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# Resolvent operator technique and iterative algorithms for system of generalized nonlinear variational inclusions and fixed point problems: Variational convergence with an application

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**Abstract.** This article is devoted to investigate the problem of finding a common point of the set of fixed points of a total uniformly L-Lipschitzian mapping and the set of solutions of a system of generalized nonlinear variational inclusions involving $P$-$\eta$-accretive mappings. For finding such an element, a new iterative algorithm is suggested. The concepts of graph convergence and the resolvent operator associated with a $P$-$\eta$-accretive mapping are used and a new equivalence relationship between graph convergence and resolvent operators convergence of a sequence of $P$-$\eta$-accretive mappings is established. As an application of the obtained equivalence relationship, we prove the strong convergence and stability of the sequence generated by our proposed iterative algorithm to a common element of the above two sets. These results are new, and can be viewed as a refinement and improvement of some known results in the literature.

## 1. Introduction

The theory of variational inequalities, which was first studied independently by Stampacchia [59] and Fichera [25] in 1964, has been widely studied and continues to be an active topic for research. One of the primary reasons for this is that a large variety of problems arising in the fields like optimization and control, engineering science, mechanics, game theory, elasticity, physics, economics, transportation equilibrium, etc., can be formulated as variational inequalities. It is to be noted that a wide class of problems arising in diverse branches of pure and applied sciences lead to mathematical models which cannot be expressed in terms of variational inequalities, but one can formulate them as generalized forms of variational inequalities.

The need to formulate and study these types of problems has motivated many authors to develop and generalize various kinds of variational inequalities in many different directions using novel and innovative techniques, see, for example, [4–6, 8, 20] and the references therein. Among these generalizations, variational inclusion has emerged as an efficient and productive mechanism for studying a large variety of problems arising in various applications. The development of solution methods and the construction of iterative algorithms by means of them for the approximation of solutions of different classes of variational

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inequalities and their generalizations have been the subjects of many research papers in the past decades. The method based on the resolvent operator technique, as a generalization of projection method, has been recognized as a strong tool for studying the approximation solvability of nonlinear variational inequalities and variational inclusions and has become more and more popular. For details, we refer the reader to [3, 9, 21–24, 38, 40, 45, 55, 62] and the references therein.

The initiation of the study of problems with monotone mappings dates back to the sixties with the pioneering studies due to Browder [14] and Minty [52, 53]. In fact, the need to study integral and partial differential equations, and as well as the theory of convex functions have given rise to the emergence of monotone operators and in particular maximal monotone ones. Due to their extraordinary utility and broad applicability in many areas of pure and applied mathematics such as optimization and nonlinear analysis, monotone operators and maximal monotone operators continue to receive great attention and the development and generalization of them are being the focus of attention of researchers coming from mathematics and many other disciplines. The notion of accretive operators was initially introduced by Browder [15] and Kato [37] independently. Those accretive operators which are $m$-accretive or satisfy the range condition play an important role in the study of nonlinear semigroups, differential equations in Banach spaces, and fully nonlinear partial differential equations. Over the last few decades, the interests have focused on extensions of maximal monotone operators and $m$-accretive mappings and there has been substantial progress made by researchers in this direction. The introduction of the class of $P$-$\eta$-accretive mappings in a real $q$-uniformly smooth Banach space setting, as a unifying framework for the classes of maximal $\eta$-monotone operators [30], $\eta$-subdifferential operators [19, 46], generalized $m$-accretive mappings [31], $H$-monotone operators [22], general $H$-monotone operators [64], $H$-accretive mappings [21] and $(H, \eta)$-monotone operators [24] was first made by Kazmi and Khan [40] in 2007. They defined the resolvent operator ($P$-$\eta$-proximal-point mapping) associated with a $P$-$\eta$-accretive mapping and deduced some properties relating to it. One year later, inspired and motivated by the results derived by the authors in [40], Peng and Zhu [55] reviewed the class of $P$-$\eta$-accretive mappings and provided the updated versions of properties concerning them. They considered a system of variational inclusions involving $P$-$\eta$-accretive mappings in real $q$-uniformly smooth Banach spaces and proved the existence of a unique solution for it under some suitable assumptions. To approximate the unique solution of the system of variational inclusions, they suggested a Mann iterative algorithm and discussed its convergence under some appropriate conditions.

The study of the concept of graph convergence for operators is due primarily to Attouch [7] in 1984. He focused on maximal monotone operators and established an equivalence between the graph convergence and resolvent operators convergence of a sequence of maximal monotone operators. Since then, many efforts have been devoted to the development and extension of this notion for other generalized monotone operators and generalized accretive mappings existing in the literature. More details along with relevant commentaries, can be found in [3, 7, 32, 41, 49, 62] and the references therein.

On the other hand, since the appearance of the theory of nonexpansive mapping in the sixties, due to the existence of a deep and close relation between the class of nonexpansive mappings and monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations, it has increasingly received much attentions by many researchers. It is well known that the study of fixed point theory, which consists of many fields of mathematics such as mathematical analysis, general topology and functional analysis, began almost a century ago in the field of algebraic topology. It is a very active field of research activity and already a vast body of literature has grown on the subject. Due to its importance, depth, applicability and usefulness, fixed point theory still attracts great attention from many mathematicians and researchers. Since 1965 considerable efforts have been done to study the fixed point theory for nonexpansive mappings in the setting of different spaces, see, for instance, [27, 42]. Besides, over the last 50 years or so, there has been considerable activity in the introduction of various classes of generalized nonexpansive mappings in the framework of different spaces. One of the first attempts in this direction has been made by Goebel and Kirk [26] in 1972, who introduced the notion of asymptotically nonexpansive mapping as a generalization of nonexpansive mapping. Afterwards, the efforts to introduce the widest class of generalized nonexpansive mappings have been continued. In 2014, Kiziltunc and Purtas [44] succeeded to introduce the notion of total uniformly L-Lipschitzian mapping which can be viewed as a unifying framework for the classes of nonexpansive mappings, asymptotically nonexpansive mappings [26], nearly asymptotically nonexpansive mappings [57], total asymptotically nonexpansive mappings [2], and several other classes of generalized nonexpansive mappings appeared in the literature. A detailed study of these generalizations can be found in [2, 9, 26, 44, 54, 57] and the references therein. It is a well-known truth that there is a strong connection between the variational inequality/inclusion problems and the fixed point problems. This fact has motivated many investigators to present a unified approach to these two different problems. We refer the reader to [4, 5, 9–13, 16, 33, 36, 56, 58, 60, 61] for more details and further information.

The rest of the paper is organized as follows. We recall some basic notions, notations, and properties of $P$-$\eta$-accretive mappings, together with some examples and preliminary results concerning them in Section 2. In Section 3, a system of generalized nonlinear variational-like inclusions (SGNVLI) involving $P$-$\eta$-accretive mappings is considered and the existence and uniqueness of its solution is demonstrated under some suitable assumptions imposed on the parameters and mappings in the SGNVLI. In Section 4, applying the notions of graph convergence and the resolvent operator associated with a $P$-$\eta$-accretive mapping, a new equivalence relationship between the graph convergence of a sequence of $P$-$\eta$-accretive mappings and their associated resolvent operators, respectively, to a given $P$-$\eta$-accretive mapping and its associated resolvent operator is established. We investigate the problem of finding a point which belongs to the intersection of the set of solutions of the SGNVLI and the set of fixed points of a total uniformly L-Lipschitzian mapping. To achieve this end, we suggest a new iterative algorithm. Finally, in Section 5, as an application of the equivalence relationship obtained in Section 4, the strong convergence and stability of the sequence generated by our proposed iterative algorithm to a common element of the above two sets are proved.

## 2. Preliminary Materials and Basic Results

In what follows, unless otherwise stated, we always let $E$ be a real Banach space with a norm $\|.\|$, $E^*$ be the topological dual space of $E$, $\langle .,. \rangle$ be the dual pair between $E$ and $E^*$, and $2^E$ denote the family of all the nonempty subsets of $E$. For the sake of simplicity, the norm of $E^*$ is also denoted by the symbol $\|.\|$. As usual, $x^*$ will stand for the weak star topology in $E^*$, and the value of a functional $x^* \in E^*$ at $x \in E$ is denoted by either $\langle x, x^* \rangle$ or $x^*(x)$, as is convenient. At the same time, the symbols $S_E$ and $B_E$ are used to represent the unit sphere and the unit ball in $E$, respectively.

For a given multi-valued mapping $M : E \to 2^E$,

(i) the set Graph($M$) defined by

$$\text{Graph}(M) := \{(x, u) \in E \times E : u \in M(x)\},$$

is called the graph of $M$;

(ii) the set Range($M$) given by the formula

$$\text{Range}(M) := \{y \in E : \exists x \in E : (x, y) \in M\} = \bigcup_{x \in E} M(x)$$

is called the range of $M$.

**Definition 2.1.** *A normed space E is called*

(i) *smooth if for every $x \in S_E$ there exists a unique $x^* \in E^*$ such that $\|x\| = \langle x, x^* \rangle = 1$;*
(ii) *strictly smooth if $S_E$ is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all $x, y \in S_E$ such that $x \neq y$.*

It is well known that $E$ is smooth if $E^*$ is strictly convex, and that $E$ is strictly convex if $E^*$ is smooth.

**Definition 2.2.** *A normed space E is said to be uniformly convex if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if x and y are unit vectors in E with $\|x - y\| \geq 2\varepsilon$, then the average $(x + y)/2$ has norm at most $1 - \delta$.*

Hence, a normed space is uniformly convex if for any two distinct points $x$ and $y$ on the unit sphere centered at the origin the midpoint of the line segment joining $x$ and $y$ is never on the sphere but is close to the sphere only if $x$ and $y$ are sufficiently close to each other.

The function $\delta_E : [0,2] \to [0,1]$ given by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}$$

is called the *modulus of convexity* of $E$. The function $\delta_E$ is continuous and increasing on the interval $[0,2]$ and $\delta_E(0) = 0$. Clearly, thanks to the definition of the function $\delta_E$, a normed space $E$ is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. In the particular case of an inner product space $\mathcal{H}$, we have $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$.

**Definition 2.3.** *A normed space E is said to be uniformly smooth if, for all $\varepsilon > 0$ there is a $\tau > 0$ such that if x and y are unit vectors in E with $\|x - y\| \leq 2\tau$, then the average $(x+y)/2$ has norm at least $1 - \varepsilon\tau$.*

The function $\rho_E : [0,+\infty) \to [0,+\infty)$ given by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+\tau y\| + \|x - \tau y\|) - 1 : x, y \in E, \|x\| = \|y\| = 1\}$$

is called the modulus of smoothness of $E$. It is worth noting that the function $\rho_E$ is convex, continuous and increasing on the interval $[0,+\infty)$ and $\rho_E(0) = 0$. In addition, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. In the light of the definition of the function $\rho_E$, a normed space $E$ is uniformly smooth if and only if $\lim\limits_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

It is important to emphasize that in the definitions of $\delta_E(\varepsilon)$ and $\rho_E(\tau)$, we can as well take the infimum and supremum over all vectors $x, y \in E$ with $\|x\|, \|y\| \leq 1$. Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space $E$ is uniformly convex (resp., uniformly smooth) if and only if $E^*$ is uniformly smooth (resp., uniformly convex). The spaces $l^p$, $L^p$ and $W^p_m$, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [18, 29, 47]. In the meanwhile, the modulus of convexity and smoothness of a Hilbert space and the spaces $l^p$, $L^p$ and $W^p_m$, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [18, 29, 47].

For an arbitrary but fixed real number $q > 1$, the multi-valued mapping $J_q : E \to 2^{E^*}$ given by

$$J_q(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

is called the *generalized duality mapping* of $E$. In particular, $J_2$ is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \neq 0$. Here it is to be noted that $J_q$ is single-valued if $E$ is uniformly smooth or equivalently $E^*$ is strictly convex. If $E$ is a Hilbert space, then $J_2$ becomes the identity mapping on $E$.

For a real constant $q > 1$, a Banach space $E$ is called $q$-uniformly smooth if there exists a constant $C > 0$ such that $\rho_E(t) \leq Ct^q$ for all $t \in [0,+\infty)$. It is well known that (see e.g. [65]) $L_q$ (or $l_q$) is $q$-uniformly smooth for $1 < q \leq 2$ and is 2-uniformly smooth if $q \geq 2$.

Concerned with the characteristic inequalities in $q$-uniformly smooth Banach spaces, Xu [65] proved the following result.

**Lemma 2.4.** *Let E be a real uniformly smooth Banach space. For a real constant $q > 1$, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We now introduce some notation and terminology and present some elementary results which will be used in later sections.

**Definition 2.5.** *Let E be a real q-uniformly smooth Banach space and let $P : E \to E$ and $\eta : E \times E \to E$ be the mappings. Then P is said to be*

(i) *$\eta$-accretive if,*

$$\langle P(x) - P(y), J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in E;$$

(ii) *strictly $\eta$-accretive if, $P$ is $\eta$-accretive and equality holds if and only if $x = y$;*

(iii) *$\gamma$-strongly $\eta$-accretive (or strongly $\eta$-accretive with a constant $\gamma > 0$) if there exists a constant $\gamma > 0$ such that*

$$\langle P(x) - P(y), J_q(\eta(x, y))\rangle \geq \gamma\|x - y\|^q, \quad \forall x, y \in E;$$

(iv) *$\mu$-Lipschitz continuous if there exists a constant $\mu > 0$ such that*

$$\|P(x) - P(y)\| \leq \mu\|x - y\|, \quad \forall x, y \in E.$$

It should be remarked that if $\eta(x, y) = x - y$, for all $x, y \in E$, then parts (i) to (iii) of Definition 2.5 reduce to the definitions of accretivity, strict accretivity and strong accretivity of the mapping $P$, respectively.

**Definition 2.6.** [21, 55] *Let $E$ be a real $q$-uniformly smooth Banach space and $P : E \to E$ be a single-valued mapping. A multi-valued mapping $M : E \to 2^E$ is said to be*

(i) *accretive if*

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(M);$$

(ii) *m-accretive if $M$ is accretive and $(I + \lambda M)(E) = E$ holds for every real constant $\lambda > 0$, where $I$ denotes the identity mapping on $E$;*

(iii) *$P$-accretive if $M$ is accretive and $(P + \lambda M)(E) = E$ holds for every $\lambda > 0$.*

Huang and Fang [31] introduced and studied the class of generalized $m$-accretive (also referred to as $m$-$\eta$-accretive and also $\eta$-$m$-accretive [17]) mappings as a generalization of $m$-accretive mappings as follows.

**Definition 2.7.** [17, 31] *Let $E$ be a real $q$-uniformly smooth Banach space and $\eta : E \times E \to E$ be a single-valued mapping. The multi-valued mapping $M : E \to 2^E$ is said to be*

(i) *$\eta$-accretive if*

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(M);$$

(ii) *generalized m-accretive if $M$ is $\eta$-accretive and $(I + \lambda M)(E) = E$ holds for every real constant $\lambda > 0$.*

We note that $M$ is a generalized $m$-accretive mapping if and only if $M$ is $\eta$-accretive and there is no other $\eta$-accretive mapping whose graph contains strictly Graph($M$). The generalized $m$-accretivity is to be understood in terms of inclusion of graphs. If $M : E \to 2^E$ is a generalized $m$-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the $\eta$-accretivity. In fact, the extended mapping is no longer $\eta$-accretive. In other words, for every pair $(x, u) \in E \times E \setminus \mathrm{Graph}(M)$ there exists $(y, v) \in \mathrm{Graph}(M)$ such that $\langle u - v, J_q(\eta(x, y))\rangle < 0$. In the light of the above-mentioned argument, a necessary and sufficient condition for a multi-valued mapping $M : E \to 2^E$ to be generalized $m$-accretive is that the property

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall (y, v) \in \mathrm{Graph}(M)$$

is equivalent to $(x, u) \in \mathrm{Graph}(M)$. The above characterization of generalized $m$-accretive mappings provides a useful and manageable way for recognizing that an element $u$ belongs to $M(x)$.

Peng and Zhu [55], and Kazmi and Khan [40] were the first to introduce and study the concept of $P$-$\eta$-accretive (also referred to as $(H, \eta)$-accretive) mapping as an extension of $H$-accretive ($P$-accretive) mapping, $(H, \eta)$-monotone operator [24], $H$-monotone operator [22], generalized $m$-accretive mapping, $m$-accretive mapping, maximal $\eta$-monotone operator [31], and maximal monotone operator as follows.

**Definition 2.8.** [40, 55] *Let $E$ be a real $q$-uniformly smooth Banach space, $P : E \to E$ and $\eta : E \times E \to E$ be single-valued mappings and $M : E \to 2^E$ be a multi-valued mapping. $M$ is said to be $P$-$\eta$-accretive if $M$ is $\eta$-accretive and $(P + \lambda M)(E) = E$ holds for every $\lambda > 0$.*

With the goal of illustrating the fact that for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a $P$-$\eta$-accretive mapping need not be $P$-accretive, Peng and Zhu presented [55, Example 2.1] as follows.

**Example 2.9.** Let $E = \mathbb{R}$ and $P : E \to E$, $\eta : E \times E \to E$, $M : E \to 2^E$ be defined as $P(x) = x^5$, $\eta(x, y) = x^4 - y^4$ and $M(x) = \{x^2, x^4, x^8\}$, for all $x, y \in E$, respectively. The authors [55] claimed that the mapping $M$ is $P$-$\eta$-accretive, but is not $P$-accretive and so it is not accretive. By careful checking, we found that there is a fatal error in the mentioned example. In fact, picking $y = 2$, $u = x^2$ and $v = y^4$, taking into account that $E$ is a 2-uniformly smooth Banach space, $y^2 > x$ and $x^4 > y^4$, we yield

$$\begin{aligned}\langle u - v, J_2(\eta(x, y))\rangle &= \langle u - v, \eta(x, y)\rangle = (x^2 - y^4)(x^4 - y^4)\\ &= -(y^2 - x)(y^2 + x)(x^4 - y^4) < 0\end{aligned}$$

and

$$\langle u - v, J_2(x - y)\rangle = \langle u - v, x - y\rangle = (x^2 - y^4)(x - y) < 0,$$

i.e., the mapping $M$ is neither $\eta$-accretive nor accretive. Thereby, contrary to the claim of the authors in [55], $M$ is neither a $P$-$\eta$-accretive mapping nor a $P$-accretive mapping.

