Filomat 38:2 (2024), 609–619
https://doi.org/10.2298/FIL2402609S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Stirling and Bell numbers of order $1/2$

Matthias Schork[a]

[a]*Institute for Mathematics, Würzburg University, Emil-Fischer Str. 40, 97074 Würzburg, Germany*

Abstract. The Stirling numbers of order $1/2$ (of the second kind) introduced by Katugampola are discussed and it is shown that they are given by a scaled subfamily of the generalized Stirling numbers introduced by Hsu and Shiue. This allows to deduce in a straightforward fashion many properties of the Stirling and Bell numbers of order $1/2$, for example, recurrence relations, generating functions, Dobiński formula, and Spivey formula. The even Bell polynomials of order $1/2$ are shown to be closely related to generalized Laguerre polynomials of order $-1/2$. Generalized Stirling numbers of order $1/2$ of the first kind are defined and studied. An analog of the Weyl algebra is introduced and proposed as a natural algebraic setting where the Stirling numbers of order $1/2$ of both kinds appear as ordering coefficients. This algebra contains the Weyl algebra as a subalgebra.

1. Introduction

The Stirling numbers of the second kind $S(n,k)$ (A008277 in [32]) count the number of set partitions of a set of n elements into k nonempty disjoint subsets and are among the most important combinatorial numbers, see, e.g., [9, 10, 22, 33]. If we denote by X and D the operators acting on functions of a real variable by $(Xf)(x) = xf(x)$ and $(Df)(x) = \frac{df}{dx}(x)$, then one has the commutation relation of the Weyl algebra,

$$DX - XD = I, \tag{1}$$

where I denotes the identity. The powers of the Euler operator (or, Mellin derivative) XD can be written in the normal ordered form

$$(XD)^n = \sum_{k=1}^{n} S(n,k) X^k D^k. \tag{2}$$

In this context, normal ordering means to bring a word in X and D into a form where all letters D stand to the right of all letters X using the commutation relation (1) (for more details concerning normal ordering, see [5, 24, 31]). The expansion (2) was already known to Scherk in 1823 [30] (but he did not recognize the coefficients as Stirling numbers). He also considered powers $(X^pD)^n$, for $p \in \mathbb{N}$, and studied the resulting normal ordering coefficients we would now call generalized Stirling numbers $S_{p,1}(n,k)$ (such that $S_{1,1}(n,k) = S(n,k)$), see the discussion in [5, Appendix A]. In [5], the combinatorial interpretation of $S_{p,1}(n,k)$

2020 *Mathematics Subject Classification*. Primary 05A10, 05A15, 11B73
Keywords. Stirling number, Bell number, Generalized Stirling number, Normal ordering, Weyl algebra
Received: 30 March 2023; Revised: 09 June 2023; Accepted: 14 July 2023
Communicated by Paola Bonacini
Email address: matthias.schork@mathematik.uni-wuerzburg.de (Matthias Schork)

in terms of trees is discussed and one can find many references to the literature where this connection was proved several times, see also the references given in [24]. More generally, Carlitz [7] and McCoy [28] considered normal ordering $(X^pD^q)^n$, for $p, q \in \mathbb{N}$, hence the generalized Stirling numbers $S_{p,q}(n, k)$. These generalized Stirling numbers have also been rediscovered several times, see, e.g., [24]. Recently, degenerate Stirling numbers as well as degenerate r-Stirling numbers were also discussed as normal ordering coefficients [14, 15, 18, 19]. Further considerations concerning normal ordering and generalized Stirling numbers can be found in [16, 17, 20, 25].

Katugampola [13, Definition 5.2] introduced the generalized Stirling numbers of order $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ to be normal ordering coefficients of $s^{\frac{1-n}{2}}(x^s \frac{d}{dx})^n$ when n is odd, and of $s^{-\frac{n}{2}}(x^{1-s}\frac{d}{dx})^n$ when n is even. For example, when $s = \frac{1}{2}$, one finds [13]

$$\begin{aligned} 2(\sqrt{x}D)^2 &= D + 2xD^2, \\ 2(\sqrt{x}D)^3 &= 3\sqrt{x}D^2 + 2\sqrt{x}^3D^3, \\ 2^2(\sqrt{x}D)^4 &= 3D^2 + 12xD^3 + 4x^2D^4, \end{aligned}$$

giving, e.g., $\mathtt{S}(4, 1) = 0, \mathtt{S}(4, 2) = 3, \mathtt{S}(4, 3) = 12, \mathtt{S}(4, 4) = 4$, where we denote the generalized Stirling numbers of order $\frac{1}{2}$ by $\mathtt{S}(n, k)$. Tables for the values of the generalized Stirling numbers for small n, k can be found for several values of s in [13]. Also, a connection to several sequences (A223168, A223523, A223524, A098503) in [32] was mentioned for $s = \frac{1}{2}$ as well as a connection to generalized Laguerre polynomials, but no systematic investigation was made. In the present work, we consider the case $s = \frac{1}{2}$ more closely.

In Section 2, we define the Stirling numbers $\mathtt{S}(n, k)$ and Bell numbers $\mathtt{B}_n$ of order $\frac{1}{2}$. In Section 3, we recall the definition and some properties of the generalized Stirling numbers $S(n, k; \alpha, \beta, r)$ due to Hsu and Shiue [11]. Using an operational interpretation of $S(n, k; \alpha, \beta, r)$ due to Kargın and Corcino [12], we identify $\mathtt{S}(n, k)$ with the scaled subfamily $S(n, k; \frac{1}{2}, 1, 0)$, allowing to deduce in a straightforward fashion many properties of $\mathtt{S}(n, k)$ and $\mathtt{B}_n$ and to introduce and study the Stirling numbers of order $\frac{1}{2}$ of the first kind, $\mathtt{s}(n, k)$. As an algebraic structure for considerations of normal ordering in this setting, an analog of the Weyl algebra is proposed in Section 4 and some ordering results analogous to those in the Weyl algebra are derived.

