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Some new characterizations of normal matrices

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Abstract. This paper mainly studies some properties of normal matrix and gives the relation between the general solution of related matrix equations and normal matrices.

1. Introduction

Let $A \in C^{n\times n}$. $B \in C^{n\times n}$ is said to be the Moore-Penrose inverse matrix of A if

$$A = ABA, \qquad B = BAB, \qquad (AB)^H = AB, \qquad (BA)^H = BA.$$

The matrix B always exists by [1, 2] and is uniquely determined by the above equations. We denote it by $A^\dagger$.

A is said to be group invertible if there exists $B \in C^{n\times n}$ such that

$$A = ABA, \qquad B = BAB, \qquad AB = BA.$$

The matrix B is called group inverse matrix of A, which is uniquely by above equations [3]. We denote it by $A^\#$.

A is said to be regular if there exists $B \in C^{n\times n}$ such that $A = ABA$. The matrix B is called an inner inverse of A. The inner inverse matrix of A is not unique , and $A\{1\}$ is used to denote the set of all inner inverses of A.

Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is called an EP matrix if $A^\# = A^\dagger$. It is known that A is EP if and only if $AA^\dagger = A^\dagger A$. For the study of EP matrices, we can also refer to [1]. A is called a SEP matrix if $A^\# = A^\dagger = A^H$. And A is called a normal matrix if $A^H A = AA^H$. In [5], some properties of normal matrices and the conditions for the establishment of SEP matrices are introduced. The rest study of normal matrix can be found in [6–8].

In this paper, we continue to study normal matrices. In Section 2, we construst inner inverse matrices to characterize normal matrix. In Section 3, with the help of EP matrices, we discuss some new characterizations of normal matrices. In Section 4, we use invertible matrices to describe normal matrices. In Section 5, we research the relationship between the consistency of matrix equations and normal matrices. In Section 6, by means of the form of the general solution of the matrix equations, we obtain some interesting results about normal matrices.

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2. Some characterizations of normal matrices and *SEP* matrices by constructing inner matrices

Theorem 2.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if*

$$AA^H(A^\#)^H = A^HA(A^\dagger)^H.$$

Proof. *" $\Longrightarrow$ " Assume that A is normal matrix. Then, by [5, Lemma 1.3.3], we have A is an EP matrix, which implies $A^\# = A^\dagger$. Since A is normal, $A^HA = AA^H$. Hence $AA^H(A^\#)^H = AA^H(A^\dagger)^H = A^HA(A^\dagger)^H$.*

" $\Longleftarrow$ " Suppose that $AA^H(A^\#)^H = A^HA(A^\dagger)^H$. Multiplying the equality on the left by $A^\dagger A$, we have

$$A^\dagger A^2A^H(A^\#)^H = AA^H(A^\#)^H.$$

So $AA^\dagger = AA^H(A^\#)^HA^\dagger = A^\dagger A^2A^H(A^\#)^HA^\dagger = A^\dagger A^2A^\dagger$, this infers A is an EP matrix. Now we have

$$AA^H = AA^H(A^\#)^HA^H = A^HA(A^\dagger)^HA^H = A^HA^2A^\dagger = A^HA.$$

Hence A is a normal matrix. $\square$

Theorem 2.1 inspires us to give the following result on normal matrix.

Theorem 2.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $\begin{pmatrix} (A^\#)^H, & A^\# - E_n \end{pmatrix} \in \begin{pmatrix} AA^H \\ A \end{pmatrix}\{1\}$.*

Proof. *" $\Longrightarrow$ " Assume that A is a normal matrix. Then, by Theorem 2.1, we have $AA^H(A^\#)^H = A^HA(A^\dagger)^H$. It follows that*

$$\begin{pmatrix} AA^H \\ A \end{pmatrix}\begin{pmatrix} (A^\#)^H, & A^\# - E_n \end{pmatrix}\begin{pmatrix} AA^H \\ A \end{pmatrix} = \begin{pmatrix} AA^H(A^\#)^HAA^H + AA^H(A^\# - E_n)A \\ A(A^\#)^HAA^H + A(A^\# - E_n)A \end{pmatrix}.$$

Noting that

$$\begin{gathered} AA^H(A^\#)^HAA^H + AA^H(A^\# - E_n)A = (A^HA(A^\dagger)^H)(A^HA) + AA^H(A^\# - E_n)A = \\ A^HA^2 + AA^HA^\#A - AA^HA = A^HA^2 + AA^H - A^HA^2 = AA^H, \end{gathered}$$

and

$$A(A^\#)^HAA^H + A(A^\# - E_n)A = A(A^\dagger)^HA^HA + A - A^2 = A^2 + A - A^2 = A.$$

Hence

$$\begin{pmatrix} AA^H \\ A \end{pmatrix}\begin{pmatrix} (A^\#)^H, & A^\# - E_n \end{pmatrix}\begin{pmatrix} AA^H \\ A \end{pmatrix} = \begin{pmatrix} AA^H \\ A \end{pmatrix}.$$

One gets $\begin{pmatrix} (A^\#)^H, & A^\# - E_n \end{pmatrix} \in \begin{pmatrix} AA^H \\ A \end{pmatrix}\{1\}$.