They further presented the following example to show the fact that for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a $P$-$\eta$-accretive mapping need not be generalized $m$-accretive (or $m$-$\eta$-accretive) mapping.

**Example 2.10.** Let $E = \mathbb{R}$ and the mappings $P : E \to E$, $\eta : E \times E \to E$ and $N : E \to 2^E$ be defined as $P(x) = x^5$, $\eta(x, y) = x^4 - y^4$ and $N(x) = \{x^2, x^2 + \frac{1}{4}, 2x^2 + 3\}$, for all $x, y \in E$, respectively. They asserted that the mapping $N$ is $P$-$\eta$-accretive, but it is not $m$-$\eta$-accretive. By a careful checking, we discovered that contrary to the claim of the authors [55, Example 2.2] is neither a $p$-$\eta$-accretive mapping nor an $m$-$\eta$-accretive mapping. In fact, taking $x = 3$, $y = 2$, $u = x^2$ and $v = 2x^2 + 3$, in virtue of the facts that $E$ is a 2-uniformly smooth Banach space and $x > y$, we obtain

$$\langle u - v, J_2(\eta(x, y))\rangle = \langle u - v, \eta(x, y)\rangle = -(x^2 + 3)(x^4 - y^4) < 0$$

and

$$\langle u - v, J_2(x - y)\rangle = \langle u - v, x - y\rangle = -(x^2 + 3)(x - y) < 0,$$

which imply that $N$ is neither $\eta$-accretive nor accretive. Accordingly, the mapping $N$ is neither $P$-$\eta$-accretive nor $m$-$\eta$-accretive.

In order to illustrate the fact that for given mappings $\eta : E \times E \to E$ and $P : E \to E$, a $P$-$\eta$-accretive mapping may be neither $P$-accretive nor generalized $m$-accretive, we now present a new example as follows.

**Example 2.11.** Let $m, n \in \mathbb{N}$ and $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{F}, i = 1, 2, \dots, m; j = 1, 2, \dots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{a}_{ij}b_{ij}, \quad \forall A \in M_{m\times n}(\mathbb{F}),$$

where $tr$ denotes the trace, that is, the sum of diagonal entries, and $A^*$ denotes the Hermitian conjugate (or adjoint) of the matrix $A$, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose $A$, and the bar denotes complex conjugation and superscript denotes the transpose of the entries. Denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the $(i, j)$-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then,

$$D_n(\mathbb{R}) = \{A = \left( \begin{array}{c} a_{ij} \end{array} \right) | a_{ij} \in \mathbb{R}, a_{ij} = 0 \text{ if } i \neq j; i, j = 1, 2, \ldots, n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$. At the same time, the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$ and the Hilbert-Schmidt norm induced by it become as

$$\langle A, B\rangle = tr(A^*B) = tr(AB)$$

and

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(AA)} = \Big(\sum_{i=1}^{n} a_{ii}^2\Big)^{\frac{1}{2}},$$

respectively. Taking into account that every finite dimensional normed space is a Banach space, it follows that $(D_n(\mathbb{R}), \|.\|)$ is a Hilbert space and so it is a 2-uniformly smooth Banach space. For any $A = \left( \begin{array}{c} a_{ij} \end{array} \right) \in D_n(\mathbb{R})$, we have $A = \sum\limits_{i=1}^{\frac{n}{2}} A_{i(n-i+1)}$, that is, every diagonal $n \times n$ matrix with real entries $A \in D_n(\mathbb{R})$ can be written as a linear combination of $\frac{n}{2}$ matrices $A_{i(n-i+1)}$, where for each $i \in \{1, 2, \ldots, \frac{n}{2}\}$, $A_{i(n-i+1)}$ is an $n \times n$ matrix such that the $(i, j)$-entry equals to $a_{ii}$, $(n - i + 1, n - i + 1)$-entry equals to $a_{(n-i+1)(n-i+1)}$, and all other entries equal to zero. For each $i \in \{1, 2, \ldots, \frac{n}{2}\}$, there are two real numbers $b_{ii}$ and $b_{(n-i+1)(n-i+1)}$ such that $b_{ii} + b_{(n-i+1)(n-i+1)} = a_{ii}$ and $b_{ii} - b_{(n-i+1)(n-i+1)} = a_{(n-i+1)(n-i+1)}$. Then for each $i \in \{1, 2, \ldots, \frac{n}{2}\}$, we have

$$A_{i(n-i+1)} = b_{ii}N_{i(n-i+1)} + b_{(n-i+1)(n-i+1)}N'_{i(n-i+1)},$$

where for each $i \in \{1, 2, \ldots, \frac{n}{2}\}$, $N_{i(n-i+1)}$ is a diagonal $n\times n$ matrix such that the $(i, i)$-entry and $(n-i+1, n-i+1)$-entry equal to 1 and all other entries equal to zero, and $N'_{i(n-i+1)}$ is a diagonal $n \times n$ matrix with the entries 1 and $-1$ at the places $(i, i)$ and $(n - i + 1, n - i + 1)$, respectively, and 0's everywhere else. Hence the set $\{N_{i(n-i+1)}, N'_{i(n-i+1)} : i = 1, 2, \ldots, \frac{n}{2}\}$ spans the Hilbert space $D_n(\mathbb{R})$. Taking $E_{i(n-i+1)} := \frac{1}{\sqrt{2}}N_{i(n-i+1)}$ and $E'_{i(n-i+1)} := \frac{1}{\sqrt{2}}N'_{i(n-i+1)}$, for $i = 1, 2, \ldots, \frac{n}{2}$, it follows that the set $\mathfrak{B} = \{E_{i(n-i+1)}, E'_{i(n-i+1)} : i = 1, 2, \ldots, \frac{n}{2}\}$ spans also $D_n(\mathbb{R})$. It can be easily seen that the set $\mathfrak{B}$ is linearly independent and orthonormal and so $\mathfrak{B}$ is an orthonormal basis for $D_n(\mathbb{R})$. Let the mappings $M : D_n(\mathbb{R}) \to 2^{D_n(\mathbb{R})}$, $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $P : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined by

$$M(A) = \begin{cases} \Phi, & A = E_{k(n-k+1)}, \\ -A + E_{k(n-k+1)}, & A \neq E_{k(n-k+1)}, \end{cases}$$

$$\eta(A, B) = \begin{cases} \alpha(B - A), & A, B \neq E_{k(n-k+1)}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $P(A) = \beta A + \gamma E_{k(n-k+1)}$, for all $A, B \in D_n(\mathbb{R})$, where

$$\Phi = \Big\{E_{i(n-i+1)} - E_{k(n-k+1)}, E'_{i(n-i+1)} - E_{k(n-k+1)} : i = 1, 2, \ldots, \frac{n}{2}\Big\},$$

$\alpha, \beta, \gamma \in \mathbb{R}$, $\beta < 0 < \alpha$, $k \in \{1, 2, \ldots, \frac{n}{2}\}$ are arbitrary but fixed, and $\mathbf{0} = \left( \begin{array}{c} 0_{ij} \end{array} \right)$ is the zero vector of the space $D_n(\mathbb{R})$, that is, the zero $n \times n$ matrix. Then for all $A, B \in D_n(\mathbb{R})$, $A \neq B \neq E_{k(n-k+1)}$, we yield

$$\begin{aligned} \langle M(A) - M(B), J_2(A - B)\rangle &= \langle M(A) - M(B), A - B\rangle \\ &= \langle -A + E_{k(n-k+1)} + B - E_{k(n-k+1)}, A - B\rangle \\ &= \langle B - A, A - B\rangle = -\|A - B\|^2 = -\sum_{i=1}^{n}(a_{ii} - b_{ii})^2 < 0, \end{aligned}$$

i.e., $M$ is not accretive and so $M$ is not $P$-accretive.

For any $A, B \in D_n(\mathbb{R})$, $A \neq B \neq E_{k(n-k+1)}$, we have

$$\begin{aligned}\langle M(A) - M(B), J_2(\eta(A, B))\rangle &= \langle M(A) - M(B), \eta(A, B)\rangle\\ &= \langle -A + E_{k(n-k+1)} + B - E_{k(n-k+1)}, \alpha(B - A)\rangle\\ &= \alpha\langle B - A, B - A\rangle = \alpha\|B - A\|^2 = \alpha\sum_{i=1}^{n}(a_{ii} - b_{ii})^2 > 0.\end{aligned}$$

Furthermore, for each of the cases, when $A \neq B = E_{k(n-k+1)}$, $B \neq A = E_{k(n-k+1)}$ and $A = B = E_{k(n-k+1)}$, clearly $\eta(A, B) = 0$ and we deduce that

$$\langle u - v, J_2(\eta(A, B))\rangle = \langle u - v, \eta(A, B)\rangle = 0, \quad \forall u \in M(A), v \in M(B).$$

Hence, $M$ is an $\eta$-accretive mapping. In the light of the fact that for all $A \in D_n(\mathbb{R})$, $A \neq E_{k(n-k+1)}$,

$$\|(I + M)(A)\|^2 = \|E_{k(n-k+1)}\|^2 = 1 > 0$$

and

$$\|(I + M)(E_{k(n-k+1)}) = \Big\{E_{i(n-i+1)}, E'_{i(n-i+1)} : i = 1, 2, \ldots, \frac{n}{2}\Big\} = \mathfrak{B},$$

where $I$ is the identity mapping on $E = D_n(\mathbb{R})$, we conclude that $\mathbf{0} \notin (I + M)(D_n(\mathbb{R}))$. Thus, $I + M$ is not surjective and so $M$ is not a generalized $m$-accretive mapping. For any $\lambda > 0$ and $A \in D_n(\mathbb{R})$, taking $B = \frac{1}{\beta-\lambda}A + \frac{\gamma+\lambda}{\lambda-\beta}E_{n(n-k+1)}$ ($\lambda \neq \beta$, because $\beta < 0$), we have

$$(P + \lambda M)(B) \quad = (P + \lambda M)\big(\tfrac{1}{\beta-\lambda}A + \tfrac{\gamma+\lambda}{\lambda-\beta}E_{n(n-k+1)}\big) = A.$$

Thereby, for any $\lambda > 0$, the mapping $P + \lambda M$ is surjective and consequently $M$ is a $P$-$\eta$-accretive mapping.

In the next example the fact that for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a generalized $m$-accretive mapping need not be $P$-$\eta$-accretive is illustrated.

**Example 2.12.** Let $H_2(\mathbb{C})$ be the set of all Hermitian matrices with complex entries. We recall that a square matrix $A$ is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In the light of the definition of a Hermitian $2 \times 2$ matrix, the condition $A^* = A$ implies that the $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Hermitian if and only if $a, d \in \mathbb{R}$ and $b = \bar{c}$. Hence,

$$H_2(\mathbb{C}) = \Big\{\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} | x, y, z, w \in \mathbb{R}\Big\}.$$

Then, $H_2(\mathbb{C})$ is a subspace of $M_2(\mathbb{C})$, the space of all $2 \times 2$ matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered as a real vector space. In other words, $H_2(\mathbb{C})$ together with the mentioned operations is a vector space over $\mathbb{R}$. By introducing the scalar product on $H_2(\mathbb{C})$ as $\langle A, B\rangle := \frac{1}{2}tr(AB)$, for all $A, B \in H_2(\mathbb{C})$, it is easy to check that $\langle ., .\rangle$ is an inner product, that is, $(H_2(\mathbb{C}), \langle ., .\rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\frac{1}{2}tr(AA)} = \sqrt{x^2 + y^2 + \frac{1}{2}(z^2 + w^2)}, \quad \forall A \in H_2(\mathbb{C}).$$

The finite dimensional normed space $(H_2(\mathbb{C}), \|.\|)$ is a Hilbert space and so it is a 2-uniformly smooth Banach space. Suppose that the mappings $M, P : H_2(\mathbb{C}) \to H_2(\mathbb{C})$ and $\eta : H_2(\mathbb{C}) \times H_2(\mathbb{C}) \to H_2(\mathbb{C})$ are defined, respectively, by

$$M(A) = M\Big(\begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w_1 \end{pmatrix}\Big) = \begin{pmatrix} \alpha\sin z_1 & x_1 - iy_1 \\ x_1 + iy_1 & \beta\cos w_1 \end{pmatrix},$$

$$P(A) = P\Big(\begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w_1 \end{pmatrix}\Big) = \begin{pmatrix} \varrho\cos z_1 & x_1^2 - iy_1^2 \\ x_1^2 + iy_1^2 & \xi\sin w_1 \end{pmatrix}$$

and

$$\begin{aligned}\eta(A,B) &= \eta\Big(\begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w_1 \end{pmatrix}, \begin{pmatrix} z_2 & x_2 - iy_2 \\ x_2 + iy_2 & w_2 \end{pmatrix}\Big) \\ &= \begin{pmatrix} \gamma(\sin z_1 - \sin z_2) & x_1 - x_2 - i(y_1 - y_2) \\ x_1 - x_2 + i(y_1 - y_2) & \theta(\cos w_1 - \cos w_2) \end{pmatrix},\end{aligned}$$

for all $A = \begin{pmatrix} z_1 & x_1 - iy_1 \\ x_1 + iy_1 & w_1 \end{pmatrix}, B = \begin{pmatrix} z_2 & x_2 - iy_2 \\ x_2 + iy_2 & w_2 \end{pmatrix} \in H_2(\mathbb{C})$, where $\varrho$ and $\xi$ are arbitrary real constants and $\alpha, \beta, \gamma, \theta$ are positive real constants. Then, for any $A, B \in H_2(\mathbb{C})$, we have

$$\begin{aligned}&\langle M(A) - M(B), J_2(\eta(A,B))\rangle = \langle M(A) - M(B), \eta(A,B)\rangle \\ &= \Big\langle \begin{pmatrix} \alpha(\sin z_1 - \sin z_2) & x_1 - x_2 - i(y_1 - y_2) \\ x_1 - x_2 + i(y_1 - y_2) & \beta(\cos w_1 - \cos w_2) \end{pmatrix}, \\ &\quad \begin{pmatrix} \gamma(\sin z_1 - \sin z_2) & x_1 - x_2 - i(y_1 - y_2) \\ x_1 - x_2 + i(y_1 - y_2) & \theta(\cos w_1 - \cos w_2) \end{pmatrix}\Big\rangle \\ &= \frac{\alpha\gamma}{2}(\sin z_1 - \sin z_2)^2 + \frac{\beta\theta}{2}(\cos w_1 - \cos w_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &\geq 0,\end{aligned}$$

Thereby, $M$ is an $\eta$-accretive mapping.

Let us now define the functions $f, g, h : \mathbb{R} \to \mathbb{R}$, respectively, as

$$f(t) = \varrho\cos t + \alpha\sin t, \ g(t) = \beta\cos t + \xi\sin t \text{ and } h(t) = t^2 + t, \quad \forall t \in \mathbb{R}.$$

Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, yields

$$(P + M)(A) = (P + M)\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\Big) = \begin{pmatrix} f(z) & h(x) - ih(y) \\ h(x) + ih(y) & g(w) \end{pmatrix}.$$

Since for arbitrary constants $a, b \in \mathbb{R}$, $-\sqrt{a^2 + b^2} \leq a\cos t + b\cos t \leq \sqrt{a^2 + b^2}$, for all $t \in \mathbb{R}$, it follows that

$$-\sqrt{\varrho^2 + \alpha^2} \leq f(t) \leq \sqrt{\varrho^2 + \alpha^2} \text{ and } -\sqrt{\beta^2 + \xi^2} \leq g(t) \leq \sqrt{\beta^2 + \xi^2}, \quad \forall t \in \mathbb{R}.$$

Moreover, for each $t \in \mathbb{R}$, we have $h(t) = t^2 + t = (t + \frac{1}{2})^2 - \frac{1}{4} \geq -\frac{1}{4}$. Consequently,

$$f(\mathbb{R}) = [-\sqrt{\varrho^2 + \alpha^2}, \sqrt{\varrho^2 + \alpha^2}] \neq \mathbb{R}, \ g(\mathbb{R}) = [-\sqrt{\beta^2 + \xi^2}, \sqrt{\beta^2 + \xi^2}] \neq \mathbb{R}$$

and $h(\mathbb{R}) = [-\frac{1}{4}, +\infty) \neq \mathbb{R}$. These facts ensure that $(P + M)(H_2(\mathbb{C})) \neq H_2(\mathbb{C})$, that is, $P + M$ is not surjective, and so $M$ is not $P$-$\eta$-accretive.

Now, let $\lambda$ be an arbitrary positive real constant and let the functions $\widehat{f}, \widehat{g}, \widehat{h} : \mathbb{R} \to \mathbb{R}$ be defined, respectively, as

$$\widehat{f}(t) = t + \lambda\alpha\sin t, \ \widehat{g}(t) = t + \lambda\beta\cos t \text{ and } \widehat{h}(t) = (1 + \lambda)t, \quad \forall t \in \mathbb{R}.$$

Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we obtain

$$(I+\lambda M)(A) = (I+\lambda M)\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} \widehat{f}(z) & \widehat{h}(x)-i\widehat{h}(y) \\ \widehat{h}(x)+i\widehat{h}(y) & \widehat{g}(w) \end{pmatrix},$$

where $I$ is the identity mapping on $H_2(\mathbb{C})$. In virtue of the fact that $\widehat{f}(\mathbb{R}) = \widehat{g}(\mathbb{R}) = \widehat{h}(\mathbb{R}) = \mathbb{R}$, we conclude that $(I+\lambda M)(H_2(\mathbb{C})) = H_2(\mathbb{C})$, that is, $I+\lambda M$ is a surjective mapping. Taking into account the arbitrariness in the choice of $\lambda > 0$, it follows that $M$ is a generalized $m$-accretive mapping.