2. Stirling and Bell numbers of order $\frac{1}{2}$

As mentioned in the Introduction, Katugampola defined [13, Definition 5.1] the *Stirling numbers of order* $\frac{1}{2}$, in the following denoted by $\mathtt{S}(n, k)$, to be the coefficients of $2^{\frac{n-1}{2}}(X^{\frac{1}{2}}\frac{d}{dx})^n$ when n is odd, and of $2^{\frac{n}{2}}(X^{\frac{1}{2}}\frac{d}{dx})^n$ when n is even. Thus, we can write

$$2^{\frac{n}{2}}\left(X^{\frac{1}{2}}D\right)^n = X^{-\frac{n}{2}}\sum_{k=0}^{n}\mathtt{S}(n, k)\, X^kD^k, \quad \text{for } n \text{ even.}$$

Note that $\mathtt{S}(n, k) = 0$ for $0 \le k < n/2$ if n is even. Thus, the summands corresponding to "small" k vanish. In a similar fashion, for n odd, we have

$$2^{\frac{n-1}{2}}\left(X^{\frac{1}{2}}D\right)^n = X^{-\frac{n}{2}}\sum_{k=0}^{n}\mathtt{S}(n, k)\, X^kD^k, \quad \text{for } n \text{ odd.}$$

For n odd, we have $\mathtt{S}(n, k) = 0$ for $0 \le k < (n + 1)/2$. Denoting by $\lfloor x \rfloor$ the greatest integer less than or equal to x, we can combine these two observations as follows,

$$\left(X^{\frac{1}{2}}D\right)^n = \sum_{k=\lfloor\frac{n+1}{2}\rfloor}^{n} 2^{-\lfloor\frac{n}{2}\rfloor}\mathtt{S}(n, k)\, X^{k-\frac{n}{2}}D^k. \tag{3}$$

By comparing this with (2), we see that $\mathtt{S}(n, k)$ should be considered as a kind of generalized Stirling number of the second kind. Note that there is a slight difference between our definition and the one of Katugampola

[13]: In his coefficients $c_{n,k}$ (see Table 7 in [13]) the second index is chosen such that $c_{n,1}$ denotes the first *nonvanishing* expansion coefficient – which corresponds to $\mathtt{S}(n, \lfloor \frac{n+1}{2} \rfloor)$. Thus, there is a shift of $\lfloor \frac{n+1}{2} \rfloor - 1$ in the second index, i.e.,

$$c_{n,k} = \mathtt{S}(n, k + \lfloor \frac{n-1}{2} \rfloor). \tag{4}$$

Let us also introduce the associated *Bell polynomials of order* $\frac{1}{2}$ by

$$\mathtt{B}_n(x) = \sum_{k=1}^{n} \mathtt{S}(n,k) x^k. \tag{5}$$

For $x = 1$, one obtains the corresponding *Bell numbers of order* $\frac{1}{2}$, $\mathtt{B}_n \equiv \mathtt{B}_n(1) = \sum_{k=1}^{n} \mathtt{S}(n,k)$. One may also introduce the *Fubini numbers* (or *ordered Bell numbers*) *of order* $\frac{1}{2}$ by $\mathtt{F}_n = \sum_{k=0}^{n} \mathtt{S}(n,k)k!$, in analogy to the conventional case (see, e.g., [9] or A000670 in [32]). The first few values of $\mathtt{S}(n,k)$, $\mathtt{B}_n$ and $\mathtt{F}_n$ are displayed in Table 1.

Remark 2.1. *The first few Bell numbers of order* $\frac{1}{2}$ *(1, 3, 5, 19, 39, 173, 407, see Table 1) coincide with the beginning of sequence A242818 in [32]. This is no coincidence, as will be shown in the next section. The sequence of Fubini numbers of order* $\frac{1}{2}$ *starts with 1, 5, 18, 174, 1050, 15.210, 128.520 (see Table 1) and is not mentioned in [32].*

$n \backslash k$	1	2	3	4	5	6	7	Bell number	Fubini number
1	1							1	1
2	1	2						3	5
3		3	2					5	18
4		3	12	4				19	174
5			15	20	4			39	1.050
6			15	90	60	8		173	15.210
7				105	210	84	8	407	128.520

Table 1: The first few Stirling, Bell and Fubini numbers of order $\frac{1}{2}$.

One could now show several proporties of the generalized Stirling numbers $\mathtt{S}(n,k)$ directly from their definition. For example, introducing the notation

$$\varepsilon_\ell = \begin{cases} 0, & \text{if } \ell \text{ is even,} \\ 1, & \text{if } \ell \text{ is odd,} \end{cases} \tag{6}$$

we have the following result.

Proposition 2.2. *The Stirling numbers of order* $\frac{1}{2}$ *satisfy, for* $n \in \mathbb{N}$ *and* $0 \le k \le n$*, the recurrence relation*

$$\mathtt{S}(n+1,k) = (1+\varepsilon_n)\mathtt{S}(n,k-1) + (1+\varepsilon_n)\left(k - \frac{n}{2}\right)\mathtt{S}(n,k), \tag{7}$$

with initial value $\mathtt{S}(1,1) = 1$ *and with* $\mathtt{S}(n,k) = 0$ *for* $0 \le k < \lfloor \frac{n+1}{2} \rfloor$.