" $\Longleftarrow$ " From the assumption, we have

$$\begin{pmatrix} AA^H \\ A \end{pmatrix}\begin{pmatrix} (A^\#)^H, & A^\# - E_n \end{pmatrix}\begin{pmatrix} AA^H \\ A \end{pmatrix} = \begin{pmatrix} AA^H \\ A \end{pmatrix},$$

this gives

$$AA^H(A^\#)^HAA^H + AA^H(A^\# - E_n)A = AA^H. \tag{1}$$

$$A(A^\#)^HAA^H + A(A^\# - E_n)A = A. \tag{2}$$

Multiplying (2) *on the right by $A^\dagger A$, we have*

$$A(A^\#)^HAA^H = A(A^\#)^HAA^HA^\dagger A.$$

Multiplying the last equality on the left by $A^\dagger A^HA^\dagger$, one yields $A^H = A^HA^\dagger A$. Hence A is EP. It follows from (1) *that $A^2A^H = AA^HA$, this gives $AA^HA^H = A^HAA^H$ and*

$$A^HA(A^\dagger)^H = A^HA(A^\#)^H = A^HAA^H(A^\#)^H(A^\#)^H = AA^HA^H(A^\#)^H(A^\#)^H = AA^H(A^\#)^H.$$

Hence A is normal by Theorem 2.1. $\square$

It is well known that A is a normal matrix if and only if A^H is a normal matrix. Hence Theorem 2.2 leads to the following corollary.

Corollary 2.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $\begin{pmatrix} A^{\#}, & (A^{\#})^H - E_n \end{pmatrix} \in \begin{pmatrix} A^H A \\ A^H \end{pmatrix}\{1\}$.*

Noting that *SEP* matrix is always normal. Hence Corollary 2.3 implies the following theorem which characterizes *SEP* matrix.

Theorem 2.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $\begin{pmatrix} A^{H}, & (A^{\#})^H - E_n \end{pmatrix} \in \begin{pmatrix} A^H A \\ A^H \end{pmatrix}\{1\}$.*

Proof. " $\Longrightarrow$ " Assume that A is *SEP*. Then A is normal and $A^{\#} = A^H$. Hence $\begin{pmatrix} A^{H}, & (A^{\#})^H - E_n \end{pmatrix} \in \begin{pmatrix} A^H A \\ A^H \end{pmatrix}\{1\}$ by Corollary 2.3.

" $\Longleftarrow$ " From the assumption, one has $\begin{pmatrix} A^H A \\ A^H \end{pmatrix}\begin{pmatrix} A^{H}, & (A^{\#})^H - E_n \end{pmatrix}\begin{pmatrix} A^H A \\ A^H \end{pmatrix} = \begin{pmatrix} A^H A \\ A^H \end{pmatrix}$, this gives

$$A^H A^H A^H A + A^H((A^{\#})^H - E_n)A^H = A^H. \tag{3}$$

e.g. $A^H A^H A^H A = A^H A^H$. Hence A is *SEP* by [5, Theorem 1.7.2]. □

Theorem 2.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $\begin{pmatrix} A, & A^{\#} - E_n \end{pmatrix} \in \begin{pmatrix} AA^H \\ A \end{pmatrix}\{1\}$.*

Proof. " $\Longrightarrow$ " Assume that A is a SEP matrix. Then A is normal and $A^{\#} = A^H$. By Theorem 2.2, we have $\begin{pmatrix} (A^{\#})^H, & A^{\#} - E_n \end{pmatrix} \in \begin{pmatrix} AA^H \\ A \end{pmatrix}\{1\}$, this gives $\begin{pmatrix} A, & A^{\#} - E_n \end{pmatrix} \in \begin{pmatrix} AA^H \\ A \end{pmatrix}\{1\}$.

" $\Longleftarrow$ " From the assumption, we have

$$\begin{pmatrix} AA^H \\ A \end{pmatrix}\begin{pmatrix} A, & A^{\#} - E_n \end{pmatrix}\begin{pmatrix} AA^H \\ A \end{pmatrix} = \begin{pmatrix} AA^H \\ A \end{pmatrix},$$

this gives

$$\begin{pmatrix} AA^H A^2 A^H + AA^H(A^{\#} - E_n)A \\ A^3 A^H + A - A^2 \end{pmatrix} = \begin{pmatrix} AA^H \\ A \end{pmatrix}.$$

Hence, we have $A^3 A^H = A^2$. Therefore A is SEP by [5, Theorem 1.7.2]. □

3. Constructing *EP* matrices to characterize normality

The following lemma can be proved by a routine verification.

Lemma 3.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then*
1) $AA^H(A^{\#})^H$ is EP with $(AA^H(A^{\#})^H)^{\dagger} = AA^{\dagger}A^{\dagger}$;
2) $A^H A(A^{\dagger})^H$ is EP with $(A^H A(A^{\dagger})^H)^{\dagger} = A^H A^{\#}(A^{\dagger})^H$.

Theorem 2.1 and Lemma 3.1 imply the following theorem.

Theorem 3.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^{\dagger}A^{\dagger} = A^H A^{\#}(A^{\dagger})^H$.*

Noting that normal matrix is EP. Then $(A^{\dagger})^H = (A^{\#})^H$, so Theorem 3.2 gives the following corollary.