**Example 2.13.** Assume that $M_{m\times n}(\mathbb{F})$ and $D_n(\mathbb{R})$ are the same as in Example 2.11. Let the mappings $P_1, P_2, M : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R})\times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined, respectively, by $P_1(A) = P_1(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} \end{pmatrix}$, $P_2(A) = P_2(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} \end{pmatrix}$, $M(A) = M(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'''_{ij} \end{pmatrix}$ and $\eta(A,B) = \eta(\begin{pmatrix} a_{ij} \end{pmatrix}, \begin{pmatrix} b_{ij} \end{pmatrix}) = \begin{pmatrix} c_{ij} \end{pmatrix}$ for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where for each $i,j \in \{1,2,\ldots,n\}$,

$$a'_{ij} = \begin{cases} \frac{\gamma^{a_{ii}}-1}{\gamma^{a_{ii}}+1} - \beta a_{ii}^k, & i=j, \\ 0, & i\neq j, \end{cases} \quad a''_{ij} = \begin{cases} \varrho a_{ii}^q, & i=j, \\ 0, & i\neq j, \end{cases}$$

$$a'''_{ij} = \begin{cases} \beta a_{ii}^k, & i=j, \\ 0, & i\neq j, \end{cases} \text{ and } c_{ij} = \begin{cases} \alpha\theta^{\varsigma a_{ii}b_{ii}}(a_{ii}^l - b_{ii}^l), & i=j, \\ 0, & i\neq j, \end{cases}$$

where $\alpha, \beta$ and $\gamma$ are arbitrary positive real constants, $\varrho$ and $\varsigma$ are two arbitrary real constants, $k$ and $l$ are two arbitrary but fixed odd natural numbers, and $q$ is an arbitrary but fixed even natural number such that $k > q$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we obtain

$$\begin{aligned} \langle M(A)-M(B), J_2(\eta(A,B))\rangle &= \langle M(A)-M(B), \eta(A,B)\rangle \\ &= tr\Big(\begin{pmatrix} a'''_{ij} - b'''_{ij} \end{pmatrix}\begin{pmatrix} c_{ij} \end{pmatrix}\Big) \\ &= \sum_{i=1}^n \alpha\beta(a_{ii}^k - b_{ii}^k)\theta^{\varsigma a_{ii}b_{ii}}(a_{ii}^l - b_{ii}^l) \\ &= \alpha\beta\sum_{i=1}^n (a_{ii}-b_{ii})^2\theta^{\varsigma a_{ii}b_{ii}} \sum_{t=1}^k a_{ii}^{k-t}b_{ii}^{t-1} \sum_{j=1}^l a_{ii}^{l-j}b_{ii}^{j-1}. \end{aligned}$$

Thanks to the fact that $k$ and $l$ are odd natural numbers, it can be easily seen that for each $i \in \{1,2,\ldots,n\}$, $\sum\limits_{t=1}^k a_{ii}^{k-t}b_{ii}^{t-1} \geq 0$ and $\sum\limits_{j=1}^l a_{ii}^{l-j}b_{ii}^{j-1} \geq 0$. These facts imply that

$$\langle M(A)-M(B), J_2(\eta(A,B))\rangle \geq 0, \quad \forall A,B \in D_n(\mathbb{R}),$$

which means that $M$ is an $\eta$-accretive mapping. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \frac{\gamma^x-1}{\gamma^x+1}$ for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we get

$$(P_1+M)(A) = (P_1+M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} + a'''_{ij} \end{pmatrix} = \begin{pmatrix} \widehat{a}_{ij} \end{pmatrix},$$

where for each $i,j \in \{1,2,\ldots,n\}$,

$$\widehat{a}_{ij} = \begin{cases} \frac{\gamma^{a_{ii}}-1}{\gamma^{a_{ii}}+1}, & i=j, \\ 0, & i\neq j, \end{cases} = \begin{cases} f(a_{ii}), & i=j, \\ 0, & i\neq j. \end{cases}$$

Taking into account that $f(\mathbb{R}) = (-1,1)$, it follows that $(P_1+M)(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, i.e., $P_1+M$ is not surjective, and so $M$ is not a $P_1$-$\eta$-accretive mapping. Now, let $\lambda > 0$ be an arbitrary real constant and let the function

$g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) := \lambda\beta x^k + \varrho x^q$, for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we yield

$$(P_2 + \lambda M)(A) = (P_2 + \lambda M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} + \lambda a'''_{ij} \end{pmatrix} = \begin{pmatrix} \widetilde{a}_{ij} \end{pmatrix},$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} \lambda\beta a_{ii}^k + \varrho a_{ii}^q, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} g(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Since $q$ is an even natural number and $k$ is an odd natural number such that $k > q$, it can be easily observed that $g(\mathbb{R}) = \mathbb{R}$, which guarantees that $(P_2 + \lambda M)(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, the mapping $P_2 + \lambda M$ is surjective. Since $\lambda > 0$ was arbitrary, we infer that $M$ is a $P_2$-accretive mapping.

It is significant to emphasize that if $P = I$, the identity mapping on $E$, then the definition of $P$-$\eta$-accretive mapping is that of generalized $m$-accretive mapping. In fact, the class of $P$-$\eta$-accretive mappings has close relation with that of generalized $m$-accretive mappings in the framework of Banach spaces. On the other hand, invoking Example 2.11, for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a $P$-$\eta$-accretive mapping may not be generalized $m$-accretive. In the following assertion, the sufficient conditions for a $P$-$\eta$-accretive mapping $M$ to be generalized $m$-accretive are stated.

**Lemma 2.14.** [55, Theorem 3.1(a)] *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly $\eta$-accretive mapping, $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping, and let $x, u \in E$ be two given points. If $\langle u - v, J_q(\eta(x, y))\rangle \geq 0$ holds for all $(y, v) \in \mathrm{Graph}(M)$, then $(x, u) \in \mathrm{Graph}(M)$.*

Regarding Example 2.12, for given mappings $P : E \to E$ and $\eta : E \times E \to E$, a generalized $m$-accretive mapping need not be $P$-$\eta$-accretive. In the next theorem, the conditions under which for given mappings $P : E \to E$ and $\eta : E \times E \to E$, every generalized $m$-accretive mapping is $P$-$\eta$-accretive are stated. Let us first recall the following concepts.

**Definition 2.15.** *Let $E$ be a real $q$-uniformly smooth Banach space. A mapping $P : E \to E$ is said to be coercive if*

$$\lim_{\|x\| \to +\infty} \frac{\langle P(x), J_q(x)\rangle}{\|x\|} = +\infty.$$

**Definition 2.16.** *Let $E$ be a real $q$-uniformly smooth Banach space and $P : E \to E$ be a single-valued mapping. $P$ is said to be*

(i) *bounded, if $P(A)$ is a bounded subset of $E$, for every bounded subset $A$ of $E$.*
(ii) *hemi-continuous if for any fixed points $x, y, z \in E$, the function $t \longmapsto \langle P(x + ty), J_q(z)\rangle$ is continuous at $0^+$.*

**Theorem 2.17.** *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, and $P : E \to E$ be a bounded, coercive, hemi-continuous and $\eta$-accretive mapping. If $M : E \to 2^E$ is a generalized $m$-accretive mapping, then $M$ is $P$-$\eta$-accretive.*

*Proof.* Since the mapping $P$ is bounded, coercive, hemi-continuous and $\eta$-accretive, using Theorem 3.1 of Guo [28, P.401], we conclude that $P + \lambda M$ is surjective for every $\lambda > 0$, i.e., $\mathrm{Range}(P + \lambda M)(E) = E$ holds for every $\lambda > 0$. Therefore, $M$ is a $P$-$\eta$-accretive mapping. This gives the desired result. □

**Theorem 2.18.** *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly $\eta$-accretive mapping, and $M : E \to 2^E$ be an $\eta$-accretive mapping. Then, the mapping $(P + \lambda M)^{-1} : \mathrm{Range}(P + \lambda M) \to E$ is single-valued for every constant $\lambda > 0$.*

*Proof.* Choose constant $\lambda > 0$ and point $u \in \text{Range}(P + \lambda M)$ arbitrarily but fixed. Then for any $x, y \in (P + \lambda M)^{-1}(u)$, we have $u = (P + \lambda M)(x) = (P + \lambda M)(y)$, which implies that

$$\lambda^{-1}(u - P(x)) \in M(x) \text{ and } \lambda^{-1}(u - P(y)) \in M(y).$$

Owing to the fact that $M$ is $\eta$-accretive, we deduce that

$$0 \leq \langle \lambda^{-1}(u - P(x)) - \lambda^{-1}(u - P(y)), J_q(\eta(x, y))\rangle = \lambda^{-1}\langle P(x) - P(y), J_q(\eta(x, y))\rangle.$$

Taking into account that the mapping $P$ is strictly $\eta$-accretive, the preceding inequality implies that $x = y$ and so the mapping $(P + \lambda M)^{-1}$ is single-valued. The proof is completed. □

As an immediate consequence of the last result, we obtain the following conclusion due to Kazmi and Khan [40].

**Lemma 2.19.** [55, Theorem 3.1(b)] *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly $\eta$-accretive mapping, and $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping. Then, the mapping $(P + \lambda M)^{-1} : E \to E$ is single-valued for every real constant $\lambda > 0$.*

Based on Lemma 2.19, one can define the resolvent operator $R^{P,\eta}_{M,\lambda}$ associated with $P, \eta, M$ and given constant $\lambda > 0$ as follows.

**Definition 2.20.** *[40, 55] Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly $\eta$-accretive mapping, $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. The resolvent operator $R^{P,\eta}_{M,\lambda} : E \to E$ associated with $P, \eta, M$ and $\lambda$ is defined by*

$$R^{P,\eta}_{M,\lambda}(u) = (P + \lambda M)^{-1}(u), \quad \forall u \in E.$$

Before dealing with the most important result of this section due to Peng and Zhu [55], we need to recall the following notion.

**Definition 2.21.** *A vector-valued mapping $\eta : E \times E \to E$ is said to be $\tau$-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$, for all $x, y \in E$.*

**Lemma 2.22.** [55, Lemma 2.4] *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a $\tau$-Lipschitz continuous mapping, $P : E \to E$ be a $\gamma$-strongly $\eta$-accretive mapping, $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. Then, the resolvent operator $R^{P,\eta}_{M,\lambda} : E \to E$ is Lipschitz continuous with a constant $\frac{\tau^{q-1}}{\gamma}$, i.e.,*

$$\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\| \leq \frac{\tau^{q-1}}{\gamma}\|u - v\|, \quad \forall u, v \in E.$$

## 3. Formulation of the Problem: Existence and Uniqueness of Solution

This section is devoted to the introduction of a new system of variational-like inclusions involving $P$-$\eta$-accretive mappings in real $q$-uniformly smooth Banach spaces and establishing the existence and uniqueness of a solution for the above mentioned system using the resolvent operator technique.

Let for each $i \in \{1, 2\}$, $E_i$ be a real $q_i$-uniformly smooth Banach space with a norm $\|.\|_i$ and $q_i > 1$, $P_i, f_i, g_i : E_i \to E_i$, $\eta_i : E_i \times E_i \to E_i$, $F : E_1 \times E_2 \to E_1$ and $G : E_1 \times E_2 \to E_2$ be the nonlinear mappings. Suppose further that $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ are two multi-valued nonlinear mappings such that for each $z \in E_1$, $M(., z) : E_1 \to 2^{E_1}$ is a $P_1$-$\eta_1$-accretive mapping with $g_1(E_1) \cap \text{dom}\, M(., z) \neq \emptyset$, and $N(., t) : E_2 \to 2^{E_2}$ is a $P_2$-$\eta_2$-accretive mapping for each $t \in E_2$ with $g_2(E_2) \cap \text{dom}\, N(., t) \neq \emptyset$. We consider the problem of finding $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F(x, y - f_2(y)) + M(g_1(x), x), \\ 0 \in G(x - f_1(x), y) + N(g_2(y), y), \end{cases} \tag{1}$$

which is called a *system of generalized nonlinear variational-like inclusions* (SGNVLI) with $P$-$\eta$-accretive mappings.

If for $i = 1, 2$, $g_i \equiv I_i$, the identity mapping on $E_i$, $f_i \equiv 0$, $M : E_1 \to 2^{E_1}$ and $N : E_2 \to 2^{E_2}$ are two unvariate multi-valued nonlinear mappings, then the SGNVLI (1) collapses to the problem of finding $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F(x, y) + M(x), \\ 0 \in G(x, y) + N(y), \end{cases} \tag{2}$$

which was introduced and studied by Peng and Zhu [55].

**Remark 3.1.** It is worth noting that for appropriate and suitable choices of the mappings $P_i, \eta_i, f_i, g_i, F, G, M, N$ and the underlying spaces $E_i$ ($i = 1, 2$), one can obtain many known classes of variational inequalities and variational inclusions as special cases of the SGNVLI (1), see different problems considered and studied in [23, 34, 51, 63, 64, 66] and the references therein.

The following conclusion which tells the SGNVLI (1) is equivalent to a fixed point problem gives a characterization of the solution of the SGNVLI (1) and plays a crucial role in the sequel.

**Lemma 3.2.** *Suppose that $E_i, P_i, \eta_i, f_i, g_i, F, G, M, N$ ($i = 1, 2$) are the same as in the* SGNVLI (1) *such that for each $i \in \{1, 2\}$, $P_i$ is a strictly $\eta_i$-accretive mapping with* $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. *Then $(x, y) \in E_1 \times E_2$ is a solution of the* SGNVLI (1) *if and only if*

$$\begin{cases} g_1(x) = R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))], \\ g_2(y) = R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho G(x - f_1(x), y)], \end{cases} \tag{3}$$

*where $R^{P_1,\eta_1}_{M(.,x),\lambda} = (P_1 + \lambda M(., x))^{-1}$, $R^{P_2,\eta_2}_{N(.,y),\rho} = (P_2 + \rho N(., y))^{-1}$, and $\lambda, \rho > 0$ are two constants.*

*Proof.* The conclusions follow directly from Definition 2.20 and some simple arguments. □

Before stating the main result of this section, we need to define the following special notions.

**Definition 3.3.** *Let $E$ be a real $q$-uniformly smooth Banach space. A mapping $T : E \to E$ is said to be $(\xi, \varsigma)$-relaxed cocoercive if there exist two constants $\xi, \varsigma > 0$ such that*

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq -\xi\|T(x) - T(y)\|^q + \varsigma\|x - y\|^q, \quad \forall x, y \in E.$$

**Definition 3.4.** *Let $E$ be a real $q$-uniformly smooth Banach space and let $F : E \times E \to E$ and $T : E \to E$ be the mappings. For a given point $(a, b) \in E \times E$, the mapping*

(i) *$F(a, .)$ is said to be $k$-strongly accretive with respect to $T$ (or $T$-strongly accretive with constant $k$) if there exists a constant $k > 0$ such that*

$$\langle F(a, x) - F(a, y), J_q(T(x) - T(y))\rangle \geq k\|x - y\|^q, \quad \forall x, y \in E;$$

(ii) *$F(., b)$ is said to be $r$-strongly accretive with respect to $T$ (or $T$-strongly accretive with constant $r$) if there exists a constant $r > 0$ such that*

$$\langle F(x, b) - F(y, b), J_q(T(x) - T(y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in E;$$

(iii) *$F(a, .)$ is said to be $\xi$-Lipschitz continuous if there exists a constant $\xi > 0$ such that*

$$\|F(a, x) - F(a, y)\| \leq \xi\|x - y\|, \quad \forall x, y \in E;$$

(iv) *$F(., b)$ is said to be $\gamma$-Lipschitz continuous if there exists a constant $\gamma > 0$ such that*

$$\|F(x, b) - F(y, b)\| \leq \gamma\|x - y\|, \quad \forall x, y \in E.$$

**Theorem 3.5.** *Let for each $i \in \{1,2\}$, $E_i$ be a real $q_i$-uniformly smooth Banach space with a norm $\|.\|_i$ and $q_i > 1$, $\eta_i : E_i \times E_i \to E_i$ be $\tau_i$-Lipschitz continuous and $P_i : E_i \to E_i$ be $\gamma_i$-strongly $\eta_i$-accretive and $\delta_i$-Lipschitz continuous. For $i = 1,2$, suppose that $f_i : E_i \to E_i$ is $(\zeta_i, \varsigma_i)$-relaxed cocoercive and $\theta_i$-Lipschitz continuous and $g_i : E_i \to E_i$ is $(\sigma_i, \nu_i)$-relaxed cocoercive and $\pi_i$-Lipschitz continuous such that* $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. *Let $F : E_1 \times E_2 \to E_1$ and $G : E_1 \times E_2 \to E_2$ be two nonlinear mappings such that for any given point $(a,b) \in E_1 \times E_2$, $F(.,b)$ is $r_1$-strongly accretive with respect to $P_1 \circ g_1$ and $s_1$-Lipschitz continuous, $F(a,.)$ is $\xi_1$-Lipschitz continuous, $G(a,.)$ is $r_2$-strongly accretive with respect to $P_2 \circ g_2$ and $s_2$-Lipschitz continuous and $G(.,b)$ is $\xi_2$-Lipschitz continuous. Assume that $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ are two multi-valued nonlinear mappings such that for each $z \in E_1$, $M(.,z) : E_1 \to 2^{E_1}$ is a $P_1$-$\eta_1$-accretive mapping with $g_1(E_1) \cap \mathrm{dom}\, M(.,z) \neq \emptyset$, and for each $t \in E_2$, $N(.,t) : E_2 \to 2^{E_2}$ is a $P_2$-$\eta_2$-accretive mapping with $g_2(E_2) \cap \mathrm{dom}\, N(.,t) \neq \emptyset$. Suppose further that there exist constants $\mu_i > 0$ $(i = 1,2)$ such that*

$$
\begin{aligned}
\|R^{P_1,\eta_1}_{M(.,u),\lambda}(w) - R^{P_1,\eta_1}_{M(.,v),\lambda}(w)\| &\leq \mu_1 \|u - v\|_1, \quad \forall u,v,w \in E_1, && (4)\\
\|R^{P_2,\eta_2}_{N(.,u),\rho}(w) - R^{P_2,\eta_2}_{N(.,v),\rho}(w)\| &\leq \mu_2 \|u - v\|_2, \quad \forall u,v,w \in E_2. && (5)
\end{aligned}
$$