Using (7), one finds $\mathtt{S}(n+1,n+1) = (1+\varepsilon_n)\mathtt{S}(n,n)$, hence

$$\mathtt{S}(n+1,n+1) = (1+\varepsilon_n)(1+\varepsilon_{n-1})\mathtt{S}(n-1,n-1) = 2\mathtt{S}(n-1,n-1).$$

Since $\mathtt{S}(1,1) = 1$, this shows that

$$\mathtt{S}(n,n) = 2^{\lfloor \frac{n}{2} \rfloor}, \tag{8}$$

see Table 1. We will refrain from giving a direct proof of Proposition 2.2 since we will show in the next section that the $\mathtt{S}(n,k)$ are given by the generalized Stirling numbers of Hsu and Shiue [11] – from which all these properties can be derived easily (see, e.g., Corollary 3.3). From Proposition 2.2, one derives the following result.

Proposition 2.3. *The Bell polynomials of order $\frac{1}{2}$ satisfy, for $n \in \mathbb{N}$, the recurrence relation*

$$\mathtt{B}_{n+1}(x) = (1+\varepsilon_n)\left(x - \frac{n}{2}\right)\mathtt{B}_n(x) + (1+\varepsilon_n)x\frac{d\mathtt{B}_n}{dx}(x), \tag{9}$$

with initial value $\mathtt{B}_1(x) = x$.

Proof. Multiplying (7) with x^k and summing over k, one obtains

$$\mathtt{B}_{n+1}(x) = (1+\varepsilon_n)\sum_{k\geq 1}\mathtt{S}(n,k-1)x^k + (1+\varepsilon_n)\sum_{k\geq 1}\left(k - \frac{n}{2}\right)\mathtt{S}(n,k)x^k.$$

The first sum on the right-hand side equals $x\mathtt{B}_n(x)$, while the second sum on the right-hand side equals $x\frac{d}{dx}\mathtt{B}_n(x) - \frac{n}{2}\mathtt{B}_n(x)$, showing the assertion. □

Before closing this section, let us make the connection to the Laguerre polynomials mentioned in the Introduction explicit. Osipov [29] considered powers of the operator $B_\alpha = X^{-\alpha}DX^{1+\alpha}D$ (where α is not a negative integer). In particular, he showed that [29, Equation (6)]

$$L_n^\alpha(x) = \frac{(-1)^n}{n!}e^x B_\alpha^n e^{-x},$$

where $L_n^\alpha(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$ is a generalized Laguerre polynomial. Choosing $\alpha = -\frac{1}{2}$, one has $B_{-\frac{1}{2}} = X^{\frac{1}{2}}DX^{\frac{1}{2}}D$, hence $(B_{-\frac{1}{2}})^n = (X^{\frac{1}{2}}D)^{2n}$. Using (3), this yields

$$L_n^{-\frac{1}{2}}(x) = \frac{(-1)^n}{n!}e^x(X^{\frac{1}{2}}D)^{2n}e^{-x} = \frac{2^{-n}}{n!}\sum_{\ell=0}^{n}\mathtt{S}(2n,n+\ell)\,(-x)^\ell. \tag{10}$$

Thus,

$$\mathtt{S}(2n,n+\ell) = 2^n\binom{n-\frac{1}{2}}{n-\ell}\frac{n!}{\ell!}.$$

In particular, $\mathtt{S}(2n,n) = 2^n n!\binom{n-\frac{1}{2}}{n} = (2n-1)!!$, where $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$ denotes the sequence of the double factorial of odd numbers (A001147 in [32]) starting with $1, 3, 15, 105$, see Table 1. Recalling (5), one infers from (10) the following result.

Proposition 2.4. *The Bell polynomials of order $\frac{1}{2}$ satisfy, for $n \in \mathbb{N}$,*

$$\mathtt{B}_{2n}(x) = (2x)^n n! L_n^{-\frac{1}{2}}(-x). \tag{11}$$

3. The connection to the generalized Stirling numbers of Hsu and Shiue

In this section, we derive and exploit the connection between the Stirling numbers of order $\frac{1}{2}$ and the generalized Stirling numbers introduced by Hsu and Shiue [11]. For the convenience of the reader, we recall their definition and those properties we will use later on. Denoting the generalized factorial by $(z|\alpha)_n = z(z-\alpha)\cdots(z-(n-1)\alpha)$, the generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ are defined as connection coefficients,

$$(z|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)(z-r|\beta)_k.$$

Here the parameters α, β, r are real or complex parameters (note that the original restriction $(\alpha, \beta, r) \neq (0, 0, 0)$ for the definition of the $S(n, k; \alpha, \beta, r)$ is unnecessary [21, Section 6]). They are given, for $\beta \neq 0$, explicitly by

$$S(n, k; \alpha, \beta, r) = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + r|\alpha)_n, \tag{12}$$

and they satisfy the recurrence relation

$$S(n+1, k; \alpha, \beta, r) = S(n, k-1; \alpha, \beta, r) + (k\beta - n\alpha + r) S(n, k; \alpha, \beta, r) \tag{13}$$

with initial value $S(n, 0; \alpha, \beta, r) = (r|\alpha)_n$. The associated generalized Bell numbers are defined by

$$B_{\alpha,\beta,r}(n) = \sum_{k=0}^{n} S(n, k; \alpha, \beta, r), \tag{14}$$

and one has the generalized Dobiński formula [11, Equation (23)]

$$B_{\alpha,\beta,r}(n) = \left(\frac{1}{e}\right)^{\frac{1}{\beta}} \sum_{k \geq 0} \frac{(1/\beta)^k}{k!} (k\beta + r|\alpha)_n. \tag{15}$$

By specializing the parameters α, β, r, many different kinds of combinatorial numbers can be recovered, see [11] or the Appendix of [4]. The numbers $S(n, k; \alpha, \beta, r)$ have been studied intensely in literature, see, e.g., [4, 21] for recent combinatorial discussions. For us, one particularly important operational property was shown by Kargın and Corcino [12, Equation (2.5)],

$$(\beta x^{1-\alpha/\beta} D)^n \left[x^{r/\beta} f(x) \right] = x^{(r-n\alpha)/\beta} \sum_{k=0}^{n} S(n, k; \alpha, \beta, r) \beta^k x^k f^{(k)}(x), \tag{16}$$

where $D = \frac{d}{dx}$ and $f^{(k)} = D^k f$. Specializing $(\alpha, \beta, r) = (\frac{1}{2}, 1, 0)$, one obtains from (16) the normal ordering result

$$(X^{\frac{1}{2}} D)^n = X^{-\frac{n}{2}} \sum_{k=0}^{n} S(n, k; \frac{1}{2}, 1, 0)\, X^k D^k. \tag{17}$$

Comparing this with (3), we have the following result.

