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A new generalization of min and max matrices and their reciprocals counterparts

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Abstract. In this paper, we introduce a new generalization of the min and max matrices called geometric min matrix and geometric max matrix, respectively. We discuss some of their properties, such as determinants, inverses, factorizations, and identities for their characteristic polynomials. Moreover, their reciprocal analogs for these newly established matrices are examined.

1. Introduction

Matrix theory is extensively used in a variety of areas including applied mathematics, computer science, economics, engineering, operations research, statistics, and others. From past to present, different types of matrices have been defined and examined such as determinant, inverse, and factorizations by the researchers [18].

The so-called *min matrix* M is defined as

$$(\min(i,j))_{i,j=1}^{n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}, \tag{1}$$

It was first introduced in the second volume of the ground-breaking book of problems by G. Pólya and Szegő [15]. In Problem 30, page 122, we may find the equalities

$$1 = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix}. \tag{2}$$

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We understand the first determinant as of a (lower triangular) matrix of order n. On the same page, in Problem 32, we find the claim for a possible generalization of (2), namely

$$\begin{vmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_1 & A_2 & A_2 & \cdots & A_2 \\ A_1 & A_2 & A_3 & \cdots & A_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_n \end{vmatrix} = a_1 a_2 \cdots a_n \, , \tag{3}$$

where $a_1, a_2, \ldots, a_n$ are given constants and $A_k = \sum_{i \leqslant k} a_i$, for $k = 1, 2, \ldots, n$. Clearly the matrix we find in (3) is a generalization for the min matrix (1). In 2009, Neudecker, Trenkler, and Liu proposed the problem of finding a necessary and sufficient condition for the matrix

$$A = \begin{pmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} \tag{4}$$

to be positive definite, its determinant and inverse, in case of being nonsingular, where $a_1, a_2, \ldots, a_n$ are real numbers [14]. These problems were soon solved by, among others, Chu, Puntanen, and Styan [6]. Namely, it was proved that

$$\det A = a_1(a_2 - a_1)(a_3 - a_2) \cdots (a_n - a_{n-1})$$

and

$$A^{-1} = \begin{pmatrix} \alpha_1 + \alpha_2 & -\alpha_2 & & & & \\ -\alpha_2 & \alpha_2 + \alpha_3 & -\alpha_3 & & & \\ & -\alpha_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & -\alpha_{n-1} & \\ & & & -\alpha_{n-1} & \alpha_{n-1} + \alpha_n & -\alpha_n \\ & & & & -\alpha_n & \alpha_n \end{pmatrix},$$

where

$$\alpha_i = \frac{1}{a_i - a_{i-1}} \, , \quad \text{for } i = 1, \ldots, n,$$

setting $a_0 = 0$.

The second equality in (2) is a consequence of the Cholesky decomposition of M:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{5}$$

Therefore, the min matrix is positive semidefinite. One can also conclude that the inverse of the min matrix is

$$M^{-1} = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}. \tag{6}$$

In the comprehensive monograph [2], Bhatia presents five other proofs for the positive semidefiniteness of M, providing distinct and sometimes surprising avenues of applying this matrix. As earlier in [3], Bhatia also generalizes the min matrix to the matrix $M(\lambda_1, \ldots, \lambda_n)$, for $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n$, where the (i, j)-entry is $\min(\lambda_i, \lambda_j)$. This matrix was coined as *generalized min matrix*. Among other related matrices we find in [2], we have the matrix W with entries

$$\left(\frac{1}{max(i,j)}\right) = \left(\min\left(\frac{1}{i}, \frac{1}{j}\right)\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/2 & 1/3 & \cdots & 1/n \\ 1/3 & 1/3 & 1/3 & \cdots & 1/n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & 1/n & \cdots & 1/n \end{pmatrix},$$

called "a loyal companion of the Hilbert matrix" by Choi in [5].

In the same fashion that we defined the min matrix, we can also consider the max matrix as

$$(\max(i,j))_{i,j=1}^{n} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{pmatrix}. \tag{7}$$

For several other extensions and applications of the min and max matrices, the reader is referred for example to [1, 4, 7–13, 16, 17].

The aim of this paper is to introduce a new generalization for the min matrix (1) and present some properties. We will also present some results for an extension of the max matrix (7) as well as the reciprocals counterparts.

2. The geometric min matrix

We start this section with the introduction of a generalization of the min matrix (1).

Definition 2.1. *For a given number r, the geometric min matrix $M_{n,r}$ of order n is defined as*

$$M_{n,r} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 2 & 2 & 2 & \cdots & 2 & 2 \\ r^2 & 2r & 3 & 3 & \cdots & 3 & 3 \\ r^3 & 2r^2 & 3r & 4 & \cdots & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-2} & 2r^{n-3} & 3r^{n-4} & 4r^{n-5} & \cdots & (n-1) & n-1 \\ r^{n-1} & 2r^{n-2} & 3r^{n-3} & 4r^{n-4} & \cdots & (n-1)\,r & n \end{pmatrix}. \tag{8}$$

In other words, (8) means that the (i,j)-entry of $M_{n,r}$ is

$$m_{ij} = r^{\max(i-j,0)} \min(i,j) = \begin{cases} i, & i \leqslant j \\ r^{i-j} j, & i > j \end{cases} .$$

The next result is a generalization of (2).