*If there exist two constants $\lambda, \rho > 0$ such that*

$$
\begin{aligned}
&\mu_1 + \sqrt[q_1]{1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}} + \frac{\tau_1^{q_1-1}}{\gamma_1} \sqrt[q_1]{\delta_1^{q_1}\pi_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}} \\
&+ \frac{\rho\xi_2\tau_2^{q_2-1}}{\gamma_2} \sqrt[q_1]{1 - q_1\varsigma_1 + (c_{q_1} + q_1\zeta_1)\theta_1^{q_1}} < 1,
\end{aligned} \tag{6}
$$

*and*

$$
\begin{aligned}
&\mu_2 + \sqrt[q_2]{1 - q_2\nu_2 + (c_{q_2} + q_2\sigma_2)\pi_2^{q_2}} + \frac{\tau_2^{q_2-1}}{\gamma_2} \sqrt[q_1]{\delta_2^{q_2}\pi_2^{q_2} - q_2\rho r_2 + \rho^{q_2} c_{q_2} s_2^{q_2}} \\
&+ \frac{\lambda\xi_1\tau_1^{q_1-1}}{\gamma_1} \sqrt[q_2]{1 - q_2\varsigma_2 + (c_{q_2} + q_2\zeta_2)\theta_2^{q_2}} < 1,
\end{aligned} \tag{7}
$$

*where $c_{q_1}$ and $c_{q_2}$ are constants guaranteed by Lemma 2.4, and for the case when $q_1$ and $q_2$ are even natural numbers, in addition to* (6) *and* (7), *the following conditions hold:*

$$
\begin{cases}
q_i\nu_i < 1 + (c_{q_i} + q_i\sigma_i)\pi_i^{q_i}, & (i = 1,2),\\
q_i\varsigma_i < 1 + (c_{q_i} + q_i\zeta_i)\theta_i^{q_i}, & (i = 1,2),\\
q_1\lambda r_1 < \delta_1^{q_1}\pi_1^{q_1} + \lambda^{q_1} c_{q_1} s_1^{q_1}, &\\
q_2\rho r_2 < \delta_2^{q_2}\pi_2^{q_2} + \rho^{q_2} c_{q_2} s_2^{q_2}, &
\end{cases} \tag{8}
$$

*then the* SGNVLI (1) *admits a unique solution.*

*Proof.* For any given $\lambda, \rho > 0$, define the mappings $S_\lambda : E_1 \times E_2 \to E_1$ and $T_\rho : E_1 \times E_2 \to E_2$, respectively, by

$$
S_\lambda(x,y) = x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] \tag{9}
$$

and

$$
T_\rho(x,y) = y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho G(x - f_1(x), y)], \tag{10}
$$

for all $(x, y) \in E_1 \times E_2$. By using (4), (9) and Lemma 2.22, for all $(x, y), (x', y') \in E_1 \times E_2$, we obtain

$$\begin{aligned}
\|S_\lambda(x,y) - S_\lambda(x',y')\|_1 \leq\; & \|x - x' - (g_1(x) - g_1(x'))\|_1 \\
& + \|R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] \\
& - R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x')) - \lambda F(x', y' - f_2(y'))]\|_1 \\
\leq\; & \|x - x' - (g_1(x) - g_1(x'))\|_1 \\
& + \|R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] \\
& - R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
& + \|R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] \\
& - R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x')) - \lambda F(x', y' - f_2(y'))]\|_1 \\
\leq\; & \|x - x' - (g_1(x) - g_1(x'))\|_1 + \mu_1\|x - x'\|_1 \\
& + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\|P_1(g_1(x)) - P_1(g_1(x')) \\
& - \lambda(F(x, y - f_2(y)) - F(x', y - f_2(y))\|_1 \\
& + \lambda\|F(x', y - f_2(y)) - F(x', y' - f_2(y'))\|_1\Big).
\end{aligned} \tag{11}$$

Thanks to Lemma 2.4, there exists a constant $c_{q_1} > 0$ such that

$$\begin{aligned}
\|x - x' - (g_1(x) - g_1(x'))\|_1^{q_1} \leq\; & \|x - x'\|_1^{q_1} - q_1\langle g_1(x) - g_1(x'), J_{q_1}(x - x')\rangle \\
& + c_{q_1}\|g_1(x) - g_1(x')\|_1^{q_1}.
\end{aligned}$$

Since $g_1$ is $(\sigma_1, \nu_1)$-relaxed cocoercive and $\pi_1$-Lipschitz continuous, it follows that

$$\begin{aligned}
\|x - x' - (g_1(x) - g_1(x'))\|_1^{q_1} &\leq \|x - x'\|_1^{q_1} - q_1\nu_1\|x - x'\|_1^{q_1} + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}\|x - x'\|_1^{q_1} \\
&= (1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1})\|x - x'\|_1^{q_1},
\end{aligned}$$

which implies that

$$\|x - x' - (g_1(x) - g_1(x'))\|_1 \leq \sqrt[q_1]{1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}}\|x - x'\|_1. \tag{12}$$

With the help of the assumptions and using Lemma 2.4, we get

$$\begin{aligned}
& \|P_1(g_1(x)) - P_1(g_1(x')) - \lambda(F(x, y - f_2(y)) - F(x', y - f_2(y)))\|_1^{q_1} \\
& \leq \|P_1(g_1(x)) - P_1(g_1(x'))\|_1^{q_1} - q_1\lambda\langle F(x, y - f_2(y)) - F(x', y - f_2(y)), \\
& \quad J_{q_1}(P_1(g_1(x)) - P_1(g_1(x')))\rangle + \lambda^{q_1}c_{q_1}\|F(x, y - f_2(y)) - F(x', y - f_2(y))\|_1^{q_1} \\
& \leq \delta_1^{q_1}\|g_1(x) - g_1(x')\|_1^{q_1} - q_1\lambda r_1\|x - x'\|_1^{q_1} + \lambda^{q_1}c_{q_1}s_1^{q_1}\|x - x'\|_1^{q_1} \\
& \leq (\delta_1^{q_1}\pi_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1}c_{q_1}s_1^{q_1})\|x - x'\|_1^{q_1},
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
& \|P_1(g_1(x)) - P_1(g_1(x')) - \lambda(F(x, y - f_2(y)) - F(x', y - f_2(y)))\|_1 \\
& \leq \sqrt[q_1]{\delta_1^{q_1}\pi_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1}c_{q_1}s_1^{q_1}}\|x - x'\|_1.
\end{aligned} \tag{13}$$

In the light of the assumptions, we yield

$$\|F(x', y - f_2(y)) - F(x', y' - f_2(y'))\|_1 \leq \xi_1\|y - y' - (f_2(y) - f_2(y'))\|_2. \tag{14}$$

Using Lemma 2.4 and the assumptions, in a similar way to that of the proof of (12), one can deduce that

$$\|y - y' - (f_2(y) - f_2(y'))\|_2 \leq \sqrt[q_2]{1 - q_2\varsigma_2 + (c_{q_2} + q_2\zeta_2)\theta_2^{q_2}}\|y - y'\|_2. \tag{15}$$

Combining (11)–(15), we get

$$\begin{aligned}\|S_\lambda(x,y) - S_\lambda(x',y')\|_1 \leq & \sqrt[q_1]{1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}}\|x - x'\|_1 + \mu_1\|x - x'\|_1 \\ & + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big( \sqrt[q_1]{\delta_1^{q_1}\pi_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}}\|x - x'\|_1 \\ & + \lambda\xi_1 \sqrt[q_2]{1 - q_2\varsigma_2 + (c_{q_2} + q_2\zeta_2)\theta_2^{q_2}}\|y - y'\|_2\Big) \\ = & \varphi_1\|x - x'\|_1 + \vartheta_1\|y - y'\|_2,\end{aligned} \tag{16}$$

where

$$\varphi_1 = \mu_1 + \sqrt[q_1]{1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}} + \frac{\tau_1^{q_1-1}}{\gamma_1}\sqrt[q_1]{\delta_1^{q_1}\pi_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}}$$

and

$$\vartheta_1 = \frac{\lambda\xi_1\tau_1^{q_1-1}}{\gamma_1}\sqrt[q_2]{1 - q_2\varsigma_2 + (c_{q_2} + q_2\zeta_2)\theta_2^{q_2}}.$$

Following the same argument, we can show that

$$\|T_\rho(x,y) - T_\rho(x',y')\|_2 \leq \varphi_2\|x - x'\|_1 + \vartheta_2\|y - y'\|_2, \tag{17}$$

where

$$\vartheta_2 = \mu_2 + \sqrt[q_2]{1 - q_2\nu_2 + (c_{q_2} + q_2\sigma_2)\pi_2^{q_2}} + \frac{\tau_2^{q_2-1}}{\gamma_2}\sqrt[q_2]{\delta_2^{q_2}\pi_2^{q_2} - q_2\rho r_2 + \rho^{q_2} c_{q_2} s_2^{q_2}}$$

and

$$\varphi_2 = \frac{\rho\xi_2\tau_2^{q_2-1}}{\gamma_2}\sqrt[q_1]{1 - q_1\varsigma_1 + (c_{q_1} + q_1\zeta_1)\theta_1^{q_1}}.$$

Let us define $\|.\|_*$ on $E_1 \times E_2$ by

$$\|(x,y)\|_* = \|x\|_1 + \|y\|_2, \quad \forall (x,y) \in E_1 \times E_2. \tag{18}$$

It can be easily seen that $(E_1 \times E_2, \|.\|_*)$ is a Banach space. For any $\lambda, \rho > 0$, define a mapping $Q_{\lambda,\rho} : E_1 \times E_2 \to E_1 \times E_2$ by

$$Q_{\lambda,\rho}(x,y) = (S_\lambda(x,y), T_\rho(x,y)), \quad \forall (x,y) \in E_1 \times E_2. \tag{19}$$

Applying (16) and (17), we obtain

$$\begin{aligned}&\|S_\lambda(x,y) - S_\lambda(x',y')\|_1 + \|T_\rho(x,y) - T_\rho(x',y')\|_2 \\ &\leq (\varphi_1 + \varphi_2)\|x - x'\|_1 + (\vartheta_1 + \vartheta_2)\|y - y'\|_2 \\ &\leq k\|(x,y) - (x',y')\|_*,\end{aligned} \tag{20}$$

where $k = \max\{\varphi_1 + \varphi_2, \vartheta_1 + \vartheta_2\}$. By virtue of (6) and (7), we know that $k \in (0,1)$ and using (20) we infer that $Q_{\lambda,\rho}$ is a contraction mapping. According to Banach fixed point theorem, there exists a unique point $(x^*, y^*) \in E_1 \times E_2$ such that $Q_{\lambda,\rho}(x^*, y^*) = (x^*, y^*)$. Employing (9), (10) and (19), it follows that

$$\begin{cases} g_1(x^*) = R^{P_1,\eta_1}_{M(.,x^*),\lambda}[P_1(g_1(x^*)) - \lambda F(x^*, y - f_2(y^*))], \\ g_2(y^*) = R^{P_2,\eta_2}_{N(.,y^*),\rho}[P_2(g_2(y^*)) - \rho G(x^* - f_1(x^*), y^*)]. \end{cases}$$

Now, Lemma 3.2 ensures that $(x^*, y^*)$ is a unique solution of the SGNVLI (1). This completes the proof. □

**Corollary 3.6.** [55, Theorem 4.1] *Let $E_1$ and $E_2$ be real $q$-uniformly smooth Banach spaces. For $i = 1, 2$, let $\eta_i : E_i \times E_i \to E_i$ be Lipschitz continuous with constant $\tau_i$, and $P_i : E_i \to E_i$ be strongly $\eta_i$-accretive and Lipschitz continuous with constants $\gamma_i$ and $\delta_i$, respectively. Let $F : E_1 \times E_2 \to E_1$ be a nonlinear operator such that for any given $(a, b) \in E_1 \times E_2$, $F(., b)$ is $P_1$-strongly accretive and Lipschitz continuous with constants $r_1$ and $s_1$, respectively, and $F(a, .)$ is Lipschitz continuous with constant $\xi_1$. Let $G : E_1 \times E_2 \to E_2$ be a nonlinear operator such that for any given $(x, y) \in E_1 \times E_2$, $G(x, .)$ is $P_2$-strongly accretive and Lipschitz continuous with constants $r_2$ and $s_2$, respectively, and $G(., y)$ is Lipschitz continuous with constant $\xi_2$. Assume that $M : E_1 \to 2^{E_1}$ is a $P_1$-$\eta_1$-accretive operator and $N : E_2 \to 2^{E_2}$ is a $P_2$-$\eta_2$-accretive operator. If there exist constants $\lambda, \rho > 0$ such that*

$$\frac{\tau_1^{q-1}}{\gamma_1}\sqrt[q]{\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q} + \frac{\xi_2\rho\tau_2^{q-1}}{\gamma_2} < 1, \tag{21}$$

*and*

$$\frac{\tau_2^{q-1}}{\gamma_2}\sqrt[q]{\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q} + \frac{\xi_1\lambda\tau_1^{q-1}}{\gamma_1} < 1, \tag{22}$$

*where $c_q$ is a constant guaranteed by Lemma 2.4, and for the case when $q$ is an even natural number, in addition to* (21) *and* (22)*, the following conditions hold:*

$$q\lambda r_1 < \delta_1^q + c_q\lambda^q s_1^q \text{ and } q\rho r_2 \leq \delta_2^q + c_q\rho^q s_2^q, \tag{23}$$

*then the problem* (2) *admits a unique solution.*

*Proof.* Since for $i = 1, 2$, $g_i \equiv I_i$, the identity mapping on $E_i$, it follows that for $i = 1, 2$, $\pi_i = 1$ and

$$\|x - x' - (g_1(x) - g_1(x'))\|_1 = \|y - y' - (g_2(y) - g_2(y'))\|_2 = 0.$$

In view of the assumptions, taking $q_i = q$, $f_i \equiv 0$ and $\mu_i = \zeta_i = \varsigma_i = \theta_i = 0$ for $i = 1, 2$, (6) and (7) reduce to (21) and (22), respectively. Now, the statement follows immediately using Theorem 3.5. □

**Remark 3.7.** Let us emphasize that by a careful reading of the proof of Theorem 4.1 in [55], we found that the conditions mentioned in the context of [55, Theorem 4.1] are not enough for guaranteeing the existence of a unique solution for the problem (2). In fact, if $q$ is an even natural number, then in addition to (21) and (22), the conditions (23) must be also added to the context of [55, Theorem 4.1], as we have done in the context of Corollary 3.6.

## 4. Variational Convergence and Iterative Algorithm

In this section, using the notions of graph convergence and the resolvent operator associated with a $P$-$\eta$-accretive mapping, we first establish a new equivalence relationship between the graph convergence of a sequence of $P$-$\eta$-accretive mappings and their associated resolvent operators, respectively, to a given $P$-$\eta$-accretive mapping and its associated resolvent operator. Then, as an application of the obtained equivalence formulation and the resolvent operator technique, a new iterative algorithm is constructed for approximating a common element of the set of solutions of the SGNVLI (1) and the set of fixed points of an $(\{a_n\}, \{b_n\}, \phi)$-total uniformly $L$-Lipschitzian mapping.

**Definition 4.1.** [3, 9] *Given multi-valued mappings $M_n, M : E \to 2^E$ $(n \geq 0)$, the sequence $\{M_n\}_{n=0}^{\infty}$ is said to be graph-convergent to $M$, denoted by $M_n \xrightarrow{G} M$, if for every point $(x, u) \in \mathrm{Graph}(M)$, there exists a sequence of points $(x_n, u_n) \in \mathrm{Graph}(M_n)$ such that $x_n \to x$ and $u_n \to u$ as $n \to \infty$.*

**Theorem 4.2.** *Let $E$ be a real $q$-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly $\eta$-accretive mapping and let $M : E \to 2^E$ be a $P$-$\eta$-accretive mapping. Assume that for each $n \geq 0$, $\eta_n : E \times E \to E$ is a $\tau_n$-Lipschitz continuous mapping, $P_n : E \to E$ is a $\gamma_n$-strongly $\eta_n$-accretive and $\delta_n$-Lipschitz continuous mapping, and $M_n : E \to 2^E$ is a $P_n$-$\eta_n$-accretive mapping. Suppose that $\lim\limits_{n\to\infty} P_n(x) = P(x)$ for any $x \in E$, and the sequences $\{\delta_n\}_{n=0}^{\infty}$, $\{\tau_n\}_{n=0}^{\infty}$ and $\{\frac{1}{\gamma_n}\}_{n=0}^{\infty}$ are bounded. Further, let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive real constants convergent to a positive real constant $\lambda$. Then $M_n \xrightarrow{G} M$ if and only if $R^{P_n,\eta_n}_{M_n,\lambda_n}(z) \to R^{P,\eta}_{M,\lambda}(z)$, for all $z \in E$, as $n \to \infty$, where for each $n \geq 0$, $R^{P_n,\eta_n}_{M_n,\lambda_n} = (P_n + \lambda_n M_n)^{-1}$ and $R^{P,\eta}_{M,\lambda} = (P + \rho M)^{-1}$.*

*Proof.* Suppose first that $M_n \xrightarrow{G} M$, and let $z \in E$ be chosen arbitrarily but fixed. Since the mapping $M$ is $P$-$\eta$-accretive, it follows that $(P + \lambda M)(E) = E$ and so, there exists a point $(x, u) \in \text{Graph}(M)$ such that $z = P(x) + \lambda u$. Invoking Definition 4.1, there exists a sequence $\{(x_n, u_n)\}_{n=0}^{\infty} \subset \text{Graph}(M_n)$ such that $x_n \to x$ and $u_n \to u$, as $n \to \infty$. In the light of the facts that $(x, u) \in \text{Graph}(M)$ and $(x_n, u_n) \in \text{Graph}(M_n)$ for all $n \geq 0$, we infer that

$$x = R^{P,\eta}_{M,\lambda}[P(x) + \lambda u] \quad \text{and} \quad x_n = R^{P_n,\eta_n}_{M_n,\lambda_n}[P_n(x_n) + \lambda_n u_n], \quad \forall n \geq 0. \tag{24}$$

Picking $z_n = P_n(x_n) + \lambda_n u_n$ for each $n \geq 0$, and by utilizing Lemma 2.22, (24) and with the help of the assumptions, for each $n \geq 0$, we obtain

$$\begin{aligned}
&\|R^{P_n,\eta_n}_{M_n,\lambda_n}(z) - R^{P,\eta}_{M,\lambda}(z)\| \\
&\leq \|R^{P_n,\eta_n}_{M_n,\lambda_n}(z) - R^{P_n,\eta_n}_{M_n,\lambda_n}(z_n)\| + \|R^{P_n,\eta_n}_{M_n,\lambda_n}(z_n) - R^{P,\eta}_{M,\lambda}(z)\| \\
&\leq \frac{\tau_n^{q-1}}{\gamma_n}\|z_n - z\| + \|R^{P_n,\eta_n}_{M_n,\lambda_n}[P_n(x_n) + \lambda_n u_n] - R^{P,\eta}_{M,\lambda}[P(x) + \lambda u]\| \\
&\leq \frac{\tau_n^{q-1}}{\gamma_n}\|z_n - z\| + \|x_n - x\| \\
&= \frac{\tau_n^{q-1}}{\gamma_n}\|P_n(x_n) + \lambda_n u_n - P(x) - \lambda u\| + \|x_n - x\| \\
&\leq \frac{\tau_n^{q-1}}{\delta_n}(\|P_n(x_n) - P(x)\| + \|\lambda_n u_n - \lambda u\|) + \|x_n - x\| \\
&\leq \frac{\tau_n^{q-1}}{\gamma_n}(\|P_n(x_n) - P_n(x)\| + \|P_n(x) - P(x)\| \\
&\quad + \|\lambda_n u_n - \lambda_n u\| + \|\lambda_n u - \lambda u\|) + \|x_n - x\| \\
&\leq (1 + \frac{\delta_n\tau_n^{q-1}}{\gamma_n})\|x_n - x\| + \frac{\tau_n^{q-1}}{\gamma_n}\|P_n(x) - P(x)\| \\
&\quad + \frac{\lambda_n\tau_n^{q-1}}{\gamma_n}\|u_n - u\| + \frac{|\lambda_n - \lambda|\tau_n^{q-1}}{\gamma_n}\|u\|.
\end{aligned}$$

Taking into account that the sequences $\{\frac{1}{\gamma_n}\}_{n=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ are bounded and $\lim\limits_{n\to\infty} \lambda_n = \lambda$, we conclude that the sequence $\{\frac{\lambda_n\tau_n^{q-1}}{\gamma_n}\}_{n=0}^{\infty}$ is also bounded. In the light of the assumptions, the right-hand side of the preceding inequality tends to zero, as $n \to \infty$, which ensures that $\lim\limits_{n\to\infty} R^{P_n,\eta_n}_{M_n,\lambda_n}(z) = R^{P,\eta}_{M,\lambda}(z)$.

