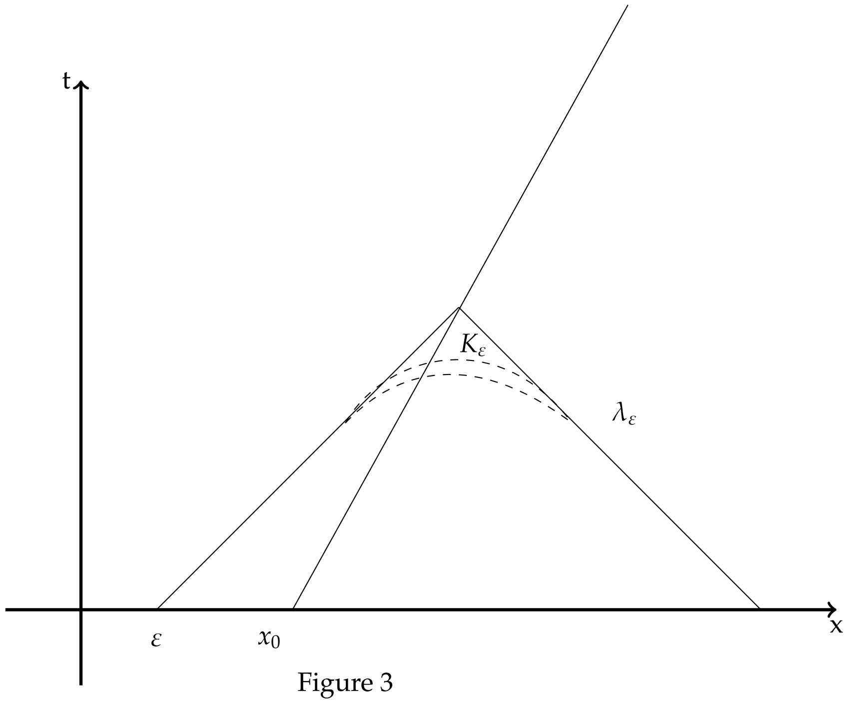

Figure 3

So the problem (2) admits as solution

$$y(t,x) = \begin{cases} y\left(\lambda_1(x,t,0)\right), & \text{on the domain (I)} \\ y\left(\lambda_2(x,t,0)\right), & \text{on the domain (II)} \end{cases}$$

λ_1 the characteristic curve associated with c_1.
λ_2 the characteristic curve associated with c_2.

Proposition 4.1. *With $y_0(x)$ is a discontinuous function in x_0 the problem (2) has a solution $y \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R})$, as:*

$$Y \approx y \quad with \quad Y = [(y_\varepsilon)] \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R})$$

Proof. According to the theorem (3) the problem (2) has a unique solution
$y \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R})$ It remains to show that

$$Y \approx y$$

On the domain (I), we fix:

$$\lambda_{1,\varepsilon} = \lambda_1 * \phi_{\eta_\varepsilon}$$

For example

$$y_0 * \phi_\varepsilon \longrightarrow y_{0,\varepsilon} \quad \text{when} \quad \varepsilon \longrightarrow 0$$

So

$$\begin{aligned} \int_I \left(y_\varepsilon(t,x) - y(t,x)\right)\psi(t,x)dxdt &= \int_I \left(y_{0,\varepsilon}\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right)\right)dxdt \\ &= \int_I y_{0,\varepsilon}\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right)dxdt \\ &\quad + \int_I y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right)dxdt \\ &= \int_I \left(y_{0,\varepsilon} - y_0\right)\left(\lambda_{1,\varepsilon}(t,x,0)\right)dxdt \\ &\quad + \int_I [y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right)]dxdt \end{aligned}$$

We know that y_0 is globally bounded, So

$$
\begin{aligned}
|\int_I y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right) dxdt| &\leq \int_I |y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right)| dxdt \\
&\leq \int_I \sup_{x\in K} |y_0||\lambda_{1,\varepsilon}(t,x,0) - \lambda_1(t,x,0)| dxdt \\
&\leq \sup_{x\in K} |y_0| \int_I |\lambda_{1,\varepsilon}(t,x,0) - \lambda_1(t,x,0)| dxdt
\end{aligned}
$$

So

$$\lim_{\varepsilon\to 0} \int_I y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_1(t,x,0)\right) dxdt = 0$$

Moreover

$$
\begin{aligned}
|\int_I (y_{0,\varepsilon}\left(\lambda_{1,\varepsilon}(t,x,0)\right)) - y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) dxdt| &\leq \int_I |y_{0,\varepsilon}\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right)| dxdt \\
&\leq \sup_{x\in K_T} |y_{0,\varepsilon} - y_0| \int_I \lambda_{1,\varepsilon}(t,x,0) dxdt \\
&\leq \sup_{x\in K_T} |y_0 * \phi_\varepsilon - y_0| \int_I \lambda_{1,\varepsilon}(t,x,0) dxdt \\
&\leq mes(K_T) \sup_{x\in K_T} |y_0 * \phi_\varepsilon - y_0|
\end{aligned}
$$

So

$$\lim_{\varepsilon\to 0} \int_I y_{0,\varepsilon}\left(\lambda_{1,\varepsilon}(t,x,0)\right) - y_0\left(\lambda_{1,\varepsilon}(t,x,0)\right) dxdt = 0$$

So

$$\int_I \left(y_\varepsilon(t,x) - y(t,x)\right) \psi(t,x) dxdt = 0$$

The same reasoning for

$$\int_{II} \left(y_\varepsilon(t,x) - y(t,x)\right) \psi(t,x) dxdt = 0$$

So

$$Y \approx y$$

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