Proposition 3.1. *The Stirling numbers of order $\frac{1}{2}$ are given, for $1 \leq k \leq n$, by*

$$\mathsf{S}(n, k) = 2^{\lfloor \frac{n}{2} \rfloor} S(n, k; \frac{1}{2}, 1, 0). \tag{18}$$

Remark 3.2. *By combining* (18) *with* (4)*, one finds for Katugampola's coefficients $c_{n,k}$ that*

$$c_{n,k} = 2^{\lfloor \frac{n}{2} \rfloor} S(n, k + \lfloor \frac{n-1}{2} \rfloor; \frac{1}{2}, 1, 0).$$

Using (18), many properties can be transferred from the generalized Stirling numbers $S(n, k; \frac{1}{2}, 1, 0)$ to $\mathsf{S}(n, k)$ (or, $c_{n,k}$). Let us give some examples.

Corollary 3.3. *The Stirling numbers of order $\frac{1}{2}$ satisfy the recurrence relation* (7)*.*

Proof. Using (18), we find $\mathsf{S}(n+1,k) = 2^{\lfloor \frac{n+1}{2} \rfloor} S(n+1,k;\frac{1}{2},1,0)$. Applying (13), this gives

$$\mathsf{S}(n+1,k) = 2^{\lfloor \frac{n+1}{2} \rfloor} S(n,k-1;\frac{1}{2},1,0) + 2^{\lfloor \frac{n+1}{2} \rfloor}\left(k-\frac{n}{2}\right) S(n,k;\frac{1}{2},1,0).$$

Using again (18), this yields

$$\mathsf{S}(n+1,k) = 2^{\left(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor\right)} \mathsf{S}(n,k-1) + 2^{\left(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor\right)}\left(k-\frac{n}{2}\right) \mathsf{S}(n,k).$$

Since $2^{\left(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor\right)} = (1+\varepsilon_n)$, this shows (7). □

Using (18) and applying (12) with $(\alpha,\beta,r) = (\frac{1}{2},1,0)$, one finds the explicit expression

$$\mathsf{S}(n,k) = \frac{1}{2^{\lfloor \frac{n+1}{2} \rfloor} k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (2j)^{\underline{n}}, \tag{19}$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ denotes the falling factorial.

In the conventional case, the Stirling numbers of the first kind $s(n,k)$ (A008275 in [32]) can be defined in various ways, and the Stirling numbers of the first and second kind satisfy orthogonality relations. In view of this property, we will define numbers $\mathsf{s}(n,k)$ which can be considered as Stirling numbers of order $\frac{1}{2}$ of the first kind. The generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ and $S(n,k;\beta,\alpha,-r)$ satisfy orthogonality relations, so with respect to (18) we define, for $k,n \in \mathbb{N}$ with $1 \le k \le n$

$$\mathsf{s}(n,k) = 2^{-\lfloor \frac{k}{2} \rfloor} S(n,k;1,\frac{1}{2},0). \tag{20}$$

Proposition 3.4. *The numbers* $\mathsf{S}(n,k)$ *and* $\mathsf{s}(n,k)$ *satisfy the orthogonality relations*

$$\sum_{\ell} \mathsf{S}(n,\ell)\mathsf{s}(\ell,m) = \sum_{\ell} \mathsf{s}(n,\ell)\mathsf{S}(\ell,m) = \delta_{n,m}. \tag{21}$$

Proof. Let us consider the first sum. Inserting (18) and (20), one obtains

$$2^{\lfloor \frac{n}{2} \rfloor} 2^{-\lfloor \frac{m}{2} \rfloor} \sum_{\ell} S(n,\ell;\frac{1}{2},1,0) S(\ell,m;1,\frac{1}{2},0) = 2^{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m}{2} \rfloor} \delta_{n,m} = \delta_{n,m},$$

where we used an orthogonality relation of the generalized Stirling numbers. The second relation is shown in the same fashion. □

Let us consider (21) for $n = m$. Only the summand $\ell = n$ remains, giving $\mathsf{s}(n,n) = (\mathsf{S}(n,n))^{-1} = 2^{-\lfloor \frac{n}{2} \rfloor}$, where we used (8) (see also Table 1). Using the same arguments as used in the proof of Corollary 3.3, one can deduce the recurrence relation

$$\mathsf{s}(n+1,k) = \frac{1}{1+\varepsilon_{k-1}} \mathsf{s}(n,k-1) + \left(\frac{k}{2} - n\right) \mathsf{s}(n,k). \tag{22}$$

This immediately implies, for $n \ge 2$, the explicit values $\mathsf{s}(n,1) = \left(-\frac{1}{2}\right)^{n-1} (2n-3)!!$. More generally, inserting (20) into (12) and simplifying the expression, one obtains the following analog of (19),

$$\mathsf{s}(n,k) = \frac{2^{\lfloor \frac{k+1}{2} \rfloor}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{j}{2}\right)^{\underline{n}}. \tag{23}$$

From (18), one obtains that the Bell numbers of order $\frac{1}{2}$ are given by

$$\mathsf{B}_n = 2^{\lfloor \frac{n}{2} \rfloor} B_{\frac{1}{2},1,0}(n). \tag{24}$$