Converse, assume that for all $z \in E$ we have $\lim\limits_{n\to\infty} R^{P_n,\eta_n}_{M_n,\lambda_n}(z) = R^{P,\eta}_{M,\lambda}(z)$. Then, for any $(x, u) \in \text{Graph}(M)$, we infer that $x = R^{P,\eta}_{M,\lambda}[P(x) + \lambda u]$ and so $R^{P_n,\eta_n}_{M_n,\lambda_n}[P(x) + \lambda u] \to x$, as $n \to \infty$. Taking $x_n = R^{P_n,\eta_n}_{M_n,\lambda_n}[P(x) + \lambda u]$ for each $n \geq 0$, it follows that $P(x) + \lambda u \in (P_n + \lambda_n M_n)(x_n)$. Hence, for each $n \geq 0$, we can choose $u_n \in M_n(x_n)$

such that $P(x) + \lambda u = P_n(x_n) + \lambda_n u_n$. Then, for each $n \geq 0$, we have

$$\begin{aligned}\|\lambda_n u_n - \lambda u\| = \|P_n(x_n) - P(x)\| &\leq \|P_n(x_n) - P_n(x)\| + \|P_n(x) - P(x)\| \\ &\leq \delta_n\|x_n - x\| + \|P_n(x) - P(x)\|.\end{aligned}$$

Owing to the fact that the sequence $\{\delta_n\}_{n=0}^{\infty}$ is bounded, $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} P_n(x) = P(x)$, it follows that $\lambda_n u_n \to \lambda u$, as $n \to \infty$. Furthermore, for each $n \geq 0$, yields

$$\begin{aligned}\lambda\|u_n - u\| = \|\lambda u_n - \lambda u\| &\leq \|\lambda_n u_n - \lambda u_n\| + \|\lambda_n u_n - \lambda u\| \\ &= |\lambda_n - \lambda|\|u_n\| + \|\lambda_n u_n - \lambda u\|.\end{aligned}$$

The facts that $\lambda_n \to \lambda$ and $\lambda_n u_n \to \lambda u$, as $n \to \infty$, imply that the right-hand side of the above inequality approaches zero, as $n \to \infty$. Accordingly, $u_n \to u$, as $n \to \infty$. Now, in view of Definition 4.1, $M_n \xrightarrow{G} M$. This completes the proof. □

Given a real normed space $E$ with a norm $\|.\|$, we recall that a nonlinear mapping $T : E \to E$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in E$. Since the appearance of the notion of nonexpansive mapping, due to the existence of a strong connection between monotone and accretive operators, two classes of operators which arise actually in the theory of differential equations, and the class of nonexpansive mappings, the fixed point theory of nonexpansive mappings has rapidly grown into an important field of study in both pure and applied mathematics. It has become one of the most essential tools in nonlinear functional analysis. For this reason, during the past few decades, many authors have shown interest in extending the concept of nonexpansive mapping in the framework of different spaces, and the study of the fixed point theory for generalized nonexpansive mappings has also attracted increasing attention. In the next definition, some classes of them are recalled.

**Definition 4.3.** *A nonlinear mapping $T : E \to E$ is said to be*

(i) *$L$-Lipschitzian if there exists a constant $L > 0$ such that*

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in E;$$

(ii) *uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\|, \quad \forall x, y \in E;$$

(iii) *asymptotically nonexpansive* [26] *if there exists a sequence $\{a_n\} \subset (0, +\infty)$ with $\lim\limits_{n\to\infty} a_n = 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall x, y \in E.$$

*Equivalently, we say that $T$ is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \quad \forall x, y \in E;$$

(iv) *total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive)* [2] *if, there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in E$,*

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

It is important to emphasize that every uniformly $L$-Lipschitzian mapping is $L$-Lipschitzian but the converse need not be true. In fact, the class of uniformly $L$-Lipschitzian mappings is essentially wider than the class of $L$-Lipschitzian mappings. This fact is illustrated in the next example.

**Example 4.4.** Consider $E = \mathbb{R}$ with the Euclidean norm $\|.\| = |.|$ and let the self-mapping $T$ of $E$ be defined by

$$T(x) = \begin{cases} \gamma_1 x, & \text{if } x \in (-\infty, 0], \\ \gamma_2 x, & \text{if } x \in [0, +\infty), \end{cases}$$

where $\gamma_1, \gamma_2 > 1$ are arbitrary constants. Taking into account that

$$|T(x) - T(y)| = \begin{cases} |\gamma_1 x - \gamma_1 y| = \gamma_1 |x - y| < \max\{\gamma_1, \gamma_2\}|x - y|, & \forall x, y \in (-\infty, 0], \\ |\gamma_2 x - \gamma_2 y| = \gamma_2 |x - y| < \max\{\gamma_1, \gamma_2\}|x - y|, & \forall x, y \in [0, +\infty), \\ |\gamma_1 x - \gamma_2 y| < |\max\{\gamma_1, \gamma_2\}x - \max\{\gamma_1, \gamma_2\}y| & \\ \qquad = \max\{\gamma_1, \gamma_2\}|x - y|, & \forall x \in (-\infty, 0], y \in [0, +\infty), \end{cases}$$

it follows that $T$ is a $\max\{\gamma_1, \gamma_2\}$-Lipschitzian mapping. But, in the light of the fact that $\gamma_1, \gamma_2 > 1$, for all $n \in \mathbb{N}\backslash\{1\}$, yields

$$|T^n(x) - T^n(y)| = \begin{cases} \gamma_1^n |x - y| > \gamma_1 |x - y|, & \forall x, y \in (-\infty, 0], \\ \gamma_2^n |x - y| > \gamma_2 |x - y|, & \forall x, y \in [0, +\infty). \end{cases}$$

If $\gamma_1 < \gamma_2$ then for all $x, y \in [0, +\infty)$, we have

$$|T^n(x) - T^n(y)| = \gamma_2^n |x - y| > \gamma_2 |x - y| = \max\{\gamma_1, \gamma_2\}|x - y|$$

and for the case when $\gamma_1 > \gamma_2$, for all $x, y \in (-\infty, 0]$, we get

$$|T^n(x) - T^n(y)| = \gamma_1^n |x - y| > \gamma_1 |x - y| = \max\{\gamma_1, \gamma_2\}|x - y|.$$

If $\gamma_1 = \gamma_2$, then for all $x, y \in E$ and $n \in \mathbb{N}\backslash\{1\}$, we obtain

$$|T^n(x) - T^n(y)| = \gamma_1^n |x - y| = \gamma_2^n |x - y| > \gamma_2 |x - y| = \gamma_1 |x - y| = \max\{\gamma_1, \gamma_2\}|x - y|.$$

These facts imply that $T$ is not a uniformly $\max\{\gamma_1, \gamma_2\}$-Lipschitzian mapping.

It is significant to mention that every nonexpansive mapping is asymptotically nonexpansive with $a_n = 0$ (or equivalently $k_n = 1$) for all $n \in \mathbb{N}$, but the converse is not true in general. The following example illustrates that the class of asymptotically nonexpansive mappings contains properly the class of nonexpansive mappings.

**Example 4.5.** For $1 \leq p < \infty$, consider

$$l^p = \Big\{x = \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\Big\},$$

the classical space consisting of all $p$-power summable sequences, with the $p$-norm $\|.\|_p$ defined on it by

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}, \quad \forall x = \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

Moreover, assume that $B$ denote the closed unit ball in the real Banach space $l^p$ and define the self-mapping $T$ of $B$ by

$$T(x_1, x_2, x_3, \dots) = (\underbrace{0, 0, \dots, 0}_{m \text{ times}}, |x_1|^{\sigma_1}, 0, a_2 \sin|x_2|^{q_1}, 0, a_3|x_3|^{\sigma_2}, 0, a_4 \sin|x_4|^{q_2}, \dots, 0,$$
$$a_k |x_k|^{\sigma_{\frac{k+1}{2}}}, 0, a_{k+1} \sin|x_{k+1}|^{q_{\frac{k+1}{2}}}, 0, a_{k+2}x_{k+2}, 0, a_{k+3}x_{k+3}, \dots),$$

where $k$ is an arbitrary but fixed odd natural number, $m \geq k+1$ is an arbitrary but fixed natural number, $\sigma_i, q_i \in \mathbb{N}\backslash\{1\}$ ($i = 1, 2, \ldots, \frac{k+1}{2}$) are arbitrary constants, and $\{a_i\}_{i=2}^{\infty}$ is a sequence of real numbers such that $0 < a_i < 1$ for each $i \geq 2$ and

$$\prod_{i=1}^{\infty} a_{(2^i-1)m+1} = \prod_{i=1}^{\infty} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1} = \prod_{i=1}^{\infty} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} = \frac{1}{\varrho}$$

for each $s \in \{1, 2, \ldots, \frac{k+1}{2}\}$ and $\widehat{s} \in \{2, 3, \ldots, \frac{k+1}{2}\}$, where $\varrho = \max\{\sigma_i, q_i : i = 1, 2, \ldots, \frac{k+1}{2}\}$. Indeed, the self-mapping $T$ of $B$ is defined for all $x = \{x_n\}_{n\in\mathbb{N}} \in B$ by $T(x) = T(\{x_n\})_{n\in\mathbb{N}} = \widehat{x} = \{\widehat{x_n}\}_{n\in\mathbb{N}}$, where $\widehat{x_i} = 0$ for all $1 \leq i \leq m$, $\widehat{x}_{m+1} = |x_1|^{\sigma_1}$, $\widehat{x}_{m+2i} = 0$ for all $i \in \mathbb{N}$,

$$\widehat{x}_{m+2i-1} = \begin{cases} a_i \sin |x_i|^{q_{\frac{i}{2}}}, & \text{if } i \in \{2r | r = 1, 2, \ldots, \frac{k+1}{2}\}, \\ a_i |x_i|^{\sigma_{\frac{i+1}{2}}}, & \text{if } i \in \{2t+1 | t = 1, 2, \ldots, \frac{k-1}{2}\}, \end{cases}$$

and $\widehat{x}_{m+2k+j} = a_{k+\frac{j+1}{2}} x_{k+\frac{j+1}{2}}$ for all $j \in \{2\varsigma + 1 | \varsigma \in \mathbb{N}\}$.

Then, it can be easily proved that for all $x, y \in B$,

$$\begin{aligned} \|T(x) - T(y)\|_p &= \Big( \big| |x_1|^{\sigma_1} - |y_1|^{\sigma_1} \big|^p + \sum_{i=2}^{\frac{k+1}{2}} a_{2i-1}^p \big| |x_{2i-1}|^{\sigma_i} - |y_{2i-1}|^{\sigma_i} \big|^p \\ &\quad + \sum_{i=1}^{\frac{k+1}{2}} a_{2i}^p \big| \sin |x_{2i}|^{q_i} - \sin |y_{2i}|^{q_i} \big|^p + \sum_{i=k+2}^{\infty} a_i^p |x_i - y_i|^p \Big)^{\frac{1}{p}} \\ &\leq \max\Big\{ \sum_{j=1}^{\sigma_i} |x_{2i-1}|^{\sigma_i - j} |y_{2i-1}|^{j-1}, \sum_{\nu=1}^{q_i} |x_{2i}|^{q_i-\nu} |y_{2i}|^{\nu-1}, 1 : \\ &\quad i = 1, 2, \ldots, \frac{k+1}{2} \Big\} \Big( \sum_{i=1}^{\infty} |x_i - y_i|^p \Big)^{\frac{1}{p}} \\ &= \max\Big\{ \sum_{j=1}^{\sigma_i} |x_{2i-1}|^{\sigma_i - j} |y_{2i-1}|^{j-1}, \sum_{\nu=1}^{q_i} |x_{2i}|^{q_i-\nu} |y_{2i}|^{\nu-1}, 1 : \\ &\quad i = 1, 2, \ldots, \frac{k+1}{2} \Big\} \|x - y\|_p. \end{aligned} \tag{25}$$

Thanks to the fact that $x, y \in B$, we deduce that $0 \leq |x_{2i-1}|^{\sigma_i - j}, |x_{2i}|^{q_i-\nu}, |y_{2i-1}|^{j-1}, |y_{2i}|^{\nu-1} \leq 1$ for each $j \in \{1, 2, \ldots, \sigma_i\}$, $\nu \in \{1, 2, \ldots, q_i\}$ and $i \in \{1, 2, \ldots, \frac{k+1}{2}\}$. This fact ensures that $0 \leq \sum_{j=1}^{\sigma_i} |x_{2i-1}|^{\sigma_i-j} |y_{2i-1}|^{j-1} \leq \sigma_i$ and $0 \leq \sum_{\nu=1}^{q_i} |x_{2i}|^{q_i-\nu} |y_{2i}|^{\nu-1} \leq q_i$ for each $i \in \{1, 2, \ldots, \frac{k+1}{2}\}$. Taking into account that $\sigma_i, q_i \in \mathbb{N}\backslash\{1\}$ for each $i \in \{1, 2, \ldots, \frac{k+1}{2}\}$, it follows from (25) that for all $x, y \in B$,

$$\|T(x) - T(y)\|_p \leq \varrho \|x - y\|_p = \|x - y\|_p + (\varrho - 1)\|x - y\|_p. \tag{26}$$

Thereby, the mapping $T$ is Lipschitzian, but not nonexpansive.

For all $n \geq 2$ and $x = \{x_n\}_{n\in\mathbb{N}} \in B$, we obtain

$$
\begin{aligned}
T^n(x) = \Big( &\underbrace{0,0,\ldots,0}_{(2^n-1)m \text{ times}}, \prod_{i=1}^{n-1} a_{(2^i-1)m+1}|x_1|^{\sigma_1}, \underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}+1} \sin|x_2|^{q_1},\\
&\underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}\times 2+1}|x_3|^{\sigma_2}, \underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}\times 3+1} \sin|x_4|^{q_2},\\
&\ldots, \underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k-1)+1}|x_k|^{\sigma_{\frac{k+1}{2}}}, \underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}},\\
&\prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}k+1} \sin|x_{k+1}|^{q_{\frac{k+1}{2}}}, \underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k+1)+1} x_{k+2},\\
&\underbrace{0,0,\ldots,0}_{(2^n-1) \text{ times}}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k+2)+1} x_{k+3}, \ldots \Big).
\end{aligned}
$$

Then, for all $x, y \in B$ and $n \geq 2$, it is easy to yield that

$$
\begin{aligned}
\|T^n(x) - T^n(y)\|_p = \Big( &\Big(\prod_{i=1}^{n-1} a_{(2^i-1)m+1}\Big)^p \Big||x_1|^{\sigma_1} - |y_1|^{\sigma_1}\Big|^p\\
&+ \sum_{s=1}^{\frac{k+1}{2}} \Big( \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}\Big)^p \Big| \sin|x_{2s}|^{q_s} - \sin|y_{2s}|^{q_s}\Big|^p\\
&+ \sum_{\widehat{s}=2}^{\frac{k+1}{2}} \Big( \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1}\Big)^p \Big||x_{2\widehat{s}-1}|^{\sigma_{\widehat{s}}} - |y_{2\widehat{s}-1}|^{\sigma_{\widehat{s}}}\Big|^p\\
&+ \sum_{\widetilde{s}=2}^{\infty} \Big( \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k+\widetilde{s}-1)+1}\Big)^p |x_{k+\widetilde{s}} - y_{k+\widetilde{s}}|^p \Big)^{\frac{1}{p}}\\
\leq \Big( &\Big(\varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1}\Big)^p |x_1 - y_1|^p\\
&+ \sum_{s=1}^{\frac{k+1}{2}} \Big( \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}\Big)^p |x_{2s} - y_{2s}|^p\\
&+ \sum_{\widehat{s}=2}^{\frac{k+1}{2}} \Big( \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1}\Big)^p |x_{2\widehat{s}-1} - y_{2\widehat{s}-1}|^p\\
&+ \sum_{\widetilde{s}=2}^{\infty} \Big( \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k+\widetilde{s}-1)+1}\Big)^p |x_{k+\widetilde{s}} - y_{k+\widetilde{s}}|^p \Big)^{\frac{1}{p}}.
\end{aligned}
\tag{27}
$$

Owing to the fact that $a_i \in (0,1)$ for each $i \geq 2$, we infer that $0 < \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(k+\widetilde{s}-1)+1} < 1$ for each $n, \widetilde{s} \geq 2$.