Proposition 2.2. *The determinant of the geometric min matrix $M_{n,r}$ in* (8) *is*

$$\det M_{n,r} = \prod_{k=1}^{n} (k-(k-1)r) . \tag{9}$$

Proof. Using an elementary column operation, we have

$$\begin{aligned} \det M_{n,r} &= \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ r & 2 & 2 & 2 & \cdots & 2 & 0 \\ r^2 & 2r & 3 & 3 & \cdots & 3 & 0 \\ r^3 & 2r^2 & 3r & 4 & \cdots & 4 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-2} & 2r^{n-3} & 3r^{n-4} & 4r^{n-5} & \cdots & (n-1) & 0 \\ r^{n-1} & 2r^{n-2} & 3r^{n-3} & 4r^{n-4} & \cdots & (n-1)r & n-(n-1)r \end{vmatrix} \\ &= \det M_{n-1,r} (n-(n-1)r) . \end{aligned}$$

A simple inductive argument provides us now (9). □

Notice that if $r = \frac{k}{k-1}$, for some $k \in \{2, \ldots, n\}$, then $M_{n,r}$ is singular. Clearly, the converse is also true. Another important topic of interest in the literature is LU factorizations for this kind of matrices.

Theorem 2.3. *An LU factorization of $M_{n,r}$ is*

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ r & 1 & 0 & \cdots & 0 & 0 \\ r^2 & r & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ r^{n-2} & r^{n-3} & r^{n-4} & \cdots & 1 & 0 \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2-r & 2-r & \cdots & 2-r & 2-r \\ \vdots & 0 & 3-2r & \cdots & 3-2r & 3-2r \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & (n-1)-(n-2)r & (n-1)-(n-2)r \\ 0 & \cdots & \cdots & \cdots & 0 & n-(n-1)r \end{pmatrix} .$$

Namely, the (i,j)-entries of $L = (l_{ij})_{i,j=1}^{n}$ and $U = (u_{ij})_{i,j=1}^{n}$ are

$$l_{ij} = \begin{cases} r^{i-j}, & i \geq j \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_{ij} = \begin{cases} i-(i-1)r, & j \geq i \\ 0, & \text{otherwise}, \end{cases}$$

respectively.

Proof. Using matrix multiplication, we get

$$\begin{aligned}
\sum_{d=1}^{\min(i,j)} l_{id}u_{dj} &= \sum_{d=1}^{\min(i,j)} r^{i-d}\left(d-(d-1)r\right)\\
&= \sum_{d=1}^{\min(i,j)} \frac{r^i}{r^d}\left(d-(d-1)r\right)\\
&= r^i\left(\frac{1}{r}+\frac{2}{r^2}-\frac{1}{r}+\frac{3}{r^3}-\frac{2}{r^2}+\frac{4}{r^4}-\frac{3}{r^3}+\ldots+\frac{\min(i,j)}{r^{\min(i,j)}}-\frac{\min(i,j)-1}{r^{\min(i,j)-1}}\right)\\
&= r^i\frac{\min(i,j)}{r^{\min(i,j)}} = r^{i-\min(i,j)}\min(i,j) = r^{\max(i-j,0)}\min(i,j)\,.
\end{aligned}$$

So the proof is complete. □

Finding the inverses of each matrix L and U is now a straightforward exercise:

$$L^{-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0\\
-r & 1 & 0 & \ddots & & \vdots\\
0 & -r & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & -r & 1
\end{pmatrix}$$

and

$$U^{-1} = \begin{pmatrix}
1 & \frac{1}{r-2} & 0 & \cdots & \cdots & 0\\
0 & -\frac{1}{r-2} & \frac{1}{2r-3} & \ddots & & \vdots\\
0 & 0 & -\frac{1}{2r-3} & \frac{1}{3r-4} & \ddots & \vdots\\
\vdots & \ddots & 0 & \ddots & \ddots & 0\\
\vdots & & \ddots & \ddots & \ddots & \frac{1}{(n-1)r-n}\\
0 & \cdots & \cdots & 0 & 0 & -\frac{1}{(n-1)r-n}
\end{pmatrix}.$$

As we could expect, the inverse of $M_{n,r}$ is a tridiagonal matrix. The next result provides an explicit formula for such matrix.

Theorem 2.4. *If $M_{n,r}$ is nonsingular, then its inverse is the tridiagonal matrix*

$$M_{n,r}^{-1} = \begin{pmatrix}
-\frac{2}{r-2} & \frac{1}{r-2} & & & & \\
\frac{r}{r-2} & -\frac{r^2-3}{(2r-3)(r-2)} & \frac{1}{2r-3} & & & \\
 & \frac{r}{2r-3} & -\frac{2r^2-4}{(3r-4)(2r-3)} & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & \ddots & -\frac{(n-2)r^2-n}{((n-1)r-n)((n-2)r-(n-1))} & \frac{1}{(n-1)r-n}\\
 & & & & \frac{r}{(n-1)r-n} & -\frac{1}{(n-1)r-n}
\end{pmatrix}.$$

Proof. For the proof, we may simply consider the product $U^{-1}L^{-1}$. □

This means that the (i, j)-entry of the inverse of $M_{n,r}$, say $\mathfrak{m}_{ij}$, is

$$\mathfrak{m}_{ij} = \begin{cases} \frac{r}{(i-1)r-i}, & i-j=1 \\ \frac{1}{(j-1)r-j}, & j-i=1 \\ -\frac{(i-1)r^2-(i+1)}{(ir-(i+1))((i-1)r-i)}, & i=j\neq n \\ -\frac{1}{(n-1)r-n}, & i=j=n \\ 0, & \text{otherwise.} \end{cases}$$

While the eigenvalues of $M_{n,r}$ are not possible to find in a "nice" closed form, even for the particular case of $r = 1$ and $n = 2$ (cf. e.g. [6]), we can provide an elegant recurrence relation for its characteristic polynomial.

