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The weighted numerical radius in Hilbert C*-modules

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Abstract. In this paper, we introduce the definition of the weighted numerical radius $\Omega_\nu(x)$ for $x \in \mathcal{E}$ by using the linking algebra of a Hilbert C*-modules $\mathcal{E}$, which extends the definition of numerical radius $\Omega(x)$ given by Zamani [Math. Inequal. Appl. 24 (2021), 1017-1030]. Among other results, we show that $\Omega_\nu(x)$ is a norm on $\mathcal{E}$ such that

$$\frac{1}{2}\|x\|_{\mathcal{E}} \leq \max\{\nu, 1-\nu\}\|x\|_{\mathcal{E}} \leq \Omega_\nu(x) \leq \|x\|_{\mathcal{E}},$$

where $0 \leq \nu \leq 1$. In addition, some relevant results are discussed.

1. Introduction and Preliminaries

Hilbert C^*-modules are generalizations of Hilbert spaces that allow the inner product to take values in a C^*-algebra instead of in the complex field. The theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and K-theory [6, 7, 11, 13].

Several mathematicians have studied the fundamental properties of numerical radius for bounded adjointable operators on Hilbert C^*-modules [5, 8, 10]. Although some inequalities in Hilbert C^*-modules can be proved using standard methods, the different structure of Hilbert C^*-modules seems to require different definitions of some concepts that are natural extensions of some standard definitions to study some inequalities in Hilbert C^*-modules.

Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{E}$ is a right $\mathcal{A}$-module. $\mathcal{E}$ is a *pre-Hilbert $\mathcal{A}$-module* if $\mathcal{E}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following properties hold:
(1) $\langle x, x\rangle_{\mathcal{E}} \geq 0$ for all $x \in \mathcal{E}$ and $\langle x, x\rangle_{\mathcal{E}} = 0$ if and only if $x = 0$.
(2) $\langle x, y+z\rangle_{\mathcal{E}} = \langle x, z\rangle_{\mathcal{E}} + \langle y, z\rangle_{\mathcal{E}}$ for every $x, y, z \in \mathcal{E}$.
(3) $\langle x, ya\rangle_{\mathcal{E}} = \langle x, y\rangle_{\mathcal{E}} a$ for every $a \in \mathcal{A}$, $x, y \in \mathcal{E}$.
(4) $\langle x, y\rangle_{\mathcal{E}} = \langle y, x\rangle_{\mathcal{E}}^*$ for every $x, y \in \mathcal{E}$.

For every $x \in \mathcal{E}$, we define $\|x\|_{\mathcal{E}} = \|\langle x, x\rangle_{\mathcal{E}}\|^{\frac{1}{2}}$. If $\mathcal{E}$ is complete with $\|\cdot\|_{\mathcal{E}}$, it is called a *Hilbert $\mathcal{A}$-module* (or a *Hilbert C^*-module* over $\mathcal{A}$). For every $a \in \mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$, and the *$\mathcal{A}$-valued norm* on $\mathcal{H}$ is defined

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by $|x| = \langle x, x\rangle_{\mathcal{E}}^{\frac{1}{2}}$. Further, it was shown in [6, Proposition 1.1] that we also have the following version of the Cauchy-Schwarz inequality:

$$\langle y, x\rangle_{\mathcal{E}}\langle x, y\rangle_{\mathcal{E}} \le \|x\|_{\mathcal{E}}^2\langle y, y\rangle_{\mathcal{E}}, \quad x, y \in \mathcal{E}.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert $\mathcal{A}$-modules. We say that the operator $T : \mathcal{E} \to \mathcal{F}$ is *adjointable* if there is another one $T^* : \mathcal{F} \to \mathcal{E}$ such that $\langle Tx, y\rangle_{\mathcal{E}} = \langle x, T^*y\rangle_{\mathcal{E}}$ for any $x \in \mathcal{E}$ and $y \in \mathcal{F}$. The operator T^* is called the *adjoint operator* of T. Note that an adjointable operator is automatically $\mathcal{A}$-linear and bounded. The family of all adjointable operators from $\mathcal{E}$ to $\mathcal{F}$ is designated as $\mathrm{End}^*_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ abbreviated as $\mathrm{End}^*_{\mathcal{A}}(\mathcal{E})$ if $\mathcal{F} = \mathcal{E}$. Let $\mathbb{K}(\mathcal{E}, \mathcal{F})$ be the closed linear subspace of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ spanned by $\{\theta_{x,y} : x \in \mathcal{E}, y \in \mathcal{F}\}$, where $\theta_{x,y}(z) = x\langle y, z\rangle_{\mathcal{E}}$. The elements of $\mathbb{K}(\mathcal{E}, \mathcal{F})$ are often referred to as "compact" operators. We write $\mathbb{K}(\mathcal{E})$ for $\mathbb{K}(\mathcal{E}, \mathcal{E})$. Given a Hilbert C^*-module $\mathcal{E}$, the *linking algebra* $\mathcal{L}(\mathcal{E})$ is defined as a matrix algebra of the following form