This fact together with (27) imply that for all $x, y \in B$ and $n \geq 2$,

$$\begin{aligned}
\|T^n(x) - T^n(y)\|_p \leq \Big( \max\Big\{ (\varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1})^p, (\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1})^p, \\
(\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1}\Big)^p, 1 : s = 1, 2, \dots, \frac{k+1}{2}; \\
\widehat{s} = 2, 3, \dots, \frac{k+1}{2} \Big\} \sum_{r=1}^{\infty} |x_r - y_r|^p \Big)^{\frac{1}{p}} \\
= \max\Big\{ \varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1}, \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}, \\
\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1}, 1 : s = 1, 2, \dots, \frac{k+1}{2}; \\
\widehat{s} = 2, 3, \dots, \frac{k+1}{2} \Big\} \|x - y\|_p.
\end{aligned} \tag{28}$$

Since $a_i \in (0, 1)$ for each $i \geq 2$, it follows that

$$0 < \prod_{i=1}^{n-1} a_{(2^i-1)m+1}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} < 1$$

for each $s \in \{1, 2, \dots, \frac{k+1}{2}\}, \widehat{s} \in \{2, 3, \dots, \frac{k+1}{2}\}$ and $n \geq 2$. Moreover, for each $n \geq 2$, we yield

$$\prod_{i=1}^{n-1} a_{(2^i-1)m+1} = a_{(2^{n-1}-1)m+1} \prod_{i=1}^{n-2} a_{(2^i-1)m+1} < \prod_{i=1}^{n-2} a_{(2^i-1)m+1},$$

i.e., $\Big\{ \prod_{i=1}^{n-1} a_{(2^i-1)m+1} \Big\}_{n=2}^{\infty}$ is a decreasing sequence. By an argument analogous to the previous one, one can show that for each $s \in \{1, 2, \dots, \frac{k+1}{2}\}$ and $\widehat{s} \in \{2, 3, \dots, \frac{k+1}{2}\}$, the sequences $\Big\{ \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1} \Big\}_{n=2}^{\infty}$ and $\Big\{ \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} \Big\}_{n=2}^{\infty}$ are also decreasing. Relying on the fact that

$$\begin{aligned}
\lim_{n\to\infty} \prod_{i=1}^{n-1} a_{(2^i-1)m+1} &= \lim_{n\to\infty} \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1} \\
&= \lim_{n\to\infty} \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} \\
&= \frac{1}{\varrho}
\end{aligned}$$

for each $s \in \Big\{1, 2, \dots, \frac{k+1}{2}\Big\}$ and $\widehat{s} \in \Big\{2, 3, \dots, \frac{k+1}{2}\Big\}$, we conclude that

$$\frac{1}{\varrho} \leq \prod_{i=1}^{n-1} a_{(2^i-1)m+1}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}, \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} < 1,$$

for each $n \geq 2$, $s \in \Big\{1, 2, \dots, \frac{k+1}{2}\Big\}$ and $\widehat{s} \in \Big\{2, 3, \dots, \frac{k+1}{2}\Big\}$, and so

$$1 \leq \varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1}, \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}, \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} < \varrho$$

for each $n \geq 2$, $s \in \left\{1, 2, \dots, \frac{k+1}{2}\right\}$ and $\widehat{s} \in \left\{2, 3, \dots, \frac{k+1}{2}\right\}$. In virtue of this fact and making use of (28), it follows that for all $x, y \in B$ and $n \geq 2$,

$$\begin{aligned}\|T^n(x) - T^n(y)\|_p \leq \max\Big\{&\varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1}, \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1}, \\ &\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1}, 1 : s = 1, 2, \dots, \frac{k+1}{2}; \\ &\widehat{s} = 2, 3, \dots, \frac{k+1}{2}\Big\}\|x - y\|_p \\ = \|x - y\|_p + \max\Big\{&\varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1} - 1, \\ &\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1} - 1, \\ &\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} - 1 : s = 1, 2, \dots, \frac{k+1}{2}; \\ &\widehat{s} = 2, 3, \dots, \frac{k+1}{2}\Big\}\|x - y\|_p.\end{aligned} \tag{29}$$

Taking $\gamma_1 = \varrho - 1$ and

$$\begin{aligned}\gamma_n = \max\Big\{&\varrho \prod_{i=1}^{n-1} a_{(2^i-1)m+1} - 1, \varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2s-1)+1} - 1, \\ &\varrho \prod_{i=1}^{n} a_{(2^{i-1}-1)m+2^{i-1}(2\widehat{s}-2)+1} - 1 : s = 1, 2, \dots, \frac{k+1}{2}; \widehat{s} = 2, 3, \dots, \frac{k+1}{2}\Big\}\end{aligned}$$

for each $n \geq 2$, we have $\gamma_n \to 0$ as $n \to \infty$. Employing (26) and (29), for all $x, y \in B$ and $n \in \mathbb{N}$, we get

$$\|T^n(x) - T^n(y)\|_p \leq \|x - y\|_p + \gamma_n\|x - y\|_p = (1 + \gamma_n)\|x - y\|_p,$$

which means that $T$ is an asymptotically nonexpansive mapping.

In recent years, many efforts have also been made to present further interesting generalizations of nonexpansive mappings and asymptotically nonexpansive mappings. In this direction, with the goal of presenting a unifying framework for generalized nonexpansive mappings appeared in the literature and verifying a general convergence theorem applicable to all these classes of nonlinear mappings, the concept of total uniformly $L$-Lipschitzian mapping was initially introduced by Kiziltunc and Purtas [44] as an extension of total asymptotically nonexpansive mapping as follows.

**Definition 4.6.** [44] *A nonlinear mapping $T : E \to E$ is said to be total uniformly L-Lipschitzian (or $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian) if, there exist a constant $L > 0$, nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq L[\|x - y\| + a_n\phi(\|x - y\|) + b_n], \quad \forall x, y \in E.$$

It should be remarked that for given nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, an $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive mapping is $(\{a_n\}, \{b_n\}, \phi)$-total uniformly $L$-Lipschitzian with $L = 1$, but the converse is not true in general. The following example shows that the class of total uniformly $L$-Lipschitzian mappings is more general than the class of total asymptotically nonexpansive mappings.

**Example 4.7.** Let $E = \mathbb{R}$ endowed with the Euclidean norm $\|.\| = |.|$ and let the self-mapping $T$ of $E$ be defined by

$$T(x) = \begin{cases} \frac{1}{\gamma}, & \text{if } x \in [0,\alpha) \cup (\alpha,\beta], \\ \gamma, & \text{if } x = \alpha, \\ 0, & \text{if } x \in (-\infty,0) \cup (\beta,+\infty), \end{cases}$$

where $\alpha > 0$ and $\frac{\alpha+\sqrt{\alpha^2+4}}{2} < \gamma \leq \beta$ are arbitrary real constants such that $\alpha\gamma > 1$. Since the mapping $T$ is discontinuous at the points $x = 0, \alpha, \beta$, it follows that $T$ is not Lipschitzian and so it is not an asymptotically nonexpansive mapping. Take $a_n = \frac{\delta}{n}$ and $b_n = \frac{\alpha}{\sigma^n}$ for each $n \in \mathbb{N}$, where $\delta > 0$ and $\sigma > 1$ are arbitrary constants. Let us now define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \lambda t^k$ for all $t \in \mathbb{R}^+$, where $k \in \mathbb{N}$ and $\lambda \in \big(0, \frac{\sigma^k(\gamma^2-\alpha\gamma-1)}{\alpha^k\beta\delta(\sigma-1)^k}\big)$ are arbitrary constants. Taking $x = \alpha$ and $y = \frac{\alpha}{\sigma}$, we have $T(x) = \gamma$ and $T(y) = \frac{1}{\gamma}$. The fact that $0 < \lambda < \frac{\sigma^k(\gamma^2-\alpha\gamma-1)}{\alpha^k\beta\delta(\sigma-1)^k}$ implies that

$$\begin{aligned} |T(x) - T(y)| = \gamma - \frac{1}{\gamma} &> \alpha + \frac{\delta\lambda(\sigma-1)^k\alpha^k}{\sigma^k} \\ &= \frac{(\sigma-1)\alpha}{\sigma} + \frac{\delta\lambda(\sigma-1)^k\alpha^k}{\sigma^k} + \frac{\alpha}{\sigma} \\ &= |x-y| + \delta\lambda|x-y|^k + \frac{\alpha}{\sigma} \\ &= |x-y| + a_1\phi(|x-y|) + b_1, \end{aligned}$$

from which we conclude that $T$ is not a $(\{\frac{\delta}{n}\}, \{\frac{\alpha}{\sigma^n}\}, \phi)$-total asymptotically nonexpansive mapping. However, for all $x, y \in E$, yields

$$\begin{aligned} |T(x) - T(y)| \leq \gamma &\leq \frac{\sigma\gamma}{\alpha}(|x-y| + \delta\lambda|x-y|^k + \frac{\alpha}{\sigma}) \\ &= \frac{\sigma\gamma}{\alpha}(|x-y| + a_1\phi(|x-y|) + b_1) \end{aligned} \tag{30}$$

and for all $n \geq 2$,

$$\begin{aligned} |T^n(x) - T^n(y)| &< \frac{\sigma\gamma}{\alpha}(|x-y| + \frac{\delta\lambda}{n}|x-y|^k + \frac{\alpha}{\sigma^n}) \\ &= \frac{\sigma\gamma}{\alpha}(|x-y| + a_n\phi(|x-y|) + b_n), \end{aligned} \tag{31}$$

because of $T^n(z) = \frac{1}{\gamma}$ for all $z \in E$ and $n \geq 2$.

Therefore, making use of (30) and (31), it follows that $T$ is a $(\{\frac{\delta}{n}\}, \{\frac{\alpha}{\sigma^n}\}, \phi)$-total uniformly $\frac{\sigma\gamma}{\alpha}$-Lipschitzian mapping.

**Lemma 4.8.** *Suppose that, for each $i \in \{1,2\}$, $E_i$ is a real Banach space with a norm $\|.\|_i$, and let $S_i : E_i \to E_i$ be an $\big(\{a_{n,i}\}_{n=1}^\infty, \{b_{n,i}\}_{n=1}^\infty, \phi_i\big)$-total uniformly $L_i$-Lipschitzian mapping. Assume further that $Q$ and $\phi$ are self-mappings of $E_1 \times E_2$ and $\mathbb{R}^+$, respectively, defined by*

$$Q(x_1,x_2) = (S_1x_1, S_2x_2), \quad \forall (x_1,x_2) \in E_1 \times E_2 \tag{32}$$

*and*

$$\phi(t) = \max\{\phi_1(t), \phi_2(t)\}, \quad \forall t \in \mathbb{R}^+. \tag{33}$$

*Then $Q$ is an $\big(\{a_{n,1} + a_{n,2}\}_{n=1}^\infty, \{b_{n,1} + b_{n,2}\}_{n=1}^\infty, \phi\big)$-total uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping.*

*Proof.* Taking into account that for each $i \in \{1,2\}$, $S_i$ is an $\big(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i\big)$-total uniformly $L$-Lipschitzian mapping and $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, for all $(x_1, x_2), (y_1, y_2) \in E_1 \times E_2$ and $n \in \mathbb{N}$, yields

$$\begin{aligned}
\|Q^n(x_1,x_2) - Q^n(y_1,y_2)\|_* &= \|(S_1^n x_1, S_2^n x_2) - (S_1^n y_1, S_2^n y_2)\|_* \\
&= \|(S_1^n x_1 - S_1^n y_1, S_2^n x_2 - S_2^n y_2)\|_* \\
&= \|S_1^n x_1 - S_1^n y_1\|_1 + \|S_2^n x_2 - S_2^n y_2\|_2 \\
&\leq L_1(\|x_1 - y_1\|_1 + a_{n,1}\phi_1(\|x_1 - y_1\|_1) + b_{n,1}) \\
&\quad + L_2(\|x_2 - y_2\|_2 + a_{n,2}\phi_2(\|x_2 - y_2\|_2) + b_{n,2}) \\
&\leq \max\{L_1, L_2\}\Big(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2 \\
&\quad + a_{n,1}\phi_1(\|x_1 - y_1\|_1) + a_{n,2}\phi_2(\|x_2 - y_2\|_2) + b_{n,1} + b_{n,2}\Big) \\
&\leq \max\{L_1, L_2\}\Big(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2 \\
&\quad + a_{n,1}\phi_1(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2) + a_{n,2}\phi_2(\|x_1 - y_1\|_1 \\
&\quad + \|x_2 - y_2\|_2) + b_{n,1} + b_{n,2}\Big) \\
&\leq \max\{L_1, L_2\}\Big(\|(x_1,x_2) - (y_1,y_2)\|_* \\
&\quad + (a_{n,1} + a_{n,2})\phi(\|(x_1,x_2) - (y_1,y_2)\|_*) + b_{n,1} + b_{n,2}\Big),
\end{aligned}$$

where $\|.\|_*$ is a norm defined on $E_1 \times E_2$ as in (18). This fact implies that $Q$ is an $\big(\{a_{n,1}+a_{n,2}\}_{n=1}^{\infty}, \{b_{n,1}+b_{n,2}\}_{n=1}^{\infty}, \phi\big)$-total uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping. The proof is completed. □

Let for each $i \in \{1,2\}$, $E_i$ be a real $q_i$-uniformly smooth Banach space with $q_i > 1$ and the norm $\|.\|_i$, and $S_i : E_i \to E_i$ be an $(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i)$-total uniformly $L_i$-Lipschitzian mapping. Suppose further that $Q$ is a self-mapping of $E_1 \times E_2$ defined as (32). Denote by $\mathrm{Fix}(S_i)$ ($i = 1,2$) and $\mathrm{Fix}(Q)$ the sets of all the fixed points of $S_i$ ($i = 1,2$) and $Q$, respectively. At the same time, denote by $\Omega_{\mathrm{SGNVLI}}$ the set of all the solutions of the SGNVLI (1) where for each $i \in \{1,2\}$, the nonlinear mapping $P_i$ is strictly $\eta_i$-accretive with $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. Making use of (32), we conclude that for any $(x_1,x_2) \in E_1 \times E_2$, $(x_1,x_2) \in \mathrm{Fix}(Q)$ if and only if $x_i \in \mathrm{Fix}(S_i)$ for each $i \in \{1,2\}$, that is, $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1,S_2) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2)$. If $(\widehat{x},\widehat{y}) \in \mathrm{Fix}(Q) \cap \Omega_{\mathrm{SGNVLI}}$, then utilizing Lemma 3.2, it can be easily seen that for each $n \in \mathbb{N}$,

$$\begin{cases}
\widehat{x} = S_1^n \widehat{x} = \widehat{x} - g_1(\widehat{x}) + R_{M(.,\widehat{x}),\lambda}^{P_1,\eta_1}[P_1(g_1(\widehat{x})) - \lambda F(\widehat{x}, \widehat{y} - f_2(\widehat{y}))] \\
\quad = S_1^n\big(\widehat{x} - g_1(\widehat{x}) + R_{M(.,\widehat{x}),\lambda}^{P_1,\eta_1}[P_1(g_1(\widehat{x})) - \lambda F(\widehat{x}, \widehat{y} - f_2(\widehat{y}))]\big), \\
\widehat{y} = S_2^n \widehat{y} = \widehat{y} - g_2(\widehat{y}) + R_{N(.,\widehat{y}),\rho}^{P_2,\eta_2}[P_2(g_2(\widehat{y})) - \rho G(\widehat{x} - f_1(\widehat{x}), \widehat{y})] \\
\quad = S_2^n\big(\widehat{y} - g_2(\widehat{y}) + R_{N(.,\widehat{y}),\rho}^{P_2,\eta_2}[P_2(g_2(\widehat{y})) - \rho G(\widehat{x} - f_1(\widehat{x}), \widehat{y})]\big).
\end{cases} \tag{34}$$

The fixed point formulation (34) enables us to construct the following iterative algorithm for finding a common element of the two sets of $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1,S_2)$ and $\Omega_{\mathrm{SGNVLI}}$.