Corollary 3.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^{\dagger}A^{\dagger} = A^H A^{\#}(A^{\#})^H$.*

Lemma 3.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then*
1) $(A^H A^{\#}(A^{\#})^H)^{+} = AA^{\dagger}A^H A^{\dagger}A^2(A^{\dagger})^H$;
2) $A^H A^{\#}(A^{\#})^H$ is group invertible with $(A^H A^{\#}(A^{\#})^H)^{\#} = A^H A^{\dagger}A^2(AA^{\#}A^{\dagger})^H$;
3) $(AA^{\#})^{\dagger} = A^{\dagger}A^2A^{\dagger}$.

Proof. 3)
$$(AA^{\#})(A^{\dagger}A^2A^{\dagger}) = AA^{\dagger} = (A^{\dagger}A^2A^{\dagger})(AA^{\#});$$
$$(AA^{\#})(A^{\dagger}A^2A^{\dagger})(AA^{\#}) = (AA^{\#});$$
$$(A^{\dagger}A^2A^{\dagger})(AA^{\#})(A^{\dagger}A^2A^{\dagger}) = (A^{\dagger}A^2A^{\dagger}).$$
Hence, we have $(AA^{\#})^{\dagger} = A^{\dagger}A^2A^{\dagger}$. □

Lemma 3.1, Corollary 3.3 and Lemma 3.4 lead to the following theorem.

Theorem 3.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the following conditions are equivalent:*
1) A is a normal matrix;
2) $AA^H(A^{\#})^H = AA^{\dagger}A^H A^{\dagger}A^2(A^{\dagger})^H$;
3) $AA^H(A^{\#})^H = A^H A^{\dagger}A^2(AA^{\#}A^{\dagger})^H$.

Observing the condition 2) of Theorem 3.5, we yield the following corollary.

Corollary 3.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H(A^{\#})^H = A^{\dagger}A^H A^{\dagger}A^2(A^{\dagger})^H$.*

Applying the involution on the equality of Corollary 3.6, we have the following corollary.

Corollary 3.7. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^{\#} = A^{\dagger}A^H A^{\dagger}A^2(A^{\dagger})^H$.*

Theorem 3.8. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^{\#} = A^{\dagger}A^H A(A^{\dagger})^H$.*

Proof. " $\Longrightarrow$ " Assume that A is normal. Then A is EP and $A^{\dagger}A^2 = A$. This infers $AA^{\#} = A^{\dagger}A^H A(A^{\dagger})^H$ by Corollary 3.7.

" $\Longleftarrow$ " If $AA^{\#} = A^{\dagger}A^H A(A^{\dagger})^H$, then A is EP because $(AA^{\#})^H = AA^{\#}$. It follows that $AA^H = A(AA^{\#})A^H = A(A^{\dagger}A^H A(A^{\dagger})^H)A^H = A^H A$. Hence A is normal. □

Noting that Lemma 3.4 and $(A^{\dagger}A^H A(A^{\dagger})^H)^{\dagger} = A^H A^{\#}AA^{\dagger}(A^{\#})^H A$. Then Theorem 3.8 leads to the following corollary.

Corollary 3.9. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^{\dagger}A^2A^{\dagger} = A^H A^{\#}AA^{\dagger}(A^{\#})^H A$.*

4. Constructing invertible matrices to characterize normality

In fact, if A is a group invertible matrix, we have $(A + E_n - AA^\#)(A^\# + E_n - AA^\#) = E_n$, then $A + E_n - AA^\#$ is invertible with $(A + E_n - AA^\#)^{-1} = A^\# + E_n - AA^\#$. Hence Lemma 3.1 gives the following lemma.

Lemma 4.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then $AA^H(A^\#)^H + E_n - AA^\dagger$ is an invertible matrix and $(AA^H(A^\#)^H + E_n - AA^\dagger)^{-1} = AA^\dagger A^\dagger + E_n - AA^\dagger$.*

Lemma 4.1 and Theorem 3.2 imply the following theorem.

Theorem 4.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $(AA^H(A^\#)^H + E_n - AA^\dagger)^{-1} = A^H A^\#(A^\dagger)^H + E_n - AA^\dagger$.*

Theorem 4.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $((A^\#)^H AA^H + E_n - AA^\dagger)^{-1} = (A^\dagger)^H A^\dagger A^\# A^H A + E_n - AA^\dagger$.*

Proof. " $\Longrightarrow$ " Assume that A is normal. Then, by Theorem 4.2. we have

$$(AA^H(A^\#)^H + E_n - AA^\dagger)^{-1} = A^H A^\#(A^\dagger)^H + E_n - AA^\dagger.$$

Noting that $AA^H(A^\#)^H + E_n - AA^\dagger = E_n - AA^H((A^\dagger)^H A^\dagger - (A^\#)^H)$. Then

$$\begin{aligned}
&(E_n - ((A^\dagger)^H A^\dagger - (A^\#)^H)AA^H)^{-1}\\
&= E_n + ((A^\dagger)^H A^\dagger - (A^\#)^H)(E_n - AA^H((A^\dagger)^H A^\dagger - (A^\#)^H))^{-1}AA^H\\
&= E_n + ((A^\dagger)^H A^\dagger - (A^\#)^H)(A^H A^\#(A^\dagger)^H + E_n - AA^\dagger)AA^H\\
&= E_n - (A^\#)^H A^H A^\#(A^\dagger)^H AA^H + (A^\dagger)^H A^\dagger A^H A^\#(A^\dagger)^H AA^H.
\end{aligned}$$