$$\mathcal{L}(\mathcal{E}) = \begin{bmatrix} \mathbb{K}(\mathcal{A}) & \mathbb{K}(\mathcal{E}, \mathcal{A}) \\ \mathbb{K}(\mathcal{A}, \mathcal{E}) & \mathbb{K}(\mathcal{E}) \end{bmatrix}.$$

Then $\mathcal{L}(\mathcal{E})$ has a canonical embedding as a closed subalgebra of the adjointable operators on the Hilbert C^*-module $\mathcal{A} \oplus \mathcal{E}$ via

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}\begin{bmatrix} a \\ \\ x \end{bmatrix} = \begin{bmatrix} Xa + Yx \\ Za + Wx \end{bmatrix},$$

which makes $\mathcal{L}(\mathcal{E})$ a C^*-algebra [12, Lemma 2.32 and Corollary 3.21]. Each $x \in \mathcal{E}$ induces the mappings $r_x \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{A}, \mathcal{E})$ and $l_x \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ given by $r_x(a) = xa$ and $l_x(y) = \langle x, y\rangle_{\mathcal{E}}$, respectively, such that $r_x^* = l_x$. For any $x, y \in \mathcal{E}$, we have $l_{x+y} = l_x + l_y$ and $r_{x+y} = r_x + r_y$. In addition, for every $a \in \mathcal{A}, x \in \mathcal{E}$, we also have $l_{\alpha x} = \overline{\alpha} l_x$ and $r_{\alpha x} = \alpha r_x$. The mapping $x \to r_x$ is an isometric linear isomorphism from $\mathcal{E}$ to $\mathbb{K}(\mathcal{A}, \mathcal{E})$ and $x \to l_x$ is an isometric conjugate linear isomorphism from $\mathcal{E}$ to $\mathbb{K}(\mathcal{E}, \mathcal{A})$. Moreover, each $a \in \mathcal{A}$ induces a mapping given by $T_a(b) = ab$ for $T_a \in \mathbb{K}(\mathcal{A})$. The mapping $a \to T_a$ defines an isomorphism between $\mathcal{A}$ and $\mathbb{K}(\mathcal{A})$. Therefore, we may write

$$\mathcal{L}(\mathcal{E}) = \left\{\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in \mathcal{E}, T \in \mathbb{K}(\mathcal{E})\right\}$$

and identify the C^*-subalgebras of compact operators with the corresponding corners in the linking algebra:

$$\mathbb{K}(\mathcal{A}) = \mathbb{K}(\mathcal{A} \oplus 0) \subseteq \mathbb{K}(\mathcal{A} \oplus \mathcal{E}) = \mathcal{L}(\mathcal{E}), \ \ \mathbb{K}(\mathcal{E}) = \mathbb{K}(0 \oplus \mathcal{E}) \subseteq \mathbb{K}(\mathcal{A} \oplus \mathcal{E}) = \mathcal{L}(\mathcal{E}).$$

For more information on Hilbert C^*-modules and linking algebras, please refer to [6, 7].