**Algorithm 4.9.** *Let $E_i, f_i, g_i, F, G$ ($i = 1,2$) be the same as in the* SGNVLI (1)*. Assume that for each $n \geq 0$ and $i \in \{1,2\}$, $\eta_{n,i} : E_i \times E_i \to E_i$ and $P_{n,i} : E_i \to E_i$ are nonlinear mappings such that for each $n \geq 0$ and $i \in \{1,2\}$, $P_{n,i}$ is a strictly $\eta_{n,i}$-accretive mapping with* $\mathrm{dom}(P_{n,i}) \cap g_i(E_i) \neq \emptyset$*. Let for all $n \geq 0$, $M_n : E_1 \times E_1 \to 2^{E_1}$ and $N_n : E_2 \times E_2 \to 2^{E_2}$ be any multi-valued nonlinear mappings such that for all $z \in E_1$ and $n \geq 0$, $M_n(.,z) : E_1 \to 2^{E_1}$ is a $P_{n,1}$-$\eta_{n,1}$-accretive mapping with $g_1(E_1) \cap \mathrm{dom}\, M_n(.,z) \neq \emptyset$, and for all $t \in E_2$ and $n \geq 0$, $N_n(.,t) : E_2 \to 2^{E_2}$ is a $P_{n,2}$-$\eta_{n,2}$-accretive mapping with $g_2(E_2) \cap \mathrm{dom}\, N_n(.,t) \neq \emptyset$. Suppose further that for each $i \in \{1,2\}$, $S_i : E_i \to E_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total uniformly $L_i$-Lipschitzian mapping. For any given $(x_0, y_0) \in E_1 \times E_2$, define the* *iterative sequence* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *in* $E_1 \times E_2$ *in the following way:*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_1^n\Big(x_n - g_1(x_n) + R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) \\ \qquad -\lambda_n F(x_n, y_n - f_2(y_n))]\Big) + (1-\alpha_n)e_n + l_n, \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)S_2^n\Big(y_n - g_2(y_n) + R^{P_{n,2},\eta_{n,2}}_{N_n(.,y_n),\rho_n}[P_{n,2}(g_2(y_n)) \\ \qquad -\rho_n G(x_n - f_1(x_n), y_n)]\Big) + (1-\alpha_n)\hat{e}_n + \hat{l}_n, \end{cases} \tag{35}$$

*where* $n = 0, 1, 2, \ldots;$ $\lambda_n, \rho_n > 0$ *are constants,* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0, 1)$ *such that* $\sum\limits_{n=0}^{\infty}(1-\alpha_n) = \infty$, *and* $\{e_n\}_{n=0}^{\infty}$, $\{l_n\}_{n=0}^{\infty}$ *and* $\{\hat{e}_n\}_{n=0}^{\infty}$, $\{\hat{l}_n\}_{n=0}^{\infty}$ *are four sequences in* $E_1$ *and* $E_2$, *respectively, to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:*

$$\begin{cases} e_n = e_n' + e_n'', \hat{e}_n = \hat{e}_n' + \hat{e}_n''; \\ \lim\limits_{n\to\infty} \|(e_n', \hat{e}_n')\|_* = 0; \\ \sum\limits_{n=0}^{\infty} \|(e_n'', \hat{e}_n'')\|_* < \infty, \ \sum\limits_{n=0}^{\infty} \|(l_n, \hat{l}_n)\|_* < \infty. \end{cases} \tag{36}$$

*Let* $\{(u_n, v_n)\}_{n=0}^{\infty}$ *be any sequence in* $E_1 \times E_2$ *and define* $\{\epsilon_n\}_{n=0}^{\infty}$ *by*

$$\begin{cases} \epsilon_n = \|(u_{n+1} - v_{n+1}) - (L_n, D_n)\|_*, \\ L_n = \alpha_n u_n + (1-\alpha_n)S_1^n\Big(u_n - g_1(u_n) + R^{P_{n,1},\eta_{n,1}}_{M_n(.,u_n),\lambda_n}[P_{n,1}(g_1(u_n)) \\ \qquad -\lambda_n F(u_n, v_n - f_2(v_n))]\Big) + (1-\alpha_n)e_n + l_n, \\ D_n = \alpha_n v_n + (1-\alpha_n)S_2^n\Big(v_n - g_2(v_n) + R^{P_{n,2},\eta_{n,2}}_{N_n(.,v_n),\rho_n}[P_{n,2}(g_2(v_n)) \\ \qquad -\rho_n G(u_n - f_1(u_n), v_n)]\Big) + (1-\alpha_n)\hat{e}_n + \hat{l}_n. \end{cases} \tag{37}$$

In the case where for each $i \in \{1, 2\}$, $S_i \equiv I_i$, the identity mapping on $E_i$, then Algorithm 4.9 reduces to the following algorithm.

**Algorithm 4.10.** *Assume that* $E_i, P_{n,i}, \eta_{n,i}, f_i, g_i, M_n, N_n, F, G$ ($i = 1, 2$ *and* $n \geq 0$) *are the same as in Algorithm 4.9. For any given* $(x_0, y_0) \times E_1 \times E_2$, *compute the iterative sequence* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *in* $E_1 \times E_2$ *by the iterative schemes*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)\Big(x_n - g_1(x_n) + R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) \\ \qquad -\lambda_n F(x_n, y_n - f_2(y_n))]\Big) + (1-\alpha_n)e_n + l_n, \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)\Big(y_n - g_2(y_n) + R^{P_{n,2},\eta_{n,2}}_{N_n(.,y_n),\rho_n}[P_{n,2}(g_2(y_n)) \\ \qquad -\rho_n G(x_n - f_1(x_n), y_n)]\Big) + (1-\alpha_n)\hat{e}_n + \hat{l}_n, \end{cases}$$

*where* $n = 0, 1, 2 \ldots;$ *and* $\lambda_n, \rho_n, \{\alpha_n\}_{n=0}^{\infty}, \{e_n\}_{n=0}^{\infty}, \{l_n\}_{n=0}^{\infty}, \{\hat{e}_n\}_{n=0}^{\infty}, \{\hat{l}_n\}_{n=0}^{\infty}$ *are the same as in Algorithm 4.9. Let* $\{(u_n, v_n)\}_{n=0}^{\infty}$ *be any sequence in* $E_1 \times E_2$ *and define* $\{\widehat{\epsilon}_n\}_{n=0}^{\infty}$ *by*

$$\begin{cases} \widehat{\epsilon}_n = \|(u_{n+1} - v_{n+1}) - (\widehat{L}_n, \widehat{D}_n)\|_*, \\ \widehat{L}_n = \alpha_n u_n + (1-\alpha_n)\Big(u_n - g_1(u_n) + R^{P_{n,1},\eta_{n,1}}_{M_n(.,u_n),\lambda_n}[P_{n,1}(g_1(u_n)) \\ \qquad -\lambda_n F(u_n, v_n - f_2(v_n))]\Big) + (1-\alpha_n)e_n + l_n, \\ \widehat{D}_n = \alpha_n v_n + (1-\alpha_n)\Big(v_n - g_2(v_n) + R^{P_{n,2},\eta_{n,2}}_{N_n(.,v_n),\rho_n}[P_{n,2}(g_2(v_n)) \\ \qquad -\rho_n G(u_n - f_1(u_n), v_n)]\Big) + (1-\alpha_n)\hat{e}_n + \hat{l}_n. \end{cases} \tag{38}$$

## 5. An Application

In this section, as an application of the notion of graph convergence for $P$-$\eta$-accretive mapping, the strong convergence of the iterative sequence generated by Algorithm 4.9 to a common element of the two sets $\Omega_{\text{SGNVLI}}$ and $\text{Fix}(Q)$, where $Q$ is a self-mapping of $E_1 \times E_2$ defined by (32), under some suitable conditions is proved. In the meanwhile, the stability of the iterative sequence generated by Algorithm 4.9 is verified. Before dealing with the convergence analysis of our proposed iterative algorithm, we need to recall the following notion and lemma.

**Definition 5.1.** *For $i = 1, 2$, let $E_i$ be a real Banach space and $T$ be a self-mapping of $E_1 \times E_2$. Suppose that $(x_0, y_0) \in E_1 \times E_2$ and $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ defines an iterative procedure which yields a sequence of points $\{(x_n, y_n)\}_{n=0}^{\infty}$ in $E_1 \times E_2$. Assume that $\text{Fix}(T) = \{(x, y) \in E_1 \times E_2 : (x, y) = T(x, y)\} \neq \emptyset$ and $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to some $(x^*, y^*) \in \text{Fix}(T)$. Further, let $\{(z_n, w_n)\}_{n=0}^{\infty}$ be an arbitrary sequence in $E_1 \times E_2$ and $\epsilon_n = \|(z_{n+1}, w_{n+1}) - f(T, z_n, w_n)\|$ for each $n \geq 0$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty}(z_n, w_n) = (x^*, y^*)$, then the iterative procedure defined by $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ is said to be $T$-stable or stable with respect to $T$.*

**Remark 5.2.** Some stability results of the iteration procedures for variational inequalities and variational inclusions have been established by various authors, see, for example, [1, 3, 9, 12, 34, 35, 38, 39, 41, 49, 50].

**Lemma 5.3.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions: there exists a natural number $n_0$ such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

*where $t_n \in [0, 1]$, $\sum\limits_{n=0}^{\infty} t_n = \infty$, $\lim\limits_{n\to\infty} b_n = 0$ and $\sum\limits_{n=0}^{\infty} c_n < \infty$.*

*Then $\lim\limits_{n\to\infty} a_n = 0$.*

*Proof.* The proof follows directly from Lemma 2 in [48]. □

**Theorem 5.4.** *For $i = 1, 2$, let $E_i, \eta_i, P_i, f_i, g_i, F, G, M$ and $N$ $(i = 1, 2)$ be the same as in Theorem 3.5 and let all the conditions of Theorem 3.5 hold. Suppose that $\eta_{n,i}, P_{n,i}, M_n$ and $N_n$ $(n \geq 0$ and $i = 1, 2)$ are the same as in Algorithm 4.9. Assume that for each $i \in \{1, 2\}$, $S_i : E_i \to E_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total uniformly $L_i$-Lipschitzian mapping and $Q$ is a self-mpping of $E_1 \times E_2$ defined by (32) such that $\text{Fix}(Q) \cap \Omega_{\text{SGNVLI}} \neq \emptyset$. Assume further that for each $n \geq 0$ and $i \in \{1, 2\}$,*

(i) *$\eta_{n,i}$ is $\tau_{n,i}$-Lipschitz continuous;*
(ii) *$P_{n,i}$ is $\gamma_{n,i}$-strongly $\eta_{n,i}$-accretive and $\delta_{n,i}$-Lipschitz continuous;*
(iii) *$\lim\limits_{n\to\infty} P_{n,i}(x_i) = P_i(x_i)$ for each $x_i \in E_i$, $M_n(., z) \xrightarrow{G} M(., z)$ and $N_n(., t) \xrightarrow{G} N(., t)$ for any $(z, t) \in E_1 \times E_2$;*
(iv) *there exist constants $\mu_{n,i} > 0$ such that*

$$\|R^{P_{n,1},\eta_{n,1}}_{M_n(.,u),\lambda_n}(w) - R^{P_{n,1},\eta_{n,1}}_{M_n(.,v),\lambda_n}(w)\| \leq \mu_{n,1}\|u - v\|, \quad \forall u, v, w \in E_1, \tag{39}$$

$$\|R^{P_{n,2},\eta_{n,2}}_{N_n(.,u),\rho_n}(w) - R^{P_{n,2},\eta_{n,2}}_{N_n(.,v),\rho_n}(w)\| \leq \mu_{n,2}\|u - v\|, \quad \forall u, v, w \in E_2; \tag{40}$$

(v) *$\gamma_{n,i} \to \gamma_i$, $\tau_{n,i} \to \tau_i$ and $\delta_{n,i} \to \delta_i$ as $n \to \infty$;*
(vi) *there exist constants $\mu_i, \lambda, \rho > 0$ $(i = 1, 2)$ satisfying (5)–(8) such that $\mu_{n,i} \to \mu_i$, $\lambda_n \to \lambda$ and $\rho_n \to \rho$ as $n \to \infty$;*
(vii) *$L_i(k + 1) < 2$ where $k$ is the same as in (20).*

*Then,*

(1) *the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.9 converges strongly to the only element $(x, y)$ of $\text{Fix}(Q) \cap \Omega_{\text{SGNVLI}}$.*
(2) *If, in addition, there exists a constant $\alpha > 0$ such that $\alpha + \alpha_n \leq 1$ for each $n \geq 0$, then $\lim\limits_{n\to\infty}(u_n, v_n) = (x, y)$ if and only if $\lim\limits_{n\to\infty} \epsilon_n = 0$, where $\{(u_n, v_n)\}_{n=0}^{\infty}$ is any sequence in $E_1 \times E_2$ satisfying (38).*

*Proof.* Since all the conditions of Theorem 3.5 hold, it ensures the existence of a unique solution $(x, y) \in E_1 \times E_2$ for the SGNVLI (1). According to Lemma 3.2, we have

$$\begin{cases} x = x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))], \\ y = y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho G(x - f_1(x), y)]. \end{cases} \tag{41}$$

Taking into account that $\Omega_{\mathrm{SGNVLI}}$ is a singleton set and $\mathrm{Fix}(Q) \cap \Omega_{\mathrm{SGNVLI}} \neq \emptyset$, it follows that $x \in \mathrm{Fix}(S_1)$ and $y \in \mathrm{Fix}(S_2)$. Thereby, in the light of this fact and making use of (41), for each $n \geq 0$, we can write

$$\begin{cases} x = \alpha_n x + (1-\alpha_n) S_1^n\Big(x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\Big), \\ y = \alpha_n y + (1-\alpha_n) S_2^n\Big(y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho G(x - f_1(x), y)]\Big). \end{cases} \tag{42}$$

Using (35), (39), (42), Lemma 2.22, we yield

$$\begin{aligned}
\|x_{n+1} - x\|_1 \leq\ & \alpha_n \|x_n - x\|_1 + (1-\alpha_n)\|S_1^n\Big(x_n - g_1(x_n) \\
& + R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))]\Big) \\
& - S_1^n\Big(x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\Big)\|_1 \\
& + (1-\alpha_n)\|e_n\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n \|x_n - x\|_1 + (1-\alpha_n) L_1\Big(\|x_n - g_1(x_n) \\
& + R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))] \\
& - (x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\Big)\|_1 \\
& + a_{n,1}\phi_1\Big(\|x_n - g_1(x_n) + R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))] \\
& - (x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\Big)\|_1\Big) \\
& + b_{n,1}\Big) + (1-\alpha_n)\|e_n\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n \|x_n - x\|_1 + (1-\alpha_n) L_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
& + \|R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))] \\
& - R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
& + a_{n,1}\phi_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
& + \|R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))] \\
& - R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1\Big) + b_{n,1}\Big) \\
& + (1-\alpha_n)(\|e'_n\|_1 + \|e''_n\|_1) + \|l_n\|_1 \\
\leq\ & \alpha_n \|x_n - x\| + (1-\alpha_n) L_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
& + \|R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))] \\
& - R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
& + \|R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] - R^{P_{n,1},\eta_{n,1}}_{M_n(.,x),\lambda_n}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
& + \|\Psi(n)\|_1 + a_{n,1}\phi_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
& + \|R^{P_{n,1},\eta_{n,1}}_{M_n(.,x_n),\lambda_n}[P_{n,1}(g_1(x_n)) - \lambda_n F(x_n, y_n - f_2(y_n))]
\end{aligned}$$

$$
\begin{aligned}
&- R_{M_n(.,x_n),\lambda_n}^{P_{n,1},\eta_{n,1}}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
&+ \|R_{M_n(.,x_n),\lambda_n}^{P_{n,1},\eta_{n,1}}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] - R_{M_n(.,x),\lambda_n}^{P_{n,1},\eta_{n,1}}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))]\|_1 \\
&+ \|\Psi(n)\|_1\Big) + b_{n,1}\Big) + (1-\alpha_n)\|e_n'\|_1 + \|e_n''\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n\|x_n - x\|_1 + (1-\alpha_n)L_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\|P_{n,1}(g_1(x_n)) - P_1(g_1(x)) - (\lambda_n F(x_n, y_n - f_2(y_n)) \\
&- \lambda F(x, y - f_2(y)))\|_1 + \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1 + a_{n,1}\phi_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\|P_{n,1}(g_1(x_n)) - P_1(g_1(x)) - (\lambda_n F(x_n, y_n - f_2(y_n)) - \lambda F(x, y - f_2(y)))\|_1 \\
&+ \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1\Big) + b_{n,1}\Big) + (1-\alpha_n)\|e_n'\|_1 + \|e_n''\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n\|x_n - x\|_1 + (1-\alpha_n)L_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\|P_{n,1}(g_1(x_n)) - P_{n,1}(g_1(x)) - \lambda_n\Big(F(x_n, y_n - f_2(y_n)) - F(x, y - f_2(y))\Big)\|_1 \\
&+ \|P_{n,1}(g_1(x)) - P_1(g_1(x))\|_1 + |\lambda_n - \lambda|\|F(x, y - f_2(y))\|_1\Big) \\
&+ \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1 + a_{n,1}\phi_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\|P_{n,1}(g_1(x_n)) - P_{n,1}(g_1(x)) - \lambda_n\Big(F(x_n, y_n - f_2(y_n)) - F(x, y - f_2(y))\Big)\|_1 \\
&+ \|P_{n,1}(g_1(x)) - P_1(g_1(x))\|_1 + |\lambda_n - \lambda|\|F(x, y - f_2(y))\|_1\Big) \\
&+ \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1\Big) + b_{n,1}\Big) + (1-\alpha_n)\|e_n'\|_1 + \|e_n''\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n\|x_n - x\|_1 + (1-\alpha_n)L_1\Big(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\Big(\|P_{n,1}(g_1(x_n)) - P_{n,1}(g_1(x)) - \lambda_n\Big(F(x_n, y_n - f_2(y_n)) - F(x, y_n - f_2(y_n))\Big)\|_1 \\
&+ \lambda_n\|F(x, y_n - f_2(y_n)) - F(x, y - f_2(y))\|_1 \\
&+ \|P_{n,1}(g_1(x)) - P_1(g_1(x))\|_1 + |\lambda_n - \lambda|\|F(x, y - f_2(y))\|_1\Big) \\
&+ \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1 + a_{n,1}\phi_1(\|x_n - x - (g_1(x_n) - g_1(x))\|_1 \\
&+ \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\Big(\|P_{n,1}(g_1(x_n)) - P_{n,1}(g_1(x)) - \lambda_n\Big(F(x_n, y_n - f_2(y_n)) - F(x, y_n - f_2(y_n))\|_1 \\
&+ \lambda_n\|F(x, y_n - f_2(y_n)) - F(x, y - f_2(y))\|_1 + \|P_{n,1}(g_1(x)) - P_1(g_1(x))\|_1 \\
&+ |\lambda_n - \lambda\|F(x, y - f_2(y))\|_1\Big) + \mu_{n,1}\|x_n - x\|_1 + \|\Psi(n)\|_1\Big) \\
&+ b_{n,1}\Big) + (1-\alpha_n)\|e_n'\|_1 + \|e_n''\|_1 + \|l_n\|_1 \\
\leq\ & \alpha_n\|x_n - x\|_1 + (1-\alpha_n)L_1\Big(\varphi_1(n)\|x_n - x\|_1 + \vartheta_1(n)\|y_n - y\|_2 \\
&+ \Delta(n) + \|\Psi(n)\|_1 + a_{n,1}\phi_1\Big(\varphi_1(n)\|x_n - x\|_1 + \vartheta_1(n)\|y_n - y\|_2 \\
&+ \Delta(n) + \|\Psi(n)\|_1\Big) + b_{n,1}\Big) + (1-\alpha_n)\|e_n'\|_1 + \|e_n''\|_1 + \|l_n\|_1,
\end{aligned} \tag{43}
$$

where for each $n \geq 0$,

$$
\begin{aligned}
\varphi_1(n) &= \mu_{n,1} + \sqrt[q_1]{1 - q_1\nu_1 + (c_{q_1} + q_1\sigma_1)\pi_1^{q_1}} + \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}} \sqrt[q_1]{\delta_{n,1}^{q_1}\pi_1^{q_1} - q_1\lambda_n r_1 + \lambda_n^{q_1} c_{q_1} s_1^{q_1}},\\
\vartheta_1(n) &= \frac{\lambda_n \xi_1 \tau_{n,1}^{q_1-1}}{\gamma_{n,1}} \sqrt[q_2]{1 - q_2\varsigma_2 + (c_{q_2} + q_2\zeta_2)\theta_2^{q_2}},\\
\Delta(n) &= \frac{\tau_{n,1}^{q_1-1}}{\gamma_{n,1}}\Big(\|P_{n,1}(g_1(x)) - P_1(g_1(x))\|_1 + |\lambda_n - \lambda|\|F(x, y - f_2(y))\|_1\Big),\\
\Psi(n) &= R^{P_{n,1},\eta_{n,1}}_{M_n(.,x),\lambda_n}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))] - R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda F(x, y - f_2(y))].
\end{aligned}
$$