Using (24) and applying (15) with $(\alpha, \beta, r) = (\frac{1}{2}, 1, 0)$, one obtains the Dobiński-like fomula

$$\mathsf{B}_n = \frac{1}{2^{\lfloor \frac{n+1}{2} \rfloor} e} \sum_{k \geq 0} \frac{(2k)^{\underline{n}}}{k!}. \tag{25}$$

Xu [35, Corollary 8] showed the following Spivey-like formula for the generalized Bell numbers,

$$B_{\alpha,\beta,r}(n+m) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} S(m, j; \alpha, \beta, r)(j\beta - m\alpha|\alpha)_{n-k} B_{\alpha,\beta,r}(k)$$

(given here in the equivalent form presented in [12]). Thus, using (24), one finds

$$\mathsf{B}_{n+m} = 2^{\lfloor \frac{n+m}{2} \rfloor} \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} S(m, j; \frac{1}{2}, 1, 0) \left(j - m\frac{1}{2} | \frac{1}{2} \right)_{n-k} B_{\frac{1}{2},1,0}(k).$$

Using (18) and (24) on the right-hand side as well as $\left(j - m\frac{1}{2} | \frac{1}{2} \right)_{n-k} = \left(\frac{1}{2} \right)^{n-k} (2j - m)^{\underline{n-k}}$, this shows the Spivey-like fomula

$$\mathsf{B}_{n+m} = 2^{\left(\lfloor \frac{n+m}{2} \rfloor - \lfloor \frac{m}{2} \rfloor - n \right)} \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \mathsf{S}(m, j)(2j - m)^{\underline{n-k}} 2^{\lfloor \frac{k+1}{2} \rfloor} \mathsf{B}_k. \tag{26}$$

Before closing this section, let us recall that Hsu and Shiue [11, Equation (15)] derived the exponential generating function for the generalized Bell numbers,

$$\sum_{n \geq 0} B_{\alpha,\beta,r}(n) \frac{x^n}{n!} = (1 + \alpha x)^{r/\alpha} \exp \left[\left((1 + \alpha x)^{\beta/\alpha} - 1 \right) / \beta \right].$$

Denoting by $[x^n] f(x)$ the coefficient of x^n in the expansion of $f(x)$, we have $B_{\frac{1}{2},1,0}(n) = n![x^n] e^{x + x^2/4}$. Combining this with (24), gives

$$\mathsf{B}_n = 2^{\lfloor \frac{n}{2} \rfloor} n! \, [x^n] e^{x + x^2/4}. \tag{27}$$

The right-hand side is the definition of sequence A242818 in [32], thereby explaining the observation of Remark 2.1. From (27), we conclude that

$$\mathsf{S}(n, k) = \frac{2^{\lfloor \frac{n}{2} \rfloor} n!}{k!} \, [x^n] \left(x + \frac{x^2}{4} \right)^k. \tag{28}$$

In a similar fashion, using (20), one finds

$$\mathsf{s}(n, k) = \frac{2^{\lfloor \frac{k+1}{2} \rfloor} n!}{k!} \, [x^n] \left(\sqrt{1 + x} - 1 \right)^k. \tag{29}$$

4. An analog of the Weyl algebra

Recall that one can define the Weyl algebra $\mathcal{W}$ as the complex unital algebra generated by letters U and V satisfying the commutation relation $UV - VU = I$, where I denotes the identity, see (1). In this setting, one can show the normal ordering result $(VU)^n = \sum_k S(n,k)V^kU^k$, recovering for the concrete representation $V \mapsto X$ and $U \mapsto D$ Scherk's result (2). Varvak [34] considered normal ordering words in letters U and V satisfying the commutation relation

$$UV - VU = hV^m, \tag{30}$$

where $m \in \mathbb{N}_0$ and $h \in \mathbb{C}$ a parameter. For such variables (generating the *generalized Weyl algebra* $\mathcal{W}_{m;h}$ with $\mathcal{W}_{0;1} = \mathcal{W}$), generalized Stirling numbers $S_{m;h}(n,k)$ were introduced as normal ordering coefficients of $(VU)^n$ [26, 27], see also [8]. Let us try to consider $(X^{\frac{1}{2}}D)^n$ in this framework. One has $DX^{\frac{1}{2}}f(x) = (DX^{\frac{1}{2}})f(x) + X^{\frac{1}{2}}Df(x)$, or $DX^{\frac{1}{2}} - X^{\frac{1}{2}}D = \frac{1}{2}(X^{\frac{1}{2}})^{-1}$. Thus, the variables U and V (with concrete representation $V \mapsto X^{\frac{1}{2}}$ and $U \mapsto D$) should satisfy

$$UV - VU = \frac{1}{2}V^{-1}. \tag{31}$$

This has a structure very close to (30), but here the inverse of the letter V appears on the right-hand side! Thus, we formally have to adjoin this inverse as a new variable W and consider the complex algebra generated by letters U, V, W. What remains to be determined is the commutation relation between U and W. Since $VW = I$ (where I denotes the identity) we can write

$$U = UVW = (UV)W = (VU + \frac{1}{2}W)W = VUW + \frac{1}{2}W^2,$$

where we used $UV = VU + \frac{1}{2}W$. Multiplying from the left with W and using $WV = I$, we obtain $UW - WU = -\frac{1}{2}W^3$. Thus, we are led to define the following object.