Theorem 2.5. *The characteristic polynomial of the geometric min matrix $M_{n,r}$ satisfies the recurrence relation*

$$P_n(\lambda) = ((n-1)\,r - n + (r+1)\,\lambda)\,P_{n-1}(\lambda) - r\lambda^2 P_{n-2}(\lambda), \tag{10}$$

with initial conditions

$$\begin{aligned} P_1(\lambda) &= \lambda - 1, \\ P_2(\lambda) &= (\lambda-1)(\lambda-2) - r. \end{aligned}$$

Proof. We will use strong induction for the proof and properties of the determinants. The cases $n = 1, 2$ are straightforward. Using elementary row operations, we can find that $P_n(\lambda)$ is given by

$$\begin{vmatrix} \lambda-1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -r & \lambda-2 & -2 & -2 & \cdots & -2 & -2 \\ 0 & -\lambda r & \lambda+2r-3 & -3+2r & \cdots & -3+2r & -3+2r \\ 0 & 0 & -\lambda r & \ddots & \ddots & -4+3r & -4+3r \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & 2-n+(n-3)r & 2-n+(n-3)r \\ \vdots & & & \ddots & -\lambda r & \lambda+(n-2)\,r-(n-1) & 1-n+(n-2)r \\ 0 & \cdots & \cdots & \cdots & 0 & -\lambda r & \lambda+(n-1)\,r-n \end{vmatrix}.$$

Now, if we subtract the $(n-1)$th column to the nth column, then we find that $P_n(\lambda)$ equals

$$\begin{vmatrix} \lambda-1 & -1 & -1 & -1 & \cdots & -1 & 0 \\ -r & \lambda-2 & -2 & -2 & \cdots & -2 & 0 \\ 0 & -\lambda r & \lambda+2r-3 & -3+2r & \cdots & -3+2r & 0 \\ 0 & 0 & -\lambda r & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 2-n+(n-3)\,r & 0 \\ \vdots & & & 0 & -\lambda r & \lambda+(n-2)\,r-(n-1) & -\lambda \\ 0 & \cdots & \cdots & \cdots & 0 & -\lambda r & (n-1)\,r-n+(r+1)\,\lambda \end{vmatrix}.$$

The Laplace expansion by the last row provides us the recurrence. □

In the last result of this section, we analyse the coefficients of the characteristic polynomial of the geometric min matrix $M_{n,r}$.

Theorem 2.6. *The coefficients of the characteristic polynomial*

$$P_n(\lambda) = \lambda^n + k_{n-1}^{(n)}\lambda^{n-1} + \cdots + k_1^{(n)}\lambda + k_0^{(n)}, \tag{11}$$

of the geometric min matrix $M_{n,r}$ satisfy

$$\begin{aligned} k_0^{(n)} &= (n-(n-1)r)\,k_0^{(n-1)} = (-1)^{n-1}\det M_{n,r} = (-1)^n\prod_{i=1}^{n}\lambda_i,\\ k_i^{(n)} &= (n-(n-1)r)\,k_i^{(n-1)} - (r+1)\,k_{i-1}^{(n-1)} + rk_{i-2}^{(n-2)}, \end{aligned}$$

with $k_{i\le 0} = 0$.

Proof. From (11), we have

$$P_{n-1}(\lambda) = \lambda^{n-1} + k_{n-2}^{(n-1)}\lambda^{n-2} + k_{n-3}^{(n-1)}\lambda^{n-3} + \cdots + k_1^{(n-1)}\lambda + k_0^{(n-1)} \tag{12}$$

and

$$P_{n-2}(\lambda) = \lambda^{n-2} + k_{n-3}^{(n-2)}\lambda^{n-3} + k_{n-4}^{(n-2)}\lambda^{n-4} + \cdots + k_1^{(n-2)}\lambda + k_0^{(n-2)}. \tag{13}$$

Substituting (11), (12), and (13) in (10), and after some calculations, we have

$$\begin{aligned} k_{n-1}^{(n)} &= (n-1)r - n + (r+1)\,k_{n-2}^{(n-1)} - rk_{n-3}^{(n-2)},\\ k_{n-2}^{(n)} &= ((n-1)r-n)\,k_{n-2}^{(n-1)} + (r+1)\,k_{n-3}^{(n-1)} - rk_{n-4}^{(n-2)},\\ &\vdots\\ k_2^{(n)} &= ((n-1)r-n)\,k_2^{(n-1)} + (r+1)\,k_1^{(n-1)} - rk_0^{(n-2)},\\ k_1^{(n)} &= ((n-1)r-n)\,k_1^{(n-1)} + (r+1)\,k_0^{(n-1)},\\ k_0^{(n)} &= ((n-1)r-n)\,k_0^{(n-1)}. \end{aligned}$$