Since A is normal, we have

$$\begin{aligned}
(A^\#)^H A^H A^\#(A^\dagger)^H AA^H &= (A^\#)^H A^H A^\dagger(A^\dagger)^H AA^H\\
&= A^\dagger(A^\dagger)^H AA^H = A^\dagger(A^\dagger)^H A^H A = A^\dagger A = AA^\dagger
\end{aligned}$$

and

$$(A^\dagger)^H A^\dagger A^H A^\#(A^\dagger)^H AA^H = (A^\dagger)^H A^\dagger A^\# A^H(A^\dagger)^H A^H A = (A^\dagger)^H A^\dagger A^\# A^H A.$$

Thus

$$((A^\#)^H AA^H + E_n - AA^\dagger)^{-1} = (E_n - ((A^\dagger)^H A^\dagger - (A^\#)^H)AA^H)^{-1} = (A^\dagger)^H A^\dagger A^\# A^H A + E_n - AA^\dagger.$$

" $\Longleftarrow$ " The assumption implies

$$\begin{aligned}
E_n &= ((A^\#)^H AA^H + E_n - AA^\dagger)((A^\dagger)^H A^\dagger A^\# A^H A + E_n - AA^\dagger) =\\
&(A^\#)^H AA^H(A^\dagger)^H A^\dagger A^\# A^H A) + E_n - AA^\dagger = (A^\#)^H A^\# A^H A + E_n - AA^\dagger,
\end{aligned}$$

it follows that

$$(A^\#)^H A^\# A^H A = AA^\dagger.$$

Multiplying the equality on the right by $A^\dagger A$, one yields

$$AA^\dagger = AA^\dagger A^\dagger A.$$

Thus A is EP, this infers

$$A^\dagger = AA^\dagger A^\dagger = (A^\#)^H A^\# A^H AA^\dagger = (A^\#)^H A^\# A^H.$$

Now we have

$$A^H A^\dagger = A^H(A^\#)^H A^\# A^H = A^H(A^\#)^H A^\dagger A^H = A^\dagger A^H.$$

Therefore A is normal by [5, Lemma 1.3.2]. □

Corollary 4.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if* $((A^{\#})^H AA^H + E_n - AA^{\dagger})^{-1} = (A^{\dagger})^H A^{\dagger} A^{\#} + E_n - AA^{\dagger}$.

Proof. $'' \Longrightarrow ''$ Assume that A is SEP. Then A is normal, by Theorem 4.3, one obtains that

$$((A^{\#})^H AA^H + E_n - AA^{\dagger})^{-1} = (A^{\dagger})^H A^{\dagger} A^{\#} A^H A + E_n - AA^{\dagger}.$$

Noting that A is SEP. Then

$$(A^{\dagger})^H A^{\dagger} A^{\#} A^H A = (A^{\dagger})^H A^{\dagger} A^{\#} A^{\dagger} A = (A^{\dagger})^H A^{\dagger} A^{\#},$$

so

$$((A^{\#})^H AA^H + E_n - AA^{\dagger})^{-1} = (A^{\dagger})^H A^{\dagger} A^{\#} + E_n - AA^{\dagger}.$$

$'' \Longleftarrow ''$ From the assumption, we have

$$E_n = ((A^{\#})^H AA^H + E_n - AA^{\dagger})((A^{\dagger})^H A^{\dagger} A^{\#} + E_n - AA^{\dagger})$$
$$= (A^{\#})^H AA^H (A^{\dagger})^H A^{\dagger} A^{\#} + E_n - AA^{\dagger} = (A^{\#})^H A^{\#} + E_n - AA^{\dagger},$$

it follows that

$$(A^{\#})^H A^{\#} = AA^{\dagger}.$$

Multiplying the equality on the left by $A^{\dagger} A^H$, one yields

$$A^{\dagger} A^{\#} = A^{\dagger} A^H.$$

Thus A is SEP by [5, Theorem 1.5.3]. □

5. Consistence of matrix equations

Theorem 2.1 leads us to construct the following equation:

$$AX(A^{\#})^H = A^H A(A^{\dagger})^H. \tag{4}$$

Theorem 5.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is normal if and only if Eq.*(5.1) *is consistent and the general solution is given by*

$$X = A^H + U - A^{\dagger} AUA^{\dagger} A, \textit{where } U \in C^{n\times n}. \tag{5}$$

Proof. $'' \Longrightarrow ''$ *If A is normal, then $AA^H(A^{\#})^H = A^H A(A^{\dagger})^H$ by Theorem 2.1, it follows that $X = A^H$ is a solution of Eq.*(4).

Noting that

$$A(A^H + U - A^{\dagger} AUA^{\dagger} A)(A^{\#})^H = AA^H (A^{\#})^H = A^H A(A^{\dagger})^H.$$

Hence Formula (5) *is the solution of Eq.*(4).