Definition 4.1. *The algebra $\mathcal{W}^{\frac{1}{2}}$ is the complex unital algebra (with identity I) generated by letters U, V, W satisfying the commutation relations*

$$UV - VU = \frac{1}{2}W, \quad VW = WV = I, \quad UW - WU = -\frac{1}{2}W^3. \tag{32}$$

Clearly, $(U, V, W) \mapsto (D, X^{\frac{1}{2}}, X^{-\frac{1}{2}})$ gives the concrete representation of $\mathcal{W}^{\frac{1}{2}}$ considered above. Similar as the Weyl algebra $\mathcal{W}$ is the abstract object behind D, X satisfying (1), the algebra $\mathcal{W}^{\frac{1}{2}}$ is the abstract object behind $D, X^{\frac{1}{2}}$. One can now consider questions of normal ordering in $\mathcal{W}^{\frac{1}{2}}$, where a word is in normal ordered form if all letters V, W stand to the left of all letters U (V and W commute, so we don't need to specify their relative order). From the commutation relations (32) one obtains, by induction, the following result.

Lemma 4.2. *For $n \in \mathbb{N}$, one has in $\mathcal{W}^{\frac{1}{2}}$ the following normal ordering formulas,*

$$UV^n = V^nU + \frac{n}{2}WV^{n-1}, \quad UW^n = W^nU - \frac{n}{2}W^{n+2}. \tag{33}$$

By linearity, this implies for any polynomial p (with derivative p') that

$$Up(V) = p(V)U + \frac{1}{2}Wp'(V), \quad Up(W) = p(W)U - \frac{1}{2}W^3p'(W).$$

Remark 4.3. *Note that $UV^2 = V^2U + I$. Thus, the subalgebra of $\mathcal{W}^{\frac{1}{2}}$ generated by $\{I, U, V^2\}$ is isomorphic to the Weyl algebra $\mathcal{W}$.*

This allows to transfer ordering results from $\mathcal{W}$ to $\mathcal{W}^{\frac{1}{2}}$, e.g., in $\mathcal{W}^{\frac{1}{2}}$ one has

$$(V^2U)^n = \sum_{k=0}^{n} S(n,k)V^{2k}U^k, \quad V^{2n}U^n = \sum_{k=1}^{n} s(n,k)(V^2U)^k, \tag{34}$$

where we used that in the Weyl algebra $\mathcal{W}$ generated by $\{I, \tilde{U}, \tilde{V}\}$ with $\tilde{U}\tilde{V} = \tilde{V}\tilde{U} + I$ one has

$$(\tilde{V}\tilde{U})^n = \sum_{k=0}^{n} S(n,k)\tilde{V}^k\tilde{U}^k, \quad \tilde{V}^n\tilde{U}^n = \sum_{k=1}^{n} s(n,k)(\tilde{V}\tilde{U})^k. \tag{35}$$

As another example for normal ordering in $\mathcal{W}^{\frac{1}{2}}$, one should consider $(VU)^n$. If we translate (3) to these variables ($D \rightsquigarrow U, X \rightsquigarrow V^2, X^{-\frac{1}{2}} \rightsquigarrow W$), we expect in $\mathcal{W}^{\frac{1}{2}}$ the following identity to be true,

$$(VU)^n = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} 2^{-\lfloor \frac{n}{2} \rfloor} \mathsf{S}(n,k) W^n V^{2k} U^k. \tag{36}$$

Proposition 4.4. *In the algebra $\mathcal{W}^{\frac{1}{2}}$ the normal ordering result* (36) *holds true for all $n \in \mathbb{N}$. Furthermore, the following analog of the second identity in* (35) *holds true in $\mathcal{W}^{\frac{1}{2}}$,*

$$V^n U^n = \sum_{k=1}^{n} 2^{\lfloor \frac{k}{2} \rfloor} \mathsf{s}(n,k) W^n V^k (VU)^k. \tag{37}$$

Proof. We show (36) by induction. Let us multiply the identity (36) on both sides by $2^{\lfloor \frac{n}{2} \rfloor}$. For $n+1$, we can then write the left-hand side as

$$2^{\lfloor \frac{n+1}{2} \rfloor}(VU)^{n+1} = 2^{\left(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor\right)}(VU)2^{\lfloor \frac{n}{2} \rfloor}(VU)^n = (1+\varepsilon_n)VUW^n \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \mathsf{S}(n,k)V^{2k}U^k,$$

where we used $2^{\left(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor\right)} = (1+\varepsilon_n)$ and the induction hypothesis for n. Using the second identity of (33), this equals

$$W^n \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)\mathsf{S}(n,k)VUV^{2k}U^k - W^{n+2} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)\frac{n}{2}\mathsf{S}(n,k)V^{2k+1}U^k.$$

Inserting into the first sum $I = WV$ and applying the first identity of (33), this gives, upon using $WV = I$ in the second sum,

$$W^{n+1} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)\mathsf{S}(n,k)V^2\left(V^{2k}U + kWV^{2k-1}\right)U^k - W^{n+1} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)\frac{n}{2}\mathsf{S}(n,k)V^{2k}U^k.$$

Note that the first of these sums equals (using $WV = I$)

$$W^{n+1} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)\mathsf{S}(n,k)V^{2(k+1)}U^{k+1} + W^{n+1} \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} (1+\varepsilon_n)k\mathsf{S}(n,k)V^{2k}U^k.$$

Thus, relabelling the index in the first sum, we obtain in total

$$W^{n+1} \sum_{k=\lfloor \frac{n+2}{2} \rfloor}^{n+1} \left\{(1+\varepsilon_n)\mathsf{S}(n,k-1) + (1+\varepsilon_n)\left(k - \frac{n}{2}\right)\mathsf{S}(n,k)\right\} V^{2k}U^k.$$