So, we can write it in the general form of

$$\begin{aligned} k_0^{(n)} &= ((n-1)r-n)\,k_0^{(n-1)},\\ k_i^{(n)} &= ((n-1)r-n)\,k_i^{(n-1)} + (r+1)\,k_{i-1}^{(n-1)} - rk_{i-2}^{(n-2)}. \end{aligned}$$

It can be easily seen from the determinant of the geometric min matrix that

$$k_0^{(n)} = ((n-1)r-n)\,k_0^{(n-1)} = (-1)^n\det M_{n,r}\,.$$

□

3. The geometric max matrix

Analogous to the geometric min matrix, in this section we introduce the *geometric max matrix* and discuss similar properties.

Definition 3.1. *For a given constant r, the geometric max matrix $\mathbb{M}_{n,r}$, of order n, is defined as*

$$\mathbb{M}_{n,r} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & (n-1) & n\\ 2r & 2 & 3 & 4 & \cdots & (n-1) & n\\ 3r^2 & 3r & 3 & 4 & \cdots & (n-1) & n\\ 4r^3 & 4r^2 & 4r & 4 & \cdots & (n-1) & n\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ (n-1)\,r^{n-2} & (n-1)\,r^{n-3} & (n-1)\,r^{n-4} & (n-1)\,r^{n-5} & \cdots & (n-1) & n\\ nr^{n-1} & nr^{n-2} & nr^{n-3} & nr^{n-4} & \cdots & nr & n \end{pmatrix}, \tag{14}$$

or, equivalently, the (i, j)*-entry is given by*

$$\begin{aligned}\mathbf{m}_{ij} &= \begin{cases} j, & i \le j \\ r^{i-j} i, & i > j \end{cases} \\ &= r^{\max(i-j,0)} \max(i, j)\,.\end{aligned}$$

The first result is related to the determinant. The proof is similar to the one given for Proposition 2.2.

Proposition 3.2. *The determinant of the matrix* $\mathbb{M}_{n,r}$ *in* (14) *is*

$$\det \mathbb{M}_{n,r} = n \prod_{s=1}^{n-1} (s - (s+1) r)\,.$$

Remark 3.3. *For* $1 \leqslant s \leqslant n-1$,

$$\det \mathbb{M}_{n,r} = 0\,,$$

when $r = \frac{s}{s+1}$. *Hence, in this case* $\mathbb{M}_{n,r}$ *is singular.*

Theorem 3.4. *An LU factorization of the matrix* $\mathbb{M}_{n,r}$ *and their corresponding inverses are*

$$L = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 2r & 1 & \ddots & & & \vdots \\ 3r^2 & \frac{3}{2}r & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ (n-1)\, r^{n-2} & \frac{n-1}{2} r^{n-3} & \frac{n-1}{3} r^{n-4} & \ddots & 1 & 0 \\ nr^{n-1} & \frac{n}{2} 2r^{n-2} & \frac{n}{3} r^{n-3} & \cdots & \frac{n}{n-1} r & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 0 & 2\,(1-2r) & 3\,(1-2r) & \cdots & (n-1)\,(1-2r) & n\,(1-2r) \\ 0 & 0 & 3\frac{2-3r}{2} & \cdots & (n-1)\,\frac{2-3r}{2} & n\frac{2-3r}{2} \\ 0 & 0 & 0 & \cdots & (n-1)\,\frac{3-4r}{3} & n\frac{3-4r}{3} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & (n-1)\,\frac{n-2-(n-1)r}{n-2} & n\frac{n-2-(n-1)r}{n-2} \\ 0 & \cdots & \cdots & \cdots & 0 & n\frac{n-1-nr}{n-1} \end{pmatrix},$$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -2r & 1 & 0 & & & \vdots \\ 0 & -\frac{3}{2}r & 1 & \ddots & & \vdots \\ 0 & 0 & -\frac{4}{3}r & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -\frac{n}{n-1}r & 1 \end{pmatrix},$$

$$U^{-1}=\begin{pmatrix} 1 & \frac{1}{2r-1} & 0 & \cdots & \cdots & 0\\ 0 & -\frac{1}{2(2r-1)} & \frac{1}{3r-2} & 0 & & \vdots\\ 0 & 0 & \frac{2}{3(3r-2)} & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & \frac{n-2}{(n-1)((n-1)r-(n-2))} & \frac{1}{nr-(n-1)}\\ 0 & \cdots & \cdots & \cdots & 0 & \frac{n-1}{n(nr-(n-1))} \end{pmatrix}.$$

We can now establish the inverse of $\mathbb{M}_{n,r}$.