Now let $X = X_0$ be any solution of Eq.(4)*. Then we have*

$$AX_0(A^{\#})^H = A^H A(A^{\dagger})^H.$$

Multiplying the equality on the left by $AA^{\dagger}$, we have

$$AA^{\dagger} A^H A(A^{\dagger})^H = A^H A(A^{\dagger})^H.$$

Multiplying the last equality on the right by $A^H AA^{\#} A^{\dagger}$, one yields

$$A^H = AA^{\dagger} A^H.$$

Hence A is EP. Now

$$A^\dagger A X_0 A^\dagger A = A^\dagger A X_0 (A^\#)^H A^H A^\dagger A = A^\dagger (A^H A (A^\dagger)^H) A^H A^\dagger A = A^\dagger A^H A^2 A^\dagger A^\dagger A$$
$$= A^\dagger A^H A A^\dagger A = A^\dagger A^H A = A^\dagger A A^H = A^H.$$

Hence

$$X_0 = A^H + X_0 - A^\dagger A X_0 A^\dagger A.$$

Hence the general solution of Eq.(4) *is given by Formula* (5).

" $\Longleftarrow$ " From the assumption, we have

$$A(A^H + U - A^\dagger A U A^\dagger A)(A^\#)^H = A^H A (A^\dagger)^H.$$

That is $AA^H(A^\#)^H = A^H A (A^\dagger)^H$. *By Theorem 2.1, we have A is normal.*
□

Now, we construct the following equation:

$$(A^\#)^H A^\dagger A X A^\dagger = A^\dagger. \tag{6}$$

Theorem 5.2. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then the general solution of Eq.*(6) *is given by* (5).

Proof. Since

$$(A^\#)^H A^\dagger A (A^H + U - A^\dagger A U A^\dagger A) A^\dagger = (A^\#)^H A^\dagger A A^H A^\dagger + (A^\#)^H A^\dagger A U A^\dagger$$
$$-(A^\#)^H A^\dagger A A^\dagger A U A^\dagger A A^\dagger = (A^\#)^H A^H A^\dagger + (A^\#)^H A^\dagger A U A^\dagger - (A^\#)^H A^\dagger A U A^\dagger = A^\dagger,$$

Formula (5) *is the solution of Eq.*(6).

Now, we assume that $X = X_0$ *is any solution of Eq.*(6). *Then*

$$(A^\#)^H A^\dagger A X_0 A^\dagger = A^\dagger.$$

Noting that

$$A^\dagger A X_0 A^\dagger A = A^H((A^\#)^H A^\dagger A X_0 A^\dagger) A = A^H A^\dagger A.$$

Then

$$A^\dagger A (X_0 - A^H) A^\dagger A = 0,$$

it follows that

$$X_0 = A^H + (X_0 - A^H) - A^\dagger A (X_0 - A^H) A^\dagger A.$$

Thus the general solution of Eq.(6) *is given by* (5). □

The following corollary is an immediate result of Theorem 5.1 and Theorem 5.2.

Corollary 5.3. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a normal matrix if and only if Eq.*(4) *and Eq.*(6) *have the same solution.*

Lemma 5.4. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is normal if and only if* $(A^\#)(A^\#)^H = (A^\#)^H A^\dagger$.

Proof. " $\Longrightarrow$ " Assume that A is a normal matrix. Then A is an EP matrix by [5, Lemma 1.3.3], this infers $A^\dagger = A^\#$. *By [5, Lemma 1.3.2], we have*

$$A^\# A^H = A^\dagger A^H = A^H A^\dagger = A^H A^\#,$$

this gives

$$A^\#(A^\#)^H = A^\# A^H (A^\#)^H (A^\#)^H = A^H A^\# (A^\#)^H (A^\#)^H = (A^\#)^H A^H A^H A^\# (A^\#)^H (A^\#)^H =$$
$$(A^\#)^H A^\# A^H A^H (A^\#)^H (A^\#)^H = (A^\#)^H A^\# A^H (A^\#)^H = (A^\#)^H A^\dagger A^H (A^\#)^H = (A^\#)^H A^\dagger.$$

" $\Longleftarrow$ " Assume that $A^{\#}(A^{\#})^H = (A^{\#})^H A^{\dagger}$. Then

$$(A^{\#})^H A^{\dagger} = AA^{\dagger}A^{\#}(A^{\#})^H = AA^{\dagger}(A^{\#})^H A^{\dagger}.$$

It follows that

$$A^{\dagger} = (A^{\#})^H A^H A^{\dagger} = (A^{\#})^H A^{\dagger}AA^H A^{\dagger} = AA^{\dagger}(A^{\#})^H A^{\dagger}AA^H A^{\dagger} = AA^{\dagger}(A^{\#})^H A^H A^{\dagger} = AA^{\dagger}A^{\dagger}.$$

Hence A is an EP matrix, and so A is normal because $A^{\#}(A^{\#})^H = (A^{\#})^H A^{\#}$. □

We construct the following equation:

$$A^{\#}(A^{\#})^H AXA^{\dagger} = A^{\dagger}. \tag{7}$$

Theorem 5.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is normal if and only if the general solution of Eq.*(7) *is given by* (5).