Applying (7), this equals $2^{\lfloor \frac{n+1}{2} \rfloor}$ times the right-hand side of (36) for $n+1$, as requested. To show (37), we could also perform an induction using (22). Instead, we can check it by inserting (36),

$$V^nU^n = \sum_{k=1}^{n} 2^{\lfloor \frac{k}{2} \rfloor}\mathsf{s}(n,k)W^nV^k(VU)^k = \sum_{k=1}^{n} \sum_{\ell=\lfloor \frac{k+1}{2} \rfloor}^{k} \mathsf{s}(n,k)\mathsf{S}(k,\ell)W^nV^{2\ell}U^\ell = \sum_{\ell=1}^{n} \delta_{n,\ell}W^nV^{2\ell}U^\ell,$$

where we changed in the last step the order of summation and used (21). The sum on the right-hand side equals $W^nV^{2n}U^n = V^nU^n$ due to $WV = I$. $\square$

Using (18) (resp., (20)), one can write (36) (resp., (37)) in the equivalent form

$$(VU)^n = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} S(n,k;\frac{1}{2},1,0)W^nV^{2k}U^k, \quad V^nU^n = \sum_{k=1}^{n} S(n,k;1,\frac{1}{2},0)W^nV^k(VU)^k.$$

Remark 4.5. *The authors of [8] considered normal ordering words in U and V satisfying* (30)*. In particular, they mentioned (see table in Section 4) that normal ordering $(VU)^n$ where $UV - VU = V^{-1}$ is related to A122848 in [32]. Denoting this sequence by $t(n,k)$, we find by comparison that $\mathsf{S}(n,k) = 2^{k-n}2^{\lfloor \frac{n}{2} \rfloor}t(n,k)$, for $1 \le k \le n \le 7$.*

Recall from the Introduction that the generalized Stirling numbers $S_{r,s}(n,k)$ are defined as normal ordering coefficients of $(\tilde{V}^r\tilde{U}^s)^n$ in the Weyl algebra $\mathcal{W}$. In particular, $S_{2,1}(n,k) = L(n,k)$, the (unsigned) Lah numbers (A271703 in [32]), see [24]. In analogy, one can define $\mathsf{S}_{r,s}(n,k)$ as normal ordering coefficients of $(V^rU^s)^n$ in $\mathcal{W}^{\frac{1}{2}}$. Due to the isomorphism mentioned in Remark 4.3 one has for r even that $\mathsf{S}_{r,s}(n,k)$ is given by $S_{\frac{r}{2},s}(n,k)$. In particular, $\mathsf{S}_{2,1}(n,k) = S_{1,1}(n,k) = S(n,k)$, see (34). On the other hand, $\mathsf{S}_{4,1}(n,k) = S_{2,1}(n,k) = L(n,k)$, i.e., normal ordering $(V^4U)^n$ in $\mathcal{W}^{\frac{1}{2}}$ involves the Lah numbers. To determine UV^n or UW^n in $\mathcal{W}^{\frac{1}{2}}$ is easy, see (33). In contrast, when higher powers of U are involved, calculations become complicated. For example, let us consider U^nV. For $n = 1$ one has $UV = VU + \frac{1}{2}W$. Using the commutation relations (33), one finds

$$\begin{aligned} U^2V &= VU^2 + WU - \frac{1}{4}W^3, \\ U^3V &= VU^3 + \frac{3}{2}WU^2 - \frac{3}{4}W^3U + \frac{3}{8}W^5, \\ U^4V &= VU^4 + 2WU^3 - \frac{3}{2}W^3U^2 + \frac{12}{8}W^5U - \frac{15}{16}W^7. \end{aligned}$$

By induction, one obtains the following result.

Proposition 4.6. *For $n \in \mathbb{N}$ with $n \ge 2$, one has in $\mathcal{W}^{\frac{1}{2}}$ the following normal ordering result,*

$$U^nV = VU^n + \frac{n}{2}WU^{n-1} + \sum_{k=2}^{n-1}(-1)^{k+1}\frac{c(n,k)}{2^k}W^{2k-1}U^{n-k} + (-1)^{n+1}\frac{(2n-3)!!}{2^n}W^{2n-1},$$

where $c(n,1) = n$, $c(n,n) = (2n-3)!!$, and $c(n+1,k) = c(n,k) + (2k-3)c(n,k-1)$ for $k = 2,\dots,n$.

The algebra $\mathcal{W}_{m;h}$ associated with commutation relation (30) has been generalized to the generalized Weyl algebras $\mathcal{W}_p$ with commutation relation $UV - VU = p(V)$ where $p \in \mathbb{C}[V]$, see [1–3, 23]. An in-depth study of $\mathcal{W}_p$ was started in [1–3] and subsequent papers, see, e.g., [6]. Let us consider $\mathcal{W}_p$ as above and, similarly, $\mathcal{W}_q$ with generators $\bar{U}$ and $\bar{V}$ satisfying $\bar{U}\bar{V} - \bar{V}\bar{U} = q(\bar{V})$ where $q \in \mathbb{C}[\bar{V}]$. Benkart, Lopes and Ondrus showed the following result ([2, Lemma 3.1] and [2, Corollary 3.2]).

Lemma 4.7 ([2]). *Suppose that $p|q$ and $q = pr$. Then the map $\mathcal{W}_q \to \mathcal{W}_p$ given by $\bar{U} \mapsto U$ and $\bar{V} \mapsto Vr(V)$ is an embedding of $\mathcal{W}_q$ into $\mathcal{W}_p$. In particular, there is an embedding of $\mathcal{W}_p$ into the Weyl algebra $\mathcal{W}$ for every nonzero $p \in \mathbb{C}[V]$.*

Thus, for any nonzero $p \in \mathbb{C}[V]$, we can combine the embedding of $\mathcal{W}_p$ into $\mathcal{W}$ and the embedding of $\mathcal{W}$ into $\mathcal{W}^{\frac{1}{2}}$ observed in Remark 4.3 to obtain an embedding of $\mathcal{W}_p$ into $\mathcal{W}^{\frac{1}{2}}$,

$$\mathcal{W}_p \hookrightarrow \mathcal{W} \hookrightarrow \mathcal{W}^{\frac{1}{2}}.$$

Acknowledgments

The author wishes to express his grateful thanks to the anonymous referee for the comments and suggestions helping to improve this paper. And many thanks to Jasmin Preißler for giving me introductory lectures on Cego.