Theorem 3.5. *If $\mathbb{M}_{n,r}$ is nonsingular, then its inverse is the tridiagonal matrix*

$$\mathbb{M}_{n,r}^{-1}=\begin{pmatrix} -\frac{1}{2r-1} & \frac{1}{2r-1} & & & & \\ \frac{r}{2r-1} & -\frac{3r^2-1}{(3r-2)(2r-1)} & \frac{1}{3r-2} & & & \\ & \frac{r}{3r-2} & -\frac{2(2r^2-1)}{(4r-3)(3r-2)} & \ddots & & \\ & & \frac{r}{4r-3} & \ddots & \ddots & \\ & & & \ddots & -\frac{nr^2-(n-2)}{((n-1)r-n)((n-2)r-(n-1))} & \frac{1}{nr-(n-1)}\\ & & & & \frac{r}{nr-(n-1)} & -\frac{n-1}{n(nr-(n-1))} \end{pmatrix}.$$

In the next result, we present an identity for the characteristic polynomial of the geometric max matrix $\mathbb{M}_{n,r}$.

Theorem 3.6. *For $n \geq 3$, the characteristic polynomial of the geometric max matrix $\mathbb{M}_{n,r}$ satisfies the following identity*

$$Q_n(\lambda)=(n-\lambda)\,B_{n-1}(\lambda)-r\lambda^2B_{n-2}(\lambda),$$

with initial conditions

$$\begin{aligned} Q_1(\lambda) &= 1-\lambda,\\ Q_2(\lambda) &= (1-\lambda)(2-\lambda)-4r, \end{aligned}$$

where $B_{n-1}(\lambda)$ is as follows:

$$\begin{vmatrix} 1-2r-(r+1)\lambda & \lambda & \cdots & 0 & 0\\ r\lambda & 2-3r-(r+1)\lambda & \cdots & 0 & 0\\ 0 & r\lambda & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ & & \ddots & (n-2)-(n-1)r-(r+1)\lambda & \lambda\\ 0 & 0 & \cdots & r\lambda & (n-1)-nr-(r+1)\lambda \end{vmatrix}.$$

Proof. It can be easily seen that

$$B_1(\lambda)=1-2r-(r+1)\lambda$$

and

$$B_{n-1}(\lambda)=((n-1)-nr-(r+1)\lambda)\,B_{n-2}(\lambda)-r\lambda^2B_{n-3}(\lambda).$$

Applying the elementary row and column operations to (14), we obtain the following determinant of a tridiagonal matrix

$$Q_n(\lambda)=\begin{vmatrix}1-2r-(r+1)\lambda & \lambda & \cdots & 0 & 0\\ r\lambda & 2-3r-(r+1)\lambda & \cdots & 0 & 0\\ 0 & r\lambda & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 0\\ 0 & 0 & \cdots & (n-1)-nr-(r+1)\lambda & \lambda\\ 0 & 0 & \cdots & r\lambda & n-\lambda\end{vmatrix}.$$

Then, we get

$$Q_n(\lambda)=(n-\lambda)\,B_{n-1}(\lambda)-r\lambda^2 B_{n-2}(\lambda)\,, \tag{15}$$

with the initial conditions $Q_1(\lambda)=1-\lambda$ and $Q_2(\lambda)=(1-\lambda)(2-\lambda)-4r$. □

Theorem 3.7. *The coefficients of the characteristic polynomial of the geometric max matrix* $\mathbb{M}_{n,r}$

$$Q_n(\lambda)=\lambda^n+\eta_{n-1}^{(n)}\lambda^{n-1}+\eta_{n-2}^{(n)}\lambda^{n-2}+\cdots+\eta_1^{(n)}\lambda+\eta_0^{(n)}, \tag{16}$$

satisfy

$$\begin{aligned}\eta_0^{(n)} &= (-1)^n\, nb_0^{(n-1)}=(-1)^n\det\mathbb{M}_{n,r},\\ \eta_1^{(n)} &= (-1)^n\left(nb_1^{(n-1)}-b_0^{(n-1)}\right),\\ \eta_n^{(n)} &= 1=(-1)^n\left(-b_{n-1}^{(n-1)}-r\eta_{n-2}^{(n-2)}\right),\\ \eta_i^{(n)} &= (-1)^n\left(nb_i^{(n-1)}-b_{i-1}^{(n-1)}-r\eta_{i-2}^{(n-2)}\right),\end{aligned}$$

where

$$B_{n-1}(\lambda)=b_{n-1}^{(n-1)}\lambda^{n-1}+b_{n-2}^{(n-1)}\lambda^{n-2}+b_{n-3}^{(n-1)}\lambda^{n-3}+\ldots+b_1^{(n-1)}\lambda+b_0^{(n-1)}. \tag{17}$$

Proof. By virtue of (15), (17), and $\eta_n^{(n)}=1$, we have

$$\begin{aligned}(-1)^n Q_n(\lambda) &= (n-\lambda)\,B_{n-1}(\lambda)-r\lambda^2 B_{n-2}(\lambda)\\ &= (n-\lambda)\left(b_{n-1}^{(n-1)}\lambda^{n-1}+b_{n-2}^{(n-1)}\lambda^{n-2}+b_{n-3}^{(n-1)}\lambda^{n-3}+\cdots+b_1^{(n-1)}\lambda+b_0^{(n-1)}\right)\\ &\quad -r\lambda^2\left(b_{n-2}^{(n-2)}\lambda^{n-2}+b_{n-3}^{(n-2)}\lambda^{n-3}+b_{n-4}^{(n-2)}\lambda^{n-4}+\cdots+b_1^{(n-2)}\lambda+b_0^{(n-2)}\right)\\ &= \lambda^n\left(-b_{n-1}^{(n-1)}-rb_{n-2}^{(n-2)}\right)+\lambda^{n-1}\left(nb_{n-1}^{(n-1)}-b_{n-2}^{(n-1)}-rb_{n-3}^{(n-2)}\right)\\ &\quad +\lambda^{n-2}\left(nb_{n-2}^{(n-1)}-b_{n-3}^{(n-1)}-rb_{n-4}^{(n-2)}\right)+\cdots+\lambda\left(nb_1^{(n-1)}-b_0^{(n-1)}\right)+nb_0^{(n-1)}.\end{aligned}$$