Proof. *" $\Longrightarrow$ " Assume that A is a normal matrix. Then $A^{\#}(A^{\#})^H = (A^{\#})^H A^{\dagger}$ by Lemma 5.4, this implies Eq.*(6) *is the same as Eq.*(7)*, so we obtain that the general solution of Eq.*(7) *is given by Formula* (5).

" $\Longleftarrow$ " From the assumption, one yields

$$A^{\#}(A^{\#})^H A(A^H + U - A^{\dagger}AUA^{\dagger}A)A^{\dagger} = A^{\dagger},$$

this gives

$$A^{\#}(A^{\#})^H AA^H A^{\dagger} = A^{\dagger}.$$

Multiplying the equality on the left by $AA^{\dagger}$, we have

$$AA^{\dagger}A^{\dagger} = A^{\dagger}.$$

Hence A is an EP matrix, one has

$$A^{\dagger}A = A^{\#}(A^{\#})^H AA^H A^{\dagger}A = A^{\#}(A^{\#})^H AA^H.$$

Multiplying the last equality on the left by A, we have

$$A = (A^{\dagger})^H AA^H,$$

one has $A^H A = A^H(A^{\dagger})^H AA^H = A^{\dagger}A^2A^H = AA^H$. Hence A is normal. □

Theorem 5.5 and Theorem 5.2 lead to the following corollary.

Corollary 5.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if Eq.*(6) *and Eq.*(7) *have the same solution.*

6. Normal matrix and the general solution of matrix equations

Observing Theorem 2.1, we can construct the following equation:

$$A(A^{\dagger})^H X = XA(A^{\#})^H. \tag{8}$$

Theorem 6.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is normal if and only if Eq.*(8) *has at least one solution in $\rho_A = \{A, A^{\#}, A^{\dagger}, A^H, (A^{\dagger})^H, (A^{\#})^H, A^{\dagger}A^3A^{\dagger}, (AA^{\#})^H A(AA^{\#})^H\}$.*

Proof. " $\Longrightarrow$ " *Assume that A is a normal matrix. Then A is an EP matrix and $AA^H(A^{\#})^H = A^HA(A^{\dagger})^H = A^HA(A^{\#})^H$ by Theorem 2.1, it follows that*

$$A(A^{\dagger})^HA^H = A^HA(A^{\#})^H.$$

Hence $X = A^H$ is a solution of Eq.(8) in ρ_A.

" $\Longleftarrow$ " *1) If $X = A$ is a solution, then $A(A^{\dagger})^HA = AA(A^{\#})^H$. Multiplying the equality on the right by $AA^{\dagger}$, we have*

$$A(A^{\dagger})^HA = A(A^{\dagger})^HA^2A^{\dagger}.$$

Multiplying the last equality on the left by $A^{\#}$, we have

$$(A^{\dagger})^HA = (A^{\dagger})^HA^2A^{\dagger}.$$

Multiplying the equality on the left by $A^{\#}A^H$, we have $A^{\#}A = AA^{\dagger}$. Hence A is an EP matrix. Now we have

$$A^{\#}(A^{\#})^H = A^{\#}A^{\#}A(A^{\dagger})^H = A^{\#}A^{\#}(A(A^{\dagger})^HA)A^{\#} = A^{\#}A^{\#}(A^2(A^{\#})^H)A^{\#}$$

$$= A^{\#}A(A^{\#})^HA^{\#} = A^{\dagger}A(A^{\#})^HA^{\dagger} = (A^{\#})^HA^{\dagger}.$$

Hence A is normal by Lemma 5.4.

2) If $X = A^{\#}$, then $A(A^{\dagger})^HA^{\#} = A^{\#}A(A^{\#})^H$. Multiplying the equality on the right by $AA^{\dagger}$, we have

$$A^{\#}A(A^{\#})^HA^{\dagger}A = A^{\#}A(A^{\#})^H.$$

Multiplying the last equality on the left by $A^{\dagger}A$, we have

$$(A^{\#})^HA^{\dagger}A = (A^{\#})^H.$$

Hence $A^{\#} = A^{\dagger}AA^{\#}$, this infers A is EP. Now we have

$$A(A^{\dagger})^HA^{\#} = A^{\#}A(A^{\#})^H = (A^{\#})^H,$$

and so

$$A^{\#}(A^{\#})^H = A^{\#}A(A^{\dagger})^HA^H = (A^{\dagger})^HA^{\#} = (A^{\#})^HA^{\dagger}.$$

Hence A is normal by Lemma 5.4.

3) If $X = A^{\dagger}$, then $A(A^{\dagger})^HA^{\dagger} = A^{\dagger}A(A^{\#})^H = (A^{\#})^H$. Multiplying the equality on the left by $AA^{\dagger}$, we have

$$(A^{\#})^H = AA^{\dagger}(A^{\#})^H,$$

this gives

$$A^{\#} = A^{\#}AA^{\dagger}.$$

Hence A is EP, this infers $X = A^{\dagger} = A^{\#}$ is a solution. Thus A is normal by 2).

4) If $X = A^H$, then $A^2A^{\dagger} = A^HA(A^{\#})^H$. Multiplying the equality on the left by $A^{\dagger}A$, we have

$$A^2A^{\dagger} = A^{\dagger}A^3A^{\dagger}.$$

Multiplying the last equality on the right by $A^{\#}$, we have

$$AA^{\#} = A^{\dagger}A.$$

Hence A is EP, one yields

$$AA^H = A^2A^{\dagger}A^H = A^HA(AA^{\#})^H = A^HA(AA^{\dagger})^H = A^HAAA^{\dagger} = A^HA.$$

Hence A is normal.