In a similar way to that of proof of (43), employing (35), (40), (42), Lemma 2.22 and the assumptions, one can show that

$$
\begin{aligned}
\|y_{n+1} - y\|_2 \leq \alpha_n\|y_n - y\|_2 &+ (1 - \alpha_n)L_2\Big(\varphi_2(n)\|x_n - x\|_1 + \vartheta_2(n)\|y_n - y\|_2\\
&+ \Upsilon(n) + \|\Phi(n)\|_2 + a_{n,2}\phi_2\Big(\varphi_2(n)\|x_n - x\|_1 + \vartheta_2(n)\|y_n - y\|_2\\
&+ \Upsilon(n) + \|\Phi(n)\|_2\Big) + b_{n,2}\Big) + (1 - \alpha_n)\|\hat{e}'_n\|_2 + \|\hat{e}''_n\|_2 + \|\hat{l}_n\|_2,
\end{aligned} \tag{44}
$$

where for each $n \geq 0$,

$$
\begin{aligned}
\vartheta_2(n) &= \mu_{n,2} + \sqrt[q_2]{1 - q_2\nu_2 + (c_{q_2} + q_2\sigma_2)\pi_2^{q_2}} + \frac{\tau_{n,2}^{q_2-1}}{\gamma_{n,2}} \sqrt[q_2]{\delta_{n,2}^{q_2}\pi_2^{q_2} - q_2\rho_n r_2 + \rho_n^{q_2} c_{q_2} s_2^{q_2}},\\
\varphi_2(n) &= \frac{\rho_n \xi_2 \tau_{n,2}^{q_2-1}}{\gamma_{n,2}} \sqrt[q_1]{1 - q_1\varsigma_1 + (c_{q_1} + q_1\varsigma_1)\theta_1^{q_1}},\\
\Upsilon(n) &= \frac{\tau_{n,2}^{q_2-1}}{\gamma_{n,2}}\Big(\|P_{n,2}(g_2(y)) - P_2(g_2(y))\|_2 + |\rho_n - \rho|\|G(x - f_1(x), y)\|_2\Big),\\
\Phi(n) &= R^{P_{n,2},\eta_{n,2}}_{N_n(.,y),\rho_n}[P_2(g_2(y)) - \rho G(x - f_1(x), y)] - R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho G(x - f_1(x), y)].
\end{aligned}
$$

Letting $L = \max\{L_1, L_2\}$ and making use of (43) and (44), we derive that

$$
\begin{aligned}
\|(x_{n+1}, y_{n+1}) - (x, y)\|_* &= \|x_{n+1} - x\|_1 + \|y_{n+1} - y\|_2\\
&\leq \alpha_n(\|x_n - x\|_1 + \|y_n - y\|_2) + (1 - \alpha_n)L\Big((\varphi_1(n) + \varphi_2(n))\|x_n - x\|_1\\
&\quad + (\vartheta_1(n) + \vartheta_2(n))\|y_n - y\|_2 + \Delta(n) + \Upsilon(n) + \|\Psi(n)\|_1\\
&\quad + \|\Phi(n)\|_2 + a_{n,1}\phi_1\Big(\varphi_1(n)\|x_n - x\|_1 + \vartheta_1(n)\|y_n - y\|_2\\
&\quad + \Delta(n) + \|\Psi(n)\|_1\Big) + a_{n,2}\phi_2\Big(\varphi_2(n)\|x_n - x\|_1\\
&\quad + \vartheta_2(n)\|y_n - y\|_2 + \Upsilon(n) + \|\Phi(n)\|_2\Big) + b_{n,1} + b_{n,2}\Big)\\
&\quad + (1 - \alpha_n)(\|e'_n\|_1 + \|\hat{e}'_n\|_2) + \|e''_n\|_1 + \|\hat{e}''_n\|_2 + \|l_n\|_1 + \|\hat{l}_n\|_2
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n\Big(\|x_n - x\|_1 + \|y_n - y\|_2\Big) + (1-\alpha_n)L\Big(k(n)(\|x_n - x\|_1 \\
&\quad + \|y_n - y\|_2) + \Delta(n) + \Upsilon(n) + \|\Psi(n)\|_1 + \|\Phi(n)\|_2 \\
&\quad + a_{n,1}\phi(k(n)(\|x_n - x\|_1 + \|y_n - y\|_2) + \Delta(n) + \|\Psi(n)\|_1) \\
&\quad + a_{n,2}\phi\Big(k(n)(\|x_n - x\|_1 + \|y_n - y\|_2) + \Upsilon(n) + \|\Phi(n)\|_2\Big) \\
&\quad + b_{n,1} + b_{n,2}\Big) + (1-\alpha_n)(\|e'_n\|_1 + \|\hat{e}'_n\|_2) + \|e''_n\|_1 + \|\hat{e}''_n\|_2 + \|l_n\|_1 + \|\hat{l}_n\|_2 \\
&= \alpha_n\|(x_n, y_n) - (x, y)\|_* + (1-\alpha_n)L\Big(k(n)\|(x_n, y_n) - (x, y)\|_* + \Delta(n) + \Upsilon(n) \\
&\quad + \|(\Psi(n), \Phi(n))\|_* + a_{n,1}\phi\Big(k(n)\|(x_n, y_n) - (x, y)\|_* + \Delta(n) + \|\Psi(n)\|_1\Big) \\
&\quad + a_{n,2}\phi\Big(k(n)\|(x_n, y_n) - (x, y)\|_* + \Upsilon(n) + \|\Phi(n)\|_2\Big) + b_{n,1} + b_{n,2}\Big) \\
&\quad + (1-\alpha_n)\|(e'_n, \hat{e}'_n)\|_* + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_* \\
&= \alpha_n\|(x_n, y_n) - (x, y)\|_* + (1-\alpha_n)Lk(n)\|(x_n, y_n) - (x, y)\|_* \\
&\quad + (1-\alpha_n)L\Big(\Gamma(n) + a_{n,1}\phi\Big(k(n)\|(x_n, y_n) - (x, y)\|_* + \Gamma_1(n)\Big) \\
&\quad + a_{n,2}\phi(k(n)\|(x_n, y_n) - (x, y)\|_* + \Gamma_2(n)\Big) + b_{n,1} + b_{n,2}\Big) \\
&\quad + (1-\alpha_n)\|(e'_n, \hat{e}'_n)\|_* + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_*,
\end{aligned} \tag{45}
$$

where $\phi$ is a self-mapping of $\mathbb{R}^+$ defined by (33), and for each $n \ge 0$,

$$
\begin{aligned}
&k(n) = \{\varphi_1(n) + \varphi_2(n), \vartheta_1(n) + \vartheta_2(n)\}, \\
&\Gamma_1(n) = \Delta(n) + \|\Psi(n)\|_1, \\
&\Gamma_2(n) = \Upsilon(n) + \|\Phi(n)\|_2, \\
&\Gamma(n) = \Gamma_1(n) + \Gamma_2(n) = \Delta(n) + \Upsilon(n) + \|(\Psi(n), \Phi(n))\|_*.
\end{aligned}
$$

Clearly, $k(n) \to k = \max\{\varphi_1 + \varphi_2, \vartheta_1 + \vartheta_2\}$ as $n \to \infty$, where $\varphi_1, \varphi_2, \vartheta_1, \vartheta_2$ are the same as in (16) and (17). Then for $\widehat{k} = \frac{k+1}{2} \in (k, 1)$, there exists $n_0 \ge 1$ such that $k(n) < \widehat{k}$ for all $n \ge n_0$. Accordingly, from (45) it follows that for all $n \ge n_0$,

$$
\begin{aligned}
\|(x_{n+1}, y_{n+1}) - (x, y)\|_* &\le \alpha_n\|(x_n, y_n) - (x, y)\|_* + (1-\alpha_n)L\widehat{k}\|(x_n, y_n) - (x, y)\|_* \\
&\quad + (1-\alpha_n)L\Big(\Gamma(n) + a_{n,1}\phi(\widehat{k}\|(x_n, y_n) - (x, y)\|_* + \Gamma_1(n)\Big) \\
&\quad + a_{n,2}\phi(\hat{k}\|(x_n, y_n) - (x, y)\|_* + \Gamma_2(n)) + b_{n,1} + b_{n,2}\Big) \\
&\quad + (1-\alpha_n)\|(e'_n, \hat{e}'_n)\|_* + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_* \\
&= \Big(1 - (1 - L\widehat{k})(1-\alpha_n)\Big)\|(x_n, y_n) - (x, y)\|_* \\
&\quad + (1 - L\hat{k})(1-\alpha_n)\frac{\Theta(n)}{1 - L\hat{k}} + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_*,
\end{aligned} \tag{46}
$$

where,

$$
\begin{aligned}
\Theta(n) = L\Big(&\Gamma(n) + a_{n,1}\phi(\widehat{k}\|(x_n, y_n) - (x, y)\|_* + \Gamma_1(n)\Big) + a_{n,2}\phi(\widehat{k}\|(x_n, y_n) - (x, y)\|_* \\
&+ \Gamma_2(n)) + b_{n,1} + b_{n,2}\Big) + \|(e'_n, \hat{e}'_n)\|_*.
\end{aligned}
$$

The condition $L_i(k+1) < 2$ ($i = 1, 2$) implies that $L\widehat{k} < 1$. Theorem 4.2 ensures that $\|\Psi(n)\|_1, \|\Phi(n)\|_2 \to 0$ as $n \to \infty$ and so $\|(\Psi(n), \Phi(n))\|_* \to 0$ as $n \to \infty$. Since $\lim\limits_{n\to\infty} P_{n,i}(x_i) = P_i(x_i)$ for each $x_i \in E_i$ and $i \in \{1, 2\}$, $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\lim\limits_{n\to\infty} \rho_n = \rho$, we infer that $\lim\limits_{n\to\infty} \Delta(n) = 0$ and $\lim\limits_{n\to\infty} \Upsilon(n) = 0$ as $n \to \infty$. Consequently, $\Gamma_1(n), \Gamma_2(n), \Gamma(n) \to 0$ as $n \to \infty$. Taking into account that $\lim\limits_{n\to\infty} a_{n,i} = \lim\limits_{n\to\infty} b_{n,i} = 0$ for $i = 1, 2$, thanks to (36) we note that all the conditions of Lemma 5.3 are satisfied. Now, lemma 5.3 and (46) guarantee that

$(x_{n+1}, y_{n+1}) \to (x, y)$ as $n \to \infty$. Therefore, the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.9 converges strongly to the unique solution of the SGNVLI (1), that is, the only element of $\mathrm{Fix}(Q) \cap \Omega_{\mathrm{SGNVLI}}$.

We now prove the conclusion (2). By (38), we obtain

$$\begin{aligned}\|(u_{n+1}, v_{n+1}) - (x, y)\|_* &\leq \|(u_{n+1}, v_{n+1}) - (L_n, D_n)\|_* + \|(L_n, D_n) - (x, y)\|_* \\ &= \epsilon_n + \|L_n - x\|_1 + \|D_n - y\|_2.\end{aligned} \tag{47}$$

By following similar arguments as in the proof of (43) and (44) with suitable modifications, we obtain

$$\begin{aligned}\|L_n - x\|_1 \leq{}& \alpha_n\|u_n - x\|_1 + (1-\alpha_n)L_1\Big(\varphi_1(n)\|u_n - x\|_1 + \vartheta_1(n)\|v_n - y\|_2 \\ &+ \Delta(n) + \|\Psi(n)\|_1 + a_{n,1}\phi_1\Big(\varphi_1(n)\|u_n - u\|_1 + \vartheta_1(n)\|v_n - y\|_2 \\ &+ \Delta(n) + \|\Psi(n)\|_1\Big) + b_{n,1}\Big) + (1-\alpha_n)\|e'_n\|_1 + \|e''_n\|_1 + \|l_n\|_1\end{aligned} \tag{48}$$

and

$$\begin{aligned}\|D_n - y\|_2 \leq{}& \alpha_n\|v_n - y\|_2 + (1-\alpha_n)L_2\Big(\varphi_2(n)\|u_n - x\|_1 + \vartheta_2(n)\|v_n - y\|_2 + \Upsilon(n) \\ &+ \|\Phi(n)\|_2 + a_{n,2}\phi_2(\varphi_2(n)\|u_n - x\|_1 + \vartheta_2(n)\|v_n - y\|_2 \\ &+ \Upsilon(n) + \|\Phi(n)\|\|_2) + b_{n,2}\Big) + (1-\alpha_n)\|\hat{e}'_n\|_2 + \|\hat{e}''_n\|_2 + \|\hat{l}_n\|_2,\end{aligned} \tag{49}$$

where for all $n \geq 0$, $\varphi_1(n), \vartheta_1(n)$ are the same as in (43) and $\varphi_2(n), \vartheta_2(n)$ are the same as in (44). Since $0 < \alpha \leq 1 - \alpha_n$ for all $n \geq 0$, making use of (47)–(49), as the proof of (46), we can conclude that

$$\begin{aligned}\|(u_{n+1}, v_{n+1}) - (x, y)\|_* \leq{}& (1 - (1 - \widehat{Lk})(1-\alpha_n))\|(u_n, v_n) - (x, y)\|_* \\ &+ (1 - \widehat{Lk})(1-\alpha_n)\frac{\Lambda(n)}{1 - \widehat{Lk}} + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_*,\end{aligned} \tag{50}$$

where

$$\begin{aligned}\Lambda(n) = {}& L\Big(\Gamma(n) + a_{n,1}\phi(\widehat{k}\|(u_n, v_n) - (x, y)\|_* + \Gamma_1(n)\Big) + a_{n,2}\phi(\widehat{k}\|(u_n, v_n) - (x, y)\|_* \\ &+ \Gamma_2(n)) + b_{n,1} + b_{n,2}\Big) + \|(e'_n, \hat{e}'_n)\|_* + \frac{\epsilon_n}{\alpha}.\end{aligned}$$

Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then it follows from (36), (50) and Lemma 5.3 that $\lim_{n\to\infty}(u_n, v_n) = (x, y)$.

Conversely, assume that $\lim_{n\to\infty}(u_n, v_n) = (x, y)$. With the help of (48) and (49), we have

$$\begin{aligned}\epsilon_n &= \|(u_{n+1}, v_{n+1}) - (L_n, D_n)\|_* \\ &\leq \|(u_{n+1}, v_{n+1}) - (x, y)\|_* + \|(L_n, D_n) - (x, y)\|_* \\ &\leq \|(u_{n+1}, v_{n+1}) - (x, y)\|_* + (1 - (1 - \widehat{Lk})(1-\alpha_n))\|(u_n, v_n) - (x, y)\|_* \\ &\quad + (1 - \widehat{Lk})(1-\alpha_n)\frac{\omega(n)}{1 - \widehat{Lk}} + \|(e''_n, \hat{e}''_n)\|_* + \|(l_n, \hat{l}_n)\|_*,\end{aligned} \tag{51}$$

where

$$\begin{aligned}\omega(n) = {}& L\Big(\Gamma(n) + a_{n,1}\phi(\widehat{k}\|(u_n, v_n) - (x, y)\|_* + \Gamma_1(n)\Big) + a_{n,2}\phi(\widehat{k}\|(u_n, v_n) - (x, y)\|_* \\ &+ \Gamma_2(n)) + b_{n,1} + b_{n,2}\Big) + \|(e'_n, \hat{e}'_n)\|_*.\end{aligned}$$

Obviously, (36) implies that $\lim_{n\to\infty}\|(e''_n, \hat{e}''_n)\|_* = \lim_{n\to\infty}\|(l_n, \hat{l}_n)\|_* = 0$. Now, the facts that $\lim_{n\to\infty} a_{n,i} = \lim_{n\to\infty} b_{n,i} = 0$ for $i = 1, 2$ and $\lim_{n\to\infty}\|(e'_n, \hat{e}'_n)\|_* = \lim_{n\to\infty}\Gamma(n) = 0$ ensure that the right-hand side of (51) tends to zero as $n \to \infty$. The proof is finished. □

Taking $S_i \equiv I_i$, the identity mapping on $E_i$, we obtain the following corollary as a direct consequence of Theorem 5.4 immediately.

**Corollary 5.5.** *Suppose that* $E_i, \eta_i, \eta_{n,i}, P_i, P_{n,i}, f_i, g_i, M_n, M, N_n, N, F$ *and* $G$ ($n \geq 0$ *and* $i = 1, 2$) *are the same as in Theorem 5.4 and let conditions* (i)–(vii) of Theorem 5.4 hold. Then

(1) the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.10 converges strongly to the unique solution $(x, y)$ of the SGNVLI (1).

(2) If, in addition, there exists a constant $\alpha > 0$ such that $\alpha + \alpha_n \leq 1$ for each $n \geq 0$, then $\lim\limits_{n\to\infty}(u_n, v_n) = (x, y)$ if and only if $\lim\limits_{n\to\infty} \widehat{\epsilon}_n = 0$, where $\{(u_n, v_n)\}_{n=0}^{\infty}$ is any sequence in $E_1 \times E_2$ defined by (38).

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