Therefore, from equality of left and right sides, we get

$$\begin{aligned}\eta_0^{(n)} &= (-1)^n\, nb_0^{(n-1)},\\ \eta_1^{(n)} &= (-1)^n\left(nb_1^{(n-1)}-b_0^{(n-1)}\right),\\ \eta_n^{(n)} &= 1=(-1)^n\left(-b_{n-1}^{(n-1)}-r\eta_{n-2}^{(n-2)}\right),\\ \eta_i^{(n)} &= (-1)^n\left(nb_i^{(n-1)}-b_{i-1}^{(n-1)}-r\eta_{i-2}^{(n-2)}\right).\end{aligned}$$

Additionally, for $\lambda = 0$, we obtain

$$B_{n-1}(0) = \begin{vmatrix} 1-2r & 0 & \cdots & 0 & 0 \\ 0 & 2-3r & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & (n-2)-(n-1)r & 0 \\ 0 & 0 & \cdots & 0 & (n-1)-nr \end{vmatrix} = \prod_{s=1}^{n-1} (s-(s+1)r),$$

and

$$b_0^{(n-1)} = \frac{1}{n} \det \mathbb{M}_{n,r}.$$

□

4. Reciprocal matrices

In this last section, we present two types of reciprocal matrices which are extensions of those discussed in the introduction.

Definition 4.1. *The reciprocal matrix of* $\mathbb{M}_{n,r}$ *is defined by*

$$\mathbb{H}_{n,r} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-2} & \frac{1}{n-1} & \frac{1}{n} \\ \frac{1}{2r} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-2} & \frac{1}{n-1} & \frac{1}{n} \\ \frac{1}{3r^2} & \frac{1}{3r} & \frac{1}{3} & \cdots & \frac{1}{n-2} & \frac{1}{n-1} & \frac{1}{n} \\ \frac{1}{4r^3} & \frac{1}{4r^2} & \frac{1}{4r} & \cdots & \frac{1}{n-2} & \frac{1}{n-1} & \frac{1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{(n-1)r^{n-2}} & \frac{1}{(n-1)r^{n-3}} & \frac{1}{(n-1)r^{n-4}} & \cdots & \frac{1}{(n-1)r} & \frac{1}{n-1} & \frac{1}{n} \\ \frac{1}{nr^{n-1}} & \frac{1}{nr^{n-2}} & \frac{1}{nr^{n-3}} & \cdots & \frac{1}{nr^2} & \frac{1}{nr} & \frac{1}{n} \end{pmatrix}.$$

We present the next results without proof in order to avoid repetitions of the previous techniques.

Theorem 4.2. *An LU factorization of the matrix* $\mathbb{H}$ *and the inverses of each factor are*

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2r} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3r^2} & \frac{2}{3r} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{4r^3} & \frac{1}{2r^2} & \frac{3}{r} & 1 & \cdots & 0 & 0 & 0 \\ 5r^4 & \frac{2}{5r^3} & \frac{3}{r^2} & \frac{4}{r} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{(n-1)r^{n-2}} & \frac{2}{(n-1)r^{n-3}} & \frac{3}{(n-1)r^{n-4}} & \frac{4}{(n-1)r^{n-5}} & \cdots & \frac{n-2}{(n-1)r} & 1 & 0 \\ \frac{1}{nr^{n-1}} & \frac{2}{nr^{n-2}} & \frac{3}{nr^{n-3}} & \frac{4}{nr^{n-4}} & \cdots & \frac{n-2}{nr^2} & \frac{n-1}{nr} & 1 \end{pmatrix},$$

$$U=\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & \frac{1}{n}\\ 0 & -\frac{1}{4}\frac{2r-1}{r} & -\frac{1}{6}\frac{2r-1}{r} & \cdots & -\frac{1}{2(n-1)}\frac{2r-1}{r} & -\frac{1}{2n}\frac{2r-1}{r}\\ 0 & 0 & -\frac{1}{9}\frac{3r-2}{r} & \cdots & -\frac{1}{3(n-1)}\frac{3r-2}{r} & -\frac{1}{3n}\frac{3r-2}{r}\\ 0 & 0 & 0 & \cdots & -\frac{1}{4(n-1)}\frac{4r-3}{r} & -\frac{1}{4n}\frac{4r-3}{r}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -\frac{1}{(n-1)^2}\frac{(n-1)r-(n-2)}{r} & -\frac{1}{(n-1)}\frac{(n-1)r-(n-2)}{r}\\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{n^2}\frac{nr-(n-1)}{r}\end{pmatrix},$$