5) If $X=(A^{\#})^H$, then $A(A^{\dagger})^H(A^{\#})^H=(A^{\#})^HA(A^{\#})^H$. Multiplying the equality on the right by A^HA^H and on the left by $A^{\dagger}A$, we have

$$A^2A^{\dagger}=A^{\dagger}A^3A^{\dagger}.$$

Hence A is an EP matrix, one obtains that

$$A=A^2A^{\dagger}=A(A^{\dagger})^HA^H=(A(A^{\dagger})^H(A^{\#})^H)(A^HA^H)=((A^{\#})^HA(A^{\#})^H(A^HA^H)=(A^{\#})^HAA^H.$$

Thus

$$A^HA=A^H(A^{\#})^HAA^H=AA^H,$$

which implies A is normal.

6) If $X=(A^{\dagger})^H$, then $A(A^{\dagger})^H(A^{\dagger})^H=(A^{\dagger})^HA(A^{\#})^H$.

Multiplying the equality on the right by $AA^{\dagger}$ and on the left by $A^{\#}$, we have

$$(A^{\dagger})^H(A^{\dagger})^HAA^{\dagger}=(A^{\dagger})^H(A^{\dagger})^H.$$

Applying the involution on the last equality, we have

$$AA^{\dagger}A^{\dagger}A^{\dagger}=A^{\dagger}A^{\dagger}.$$

By [4, Lemma 2.11], we have

$$AA^{\dagger}A^{\dagger}=A^{\dagger}.$$

Hence A is an EP matrix, this infers $X=(A^{\#})^H$ is a solution. Thus A is normal by 5).

7) If $X=A^{\dagger}A^3A^{\dagger}$, then $A(A^{\dagger})^HA^{\dagger}A^3A^{\dagger}=A^{\dagger}A^3A^{\dagger}A(A^{\#})^H$,

e.g.,

$$A(A^{\dagger})^HA^2A^{\dagger}=A^{\dagger}A^3(A^{\#})^H.$$

Multiplying the last equality on the left by $A^{\dagger}A$, one obtains

$$A(A^{\dagger})^HA^2A^{\dagger}=A^{\dagger}A^2(A^{\dagger})^HA^2A^{\dagger}.$$

This gives

$$A(A^{\dagger})^H=A(A^{\dagger})^HA^2A^{\dagger}A^{\#}=A^{\dagger}A^2(A^{\dagger})^HA^2A^{\dagger}A^{\#}=A^{\dagger}A^2(A^{\dagger})^H,$$

and

$$AA^{\#}=A^2A^{\dagger}A^{\#}=A(A^{\dagger})^HA^HA^{\#}=A^{\dagger}A^2(A^{\dagger})^HA^HA^{\#}=A^{\dagger}A.$$

Hence A is EP, which implies $X=A^{\dagger}A^3A^{\dagger}=A$ is a solution. Hence A is normal by 1).

8) If $X=(AA^{\#})^HA(AA^{\#})^H$, then

$$A(A^{\dagger})^H(AA^{\#})^HA(AA^{\#})^H=(AA^{\#})^HA(AA^{\#})^HA(A^{\#})^H.$$

Multiplying the equality on the left by $A^{\dagger}A$ and on the right by $A^{\dagger}A^{\dagger}A$, one yields

$$A(A^{\dagger})^H=A^{\dagger}A^2(A^{\dagger})^H.$$

Hence A is EP by 7)*, which implies*

$$X=(AA^{\#})^HA(AA^{\#})^H=(AA^{\dagger})^HA(AA^{\dagger})^H=AA^{\dagger}A(AA^{\dagger})=A.$$

Thus A is normal by 1).

□

We can generalize Eq.(8) as follows.

$$A(A^{\dagger})^HX=YA(A^{\#})^H. \tag{9}$$

Clearly, the general solution of Eq.(9) can be represented by the following theorem.

Theorem 6.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.*(9) *is given by*

$$\begin{cases} X = PA(A^{\#})^H + U - A^{\dagger}AU \\ Y = A(A^{\dagger})^H P + V - VAA^{\dagger} \end{cases}, \text{where } P,\ U,\ V \in C^{n\times n}. \tag{10}$$

Theorem 6.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if the general solution of Eq.*(9) *is given by*

$$\begin{cases} X = P(A^{\#})^H A + U - A^{\dagger}AU \\ Y = A(A^{\dagger})^H P + V - VAA^{\dagger} \end{cases}, \text{where } P,\ U,\ V \in C^{n\times n}. \tag{11}$$

Proof. " $\Longrightarrow$ " Assume that A is normal. Then $A(A^{\#})^H = (A^{\#})^H A$. Hence Formula (10) is the same as Formula (11). By Theorem 6.2, we obtains the general solution of Eq.(9) is given by Formula (11).