References

[1] G. Benkart, S. A. Lopes and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra. II: Irreducible modules*, Recent developments in algebraic and combinatorial aspects of representation theory. Contemp. Math. **602**, Amer. Math. Soc., Providence, RI (2013), 73–98.

[2] G. Benkart, S. A. Lopes and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra. I: Structure and automorphisms*, Trans. Amer. Math. Soc. **367** (2015), 1993–2021.

[3] G. Benkart, S. A. Lopes and M. Ondrus, *Derivations of a parametric family of subalgebras of the Weyl algebra*, J. Algebra **424** (2015), 46–97.

[4] B. Bényi, S. Nkonkobe and M. Shattuck, *Unfair distributions counted by the generalized Stirling numbers*, Integers **22** (2022), Art. A79.

[5] P. Blasiak and P. Flajolet, *Combinatorial models of creation-annihilation*, Sém. Lothar. Combin. **65** (2011), Art. B65c.

[6] E. Briand, S. A. Lopes and M. Rosas, *Normally ordered forms of powers of differential operators and their combinatorics*, J. Pure Appl. Algebra **224** (2020), Art. 106312.

[7] L. Carlitz, *On arrays of numbers*, Amer. J. Math. **54** (1932), 739–752.

[8] R. O. Celeste, R. B. Corcino and K. J. M. Gonzales, *Two approaches to normal order coefficients*, J. Integer Seq. **20** (2017), Art. 17.3.5.

[9] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, 1974.

[10] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete mathematics: a foundation for computer science*, (2nd edition), Addison-Wesley Publishing Group, Amsterdam, 1994.

[11] L. C. Hsu and P. J.-S. Shiue, *A unified approach to generalized Stirling numbers*, Adv. Appl. Math. **20** (1998), 366–384.

[12] L. Kargın and R. B. Corcino, *Generalization of Mellin derivative and its applications*, Integral Transforms Spec. Funct. **27** (2016), 620–631.

[13] U. N. Katugampola, *Mellin transforms of generalized fractional integrals and derivatives*, Appl. Math. Comput. **257** (2015), 566–580.

[14] T. Kim and D. S. Kim, *Some identities involving degenerate Stirling numbers arising from normal ordering*, AIMS Math. **7** (2022), 17357–17368.

[15] T. Kim and D. S. Kim, *Some Identities on Degenerate r-Stirling Numbers via Boson Operators*, Russ. J. Math. Phys. **29** (2022), 508–517.

[16] T. Kim and D. S. Kim, *Degenerate r-Whitney numbers and degenerate r-Dowling polynomials via boson operators*, Adv. Appl. Math. **140** (2022), Art. 102394.

[17] T. Kim and D. S. Kim, *Combinatorial identities involving degenerate harmonic and hyperharmonic numbers*, Adv. Appl. Math. **148** (2023), Art. 102535.

[18] T. Kim, D. S. Kim and H. K. Kim, *Degenerate r-Bell Polynomials Arising from Degenerate Normal Ordering*, J. Math. **2022** (2022), Art. 2626249.

[19] T. Kim, D. S. Kim and H. K. Kim, *Normal ordering of degenerate integral powers of number operator and its applications*, Appl. Math. Sci. Eng. **30** (2022), 440–447.

[20] T. Kim, D. S. Kim and H. K. Kim, *Normal ordering associated with λ-Stirling numbers in λ-shift algebra*, Demonstr. Math. **56** (2023), Art. 20220250.

[21] M. Maltenfort, *New definitions of the generalized Stirling numbers*, Aequationes Math. **94** (2020), 169–200.

[22] T. Mansour, *Combinatorics of set partitions*, CRC Press, Boca Raton, 2012.

[23] T. Mansour and M. Schork, *The commutation relation $xy = qyx + hf(y)$ and Newton's binomial formula*, Ramanujan J. **25** (2011), 405–445.

[24] T. Mansour and M. Schork, *Commutation relations, normal ordering, and Stirling numbers*, CRC Press, Boca Raton, 2016.

[25] T. Mansour and M. Schork, *On Ore-Stirling numbers defined by normal ordering in the Ore algebra*, Filomat **37** (2023), 6115–6131.

[26] T. Mansour, M. Schork and M. Shattuck, *On a new family of generalized Stirling and Bell numbers*, Electron. J. Combin. **18** (2011), Art. 77.

[27] T. Mansour, M. Schork and M. Shattuck, *The generalized Stirling and Bell numbers revisited*, J. Integer Seq. **15** (2012), Art. 12.8.3.

[28] N. H. McCoy, *Expansions of certain differential operators*, Tôhoku Math. J. **39** (1934), 181–186.

[29] S. Osipov, *On the expansion of a polynomial of the operator B_α*, U.S.S.R. Comput. Math. Phys. **3** (1963), 250–256.

[30] H. Scherk, *De evolvenda functione $(yd \cdot yd \cdot yd \dots yd\,X)/dx^n$ disquisitiones nonnullae analyticae*, (Ph.D. Thesis), University of Berlin, 1823.

[31] M. Schork, *Recent developments in combinatorial aspects of normal ordering*, Enumer. Combin. Appl. **1** (2021), Article S2S2.

[32] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, http://oeis.org/

[33] R. P. Stanley, *Enumerative Combinatorics* (2 Volumes), Cambridge University Press, Cambridge, 1999.

[34] A. Varvak, *Rook numbers and the normal ordering problem*, J. Combin. Theory Ser. A **112** (2005), 292–307.

[35] A. Xu, *Extensions of Spivey's Bell number formula*, Electron J Combin. **19** (2012), #P6 .