$$L^{-1}=\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0\\ -\frac{1}{2r} & 1 & 0 & \cdots & 0 & 0\\ 0 & -\frac{2}{3r} & 1 & \ddots & 0 & 0\\ 0 & 0 & -\frac{3}{4r} & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -\frac{n-1}{nr} & 1\end{pmatrix},$$

$$U^{-1}=\begin{pmatrix}1 & -\frac{2r}{2r-1} & 0 & \cdots & 0 & 0\\ 0 & \frac{4r}{2r-1} & -\frac{6r}{3r-2} & \cdots & 0 & 0\\ 0 & 0 & \frac{9r}{3r-2} & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & \frac{(n-1)^2r}{(n-1)r-(n-2)} & -\frac{n(n-1)r}{nr-(n-1)}\\ 0 & 0 & 0 & \cdots & 0 & \frac{n^2r}{nr-(n-1)}\end{pmatrix}.$$

Theorem 4.3. *The inverse matrix of $\mathbb{H}_{n,r}$ is of the form*

$$\mathbb{H}_{n,r}^{-1}=\begin{pmatrix}\frac{2r}{2r-1} & -\frac{2r}{2r-1} & & & & \\ -\frac{2}{2r-1} & \frac{4(3r^2-1)}{(3r-2)(2r-1)} & -\frac{6r}{3r-2} & & & \\ & -\frac{6}{3r-2} & \frac{18(2r^2-1)}{(4r-3)(3r-2)} & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \frac{(n-1)^2(nr^2-(n-2))}{(nr-(n-1))((n-1)r-(n-2))} & -\frac{n(n-1)r}{nr-(n-1)}\\ & & & & -\frac{n(n-1)}{nr-(n-1)} & \frac{n^2r}{nr-(n-1)}\end{pmatrix}.$$

We may also define an reciprocal analogous concept for $M_{n,r}$.

Definition 4.4. *The reciprocal matrix of $M_{n,r}$ is defined as*

$$\mathcal{H}_{n,r}=\begin{pmatrix}1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\ \frac{1}{r} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{r^2} & \frac{1}{2r} & \frac{1}{3} & \frac{1}{3} & \cdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{r^3} & \frac{1}{2r^2} & \frac{1}{3r} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{r^4} & \frac{1}{2r^3} & \frac{1}{3r^2} & \frac{1}{4r} & \ddots & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{1}{r^{n-2}} & \frac{1}{2r^{n-3}} & \frac{1}{3r^{n-4}} & \frac{1}{4r^{n-5}} & \cdots & \frac{1}{(n-2)r} & \frac{1}{(n-1)} & \frac{1}{(n-1)}\\ \frac{1}{r^{n-1}} & \frac{1}{2r^{n-2}} & \frac{1}{3r^{n-3}} & \frac{1}{4r^{n-4}} & \cdots & \frac{1}{(n-2)r^2} & \frac{1}{(n-1)r} & \frac{1}{n}\end{pmatrix}, \tag{18}$$

and with another representation

$$h_{ij} = \begin{cases} \frac{1}{i}, & i \le j \\ \frac{1}{r^{i-j} j}, & i > j \end{cases} = \frac{1}{r^{\max(i-j,0)} \min(i,j)}.$$

Theorem 4.5. *The determinant of $\mathcal{H}_{n,r}$ is*

$$\det \mathcal{H} = \prod_{s=2}^{n} \left(\frac{(s-1)r - s}{s(s-1)r} \right).$$

Notice that, for $2 \le s \le n$,

$$\det \mathcal{H} = 0,$$

when $r = \frac{s}{s-1}$. Hence, in this case $\mathcal{H}$ is singular.

Theorem 4.6. *An LU factorization of $\mathcal{H}_n$ and the inverses of each factor are:*

$$L = \left(x_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{r} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{r^2} & \frac{1}{r} & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{r^3} & \frac{1}{r^2} & \frac{1}{r} & 1 & \cdots & 0 & 0 & 0 \\ \frac{1}{r^4} & \frac{1}{r^3} & \frac{1}{r^2} & \frac{1}{r} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{r^{n-2}} & \frac{1}{r^{n-3}} & \frac{1}{r^{n-4}} & \frac{1}{r^{n-5}} & \cdots & \frac{1}{r} & 1 & 0 \\ \frac{1}{r^{n-1}} & \frac{1}{r^{n-2}} & \frac{1}{r^{n-3}} & \frac{1}{r^{n-4}} & \cdots & \frac{1}{r^2} & \frac{1}{r} & 1 \end{pmatrix},$$

$$U = \left(y_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -\frac{1}{2}\frac{2-r}{r} & -\frac{1}{2}\frac{2-r}{r} & \cdots & -\frac{1}{2}\frac{2-r}{r} & -\frac{1}{2}\frac{2-r}{r} \\ 0 & 0 & -\frac{1}{6}\frac{3-2r}{r} & \cdots & -\frac{1}{6}\frac{3-2r}{r} & -\frac{1}{6}\frac{3-2r}{r} \\ 0 & 0 & 0 & \cdots & -\frac{1}{12}\frac{4-3r}{r} & -\frac{1}{12}\frac{4-3r}{r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{(n-1)(n-2)}\frac{(n-1)-(n-2)r}{r} & -\frac{1}{(n-1)(n-2)}\frac{(n-1)-(n-2)r}{r} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{n(n-1)}\frac{n-(n-1)r}{r} \end{pmatrix},$$