" $\Longleftarrow$ " From the assumption, we have

$$A(A^{\dagger})^H(P(A^{\#})^H A + U - A^{\dagger}AU) = (A(A^{\dagger})^H P + V - VAA^{\dagger})A(A^{\#})^H,$$

e.g.,

$$A(A^{\dagger})^H P(A^{\#})^H A = A(A^{\dagger})^H PA(A^{\#})^H$$

for all $P \in C^{n\times n}$.

Especially, choose $P = E_n$. Then we have

$$A(A^{\dagger})^H(A^{\#})^H A = A(A^{\dagger})^H A(A^{\#})^H.$$

Multiplying the equality on the left by $A^H A^{\#}$ and on the right by $AA^{\dagger}$, we have

$$(A^{\#})^H A = (A^{\#})^H A^2 A^{\dagger}.$$

This gives

$$A = AA^{\dagger}A^H(A^{\#})^H A = AA^{\dagger}A^H(A^{\#})^H A^2 A^{\dagger} = A^2 A^{\dagger},$$

which implies that A is EP. Hence

$$(A^{\dagger})^H(A^{\#})^H A = A^{\#}A(A^{\dagger})^H(A^{\#})^H A = A^{\#}A(A^{\dagger})^H A(A^{\#})^H = (A^{\dagger})^H A(A^{\#})^H,$$

it follows that

$$(A^{\#})^H A = A^H(A^{\dagger})^H(A^{\#})^H A = A^H(A^{\dagger})^H A(A^{\#})^H = A^{\dagger}A^2(A^{\#})^H = A(A^{\#})^H.$$

Thus A is normal.

□

Theorem 6.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then*

1) $(A(A^{\dagger})^H X)^{\dagger} = X^{\dagger}AA^{\#}A^H AA^{\#}A^{\dagger}$, where $X \in \rho_A$;

2) $(A(A^{\dagger})^H X)^{\#} = X^{\#}A^H A^{\#}$, where $X \in \tau_A = \{A, A^{\#}, (A^{\dagger})^H\}$;

3) $(A(A^{\dagger})^H X)^{\#} = X^{\dagger}AA^{\#}A^H AA^{\#}A^{\dagger}$, where $X \in \nu_A = \{A^{\dagger}, A^H, (A^{\#})^H, A^{\dagger}A^3A^{\dagger}, (AA^{\#})^H A(AA^{\#})^H\}$;

4) $(XA(A^{\#})^H)^{\dagger} = AA^{\dagger}A^H A^{\dagger}AA^{\#}X^{\dagger}$, where $X \in \rho_A$;

5) $(XA(A^{\#})^H)^{\#} = AA^{\dagger}A^H A^{\dagger}AA^{\#}X^{\dagger}$, where $X \in \tau_A$;

6) $(XA(A^{\#})^H)^{\#} = A^H A^{\dagger}X^{\#}$, where $X \in \nu_A$.

Proof. Since when $X \in \tau_A$,

$$XX^{\dagger} = aa^{\dagger}$$

and when $X \in \nu_A$,

$$XX^{\dagger} = A^{\dagger}A, (A^{\dagger})^H XX^{\dagger}AA^{\#} = (A^{\dagger})^H.$$

This gives

$$(A(A^{\dagger}X)(X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger}) = A((A^{\dagger})^{H}XX^{\dagger}AA^{\#})A^{H}AA^{\#}A^{\dagger} = A(A^{\dagger})^{H}A^{H}AA^{\#}A^{\dagger} = AA^{\dagger}.$$

Also when $X \in \tau_A$,

$$X^{\dagger}AA^{\#}X = A^{\dagger}A$$

and when $X \in \nu_A$,

$$X^{\dagger}AA^{\#}X = A^{\dagger}A,$$

it follows that

$$\begin{aligned}(X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger})(A(A^{\dagger}X) &= X^{\dagger}(AA^{\#}A^{H}AA^{\#}A^{\dagger}A(A^{\dagger})^{H})X = X^{\dagger}AA^{\#}X \\ &= (X^{\dagger}AA^{\#}X)^{H} = ((X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger})(A(A^{\dagger}X))^{H};\end{aligned}$$

Noting that $AA^{\#}XX^{\dagger}AA^{\#} = AA^{\#}$. *Then*

$$\begin{aligned}(X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger})(A(A^{\dagger}X)(X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger}) &= (X^{\dagger}AA^{\#}X)(X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger}) \\ &= X^{\dagger}(AA^{\#}XX^{\dagger}AA^{\#})A^{H}AA^{\#}A^{\dagger} = X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger}.\end{aligned}$$

Thus

$$(A(A^{\dagger})^{H}X)^{\dagger} = X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger}.$$

Similarly, we can show (2) (6) □

By Theorem 6.1 and Theorem 6.4, we have the following theorem.

Theorem 6.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if one of the following conditions hold.*

1) Equation $X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger} = AA^{\dagger}A^{H}A^{\dagger}AA^{\#}X^{\dagger}$ has at least one solution in ρ_A;

2) Equation $X^{\#}A^{H}A^{\#} = AA^{\dagger}A^{H}A^{\dagger}AA^{\#}X^{\dagger}$ has at least one solution in τ_A;

3) Equation $X^{\dagger}AA^{\#}A^{H}AA^{\#}A^{\dagger} = A^{H}A^{\dagger}A^{\#}$ has at least one solution in ν_A.

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