$$x_{ij} = \begin{cases} \frac{1}{r^{i-j}}, & i \ge j \\ 0, & \textit{otherwise} \end{cases},$$

$$y_{ij} = \begin{cases} -\frac{1}{i(i-1)}\frac{i-(i-1)r}{r}, & j \ge i > 1 \\ 1, & i = j = 1 \\ 0, & \textit{otherwise} \end{cases},$$

$$L^{-1} = \left(\mathbf{x}_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{r} & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{r} & 1 & \ddots & 0 & 0 \\ 0 & 0 & -\frac{1}{r} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{r} & 1 \end{pmatrix},$$

$$U^{-1}=\left(\mathbf{y}_{ij}\right)_{i,j=1}^{n}=\begin{pmatrix}1 & -\frac{2r}{r-2} & 0 & \cdots & 0 & 0\\ 0 & \frac{2r}{r-2} & -\frac{6r}{2r-3} & \cdots & 0 & 0\\ 0 & 0 & \frac{6r}{2r-3} & \ddots & 0 & 0\\ \vdots & \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & \frac{(n-1)(n-2)r}{(n-2)r-(n-1)} & -\frac{n(n-1)r}{(n-1)r-n}\\ 0 & 0 & 0 & \cdots & 0 & \frac{n(n-1)r}{(n-1)r-n}\end{pmatrix}.$$

That is

$$\mathbf{x}_{ij} = \begin{cases} -\frac{1}{r}, & i-j=1\\ 1, & i=j\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathbf{y}_{ij} = \begin{cases} \frac{i(i-1)r}{(i-1)r-i}, & i=j\neq 1\\ 1, & i=j=1\\ \frac{i(i-1)r}{(i-1)r-i}, & j-i=1\\ 0, & \text{otherwise.}\end{cases}$$

Proof. From the matrix multiplication, we have

$$\begin{aligned} LU &= \sum_{d=1}^{\min(i,j)} x_{id}y_{dj} = x_{i1}y_{1j} + \sum_{d=2}^{\min(i,j)} x_{id}y_{dj}\\ &= \frac{1}{r^{i-1}}1 - \sum_{d=2}^{\min(i,j)} \frac{1}{r^{i-d}}\frac{1}{d(d-1)}\frac{d-(d-1)r}{r}\\ &= \frac{1}{r^{i-1}} - \sum_{d=2}^{\min(i,j)} \frac{r^d}{r^i}\left(\frac{d-(d-1)r}{d(d-1)r}\right)\\ &= \frac{1}{r^{i-1}} - \frac{1}{r^i}\sum_{d=2}^{\min(i,j)} r^d\left(\frac{1}{(d-1)r}-\frac{r}{d}\right)\\ &= \frac{1}{r^{i-1}} - \frac{1}{r^i}\sum_{d=2}^{\min(i,j)} \left(\frac{r^{d-1}}{(d-1)}-\frac{r^{d+1}}{d}\right). \end{aligned}$$

So our first assertion is proven and the other equations can be shown similarly. □

Theorem 4.7. *If $\mathcal{H}_{n,r}$ is nonsingular, the inverse of the matrix $\mathcal{H}_{n,r}$ is the tridiagonal matrix:*

$$\left(\mathbf{h}_{ij}\right)_{i,j=1}^{n}=\begin{pmatrix}\frac{r}{r-2} & -\frac{2r}{r-2} & & & & \\ -\frac{2}{r-2} & \frac{4(r^2-3)}{(2r-3)(r-2)} & -\frac{6r}{2r-3} & & & \\ & -\frac{6}{2r-3} & \frac{18(r^2-2)}{(3r-4)(2r-3)} & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \frac{(n-1)^2((n-2)r^2-n)}{((n-1)r-n)((n-2)r-(n-1))} & -\frac{n(n-1)r}{(n-1)r-n}\\ & & & & -\frac{n(n-1)}{((n-1)r-n)} & \frac{n(n-1)r}{((n-1)r-n)}\end{pmatrix},$$

or, in other words,

$$\mathbf{h}_{ij} = \begin{cases} -\frac{ij}{(i-1)r-i}, & i-j=1 \\ -\frac{ijr}{(j-1)r-j}, & j-i=1 \\ \frac{r}{r-2}, & i=j=1 \\ \frac{i^2\left((i-1)r^2-(i+1)\right)}{(ir-(i+1))((i-1)r-i)}, & 1<i=j<n \\ \frac{n(n-1)r}{((n-1)r-n)}, & i=j=n \\ 0, & \textit{otherwise} \end{cases}.$$

From Theorems 2.4 and 4.7, one can get the following relation between the matrices $\mathcal{H}^{-1}$ and M^{-1}.

Corollary 4.8. *The matrices $\mathcal{H}^{-1}$ and M^{-1} are related as:*

$$\mathbf{h}_{ij} = \begin{cases} -\frac{r}{2}\mathfrak{m}_{ji}, & i=j=1 \\ -ij\mathfrak{m}_{ji}, & \textit{otherwise.} \end{cases}$$

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