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Every operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$ are the real and imaginary parts of T, respectively. For $0 \le \nu \le 1$, we define the weighted real and imaginary parts of $T \in \mathcal{B}(\mathcal{H})$ by $\Re_\nu(T) = \nu T + (1-\nu)T^*$. The *numerical radius* and *operator norm* of an element $T \in \mathcal{B}(\mathcal{H})$ are defined by

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}, \ \ \|T\| = \sup_{\|x\|=1} \|Tx\|,$$

where $\|\cdot\|$ is the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$. These concepts have proven useful in some cases [1, 2, 4, 9, 16, 18]. An important and useful identity for the numerical radius [15] is as follows:

$$\omega(T) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}T)\|.$$

Recently, Sheikhhosseini et al. [14] introduced the so-called *weighted numerical radius*: If $0 \le \nu \le 1$, the weighted numerical radius for $T \in \mathcal{B}(\mathcal{H})$ denoted by $\omega_\nu(T)$ is introduced by

$$\omega_\nu(T) = \sup_{\theta\in\mathbb{R}} \|\Re_\nu(e^{i\theta}T)\|. \tag{1}$$

Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{E}$ is a right $\mathcal{A}$-module. Let $\mathcal{A}^*$ be the *dual space* of $\mathcal{A}$. A *positive linear functional* of $\mathcal{A}$ is a map $\phi \in \mathcal{A}^*$ such that $\phi(a) \geq 0$ whenever $a \geq 0$. Let $\mathcal{S}(\mathcal{A})$ be the set of all positive linear functionals on $\mathcal{A}$ of norm 1. Recall [17] that the *numerical radius* $\Omega(x)$ of $x \in \mathcal{E}$ is defined by

$$\Omega(x) = \sup\left\{\left|\phi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| : \phi \in \mathcal{S}(\mathcal{L}(\mathcal{E}))\right\}.$$

It is known that $\Omega(\cdot)$ is a norm on Hilbert C^*-module $\mathcal{E}$, which is equivalent to the norm $\|\cdot\|_{\mathcal{E}}$. In fact, for every $x \in \mathcal{E}$,

$$\frac{1}{2}\|x\|_{\mathcal{E}} \leq \Omega(x) \leq \|x\|_{\mathcal{E}}. \tag{2}$$

An important and useful identity for the numerical radius is as follows.

Proposition 1.1. *[17, Theorem 2.6] Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Then*

$$\Omega(x) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 0 & e^{-i\theta}l_x \\ e^{i\theta}r_x & 0 \end{bmatrix}\right\|$$

for every $x \in \mathcal{E}$.

Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. For $0 \leq \nu \leq 1$, we defined the *weighted real and imaginary parts* of $\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}$ by

$$\mathfrak{R}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}$$

and

$$\mathfrak{I}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & (1-\nu)il_x \\ -\nu i r_x & 0 \end{bmatrix}.$$

Inspired by (1) and Proposition 1.1, we introduce the following definition, which we call the *weighted numerical radius*.

Definition 1.2. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. For $0 \leq \nu \leq 1$ and $x \in \mathcal{E}$, the weighted numerical radius of x is denoted by $\Omega_\nu(x)$ and is defined as*

$$\Omega_\nu(x) = \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta}l_x \\ \nu e^{i\theta}r_x & 0 \end{bmatrix}\right\|.$$

Since $\mathcal{L}(\mathcal{E})$ is a C^*-algebra, the operator norm $\|\cdot\|$ on $\mathcal{L}(\mathcal{E})$ is self-adjoint. For any $x \in \mathcal{E}$, we have

$$\Omega_0(x) = \Omega_1(x) = \left\|\begin{bmatrix} 0 & e^{-i\theta}l_x \\ 0 & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 & e^{-i\theta}l_x \\ 0 & 0 \end{bmatrix}^*\right\| = \left\|\begin{bmatrix} 0 & 0 \\ e^{i\theta}r_x & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\| = \|x\|_{\mathcal{E}}.$$

For every $x \in \mathcal{E}$, it is easy to see that

$$\Omega_\nu(x) = \sup_{\theta\in\mathbb{R}}\left\|\mathfrak{I}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_{ie^{i\theta}x} & 0 \end{bmatrix}\right)\right\|.$$

In this paper, we first use the linking algebra $\mathcal{L}(\mathcal{E})$ of a Hilbert $\mathcal{A}$-module $\mathcal{E}$ to define the weighted numerical radius of $x \in \mathcal{E}$ and denote it by $\Omega_\nu(x)$. We then show that $\Omega_\nu(\cdot)$ is a norm on $\mathcal{E}$, which is equivalent to the norm $\|\cdot\|_{\mathcal{E}}$ and the following inequalities

$$\frac{1}{2}\|x\|_{\mathcal{E}} \leq \max\{\nu, 1-\nu\}\|x\|_{\mathcal{E}} \leq \Omega_\nu(x) \leq \|x\|_{\mathcal{E}}$$

hold for every $x \in \mathcal{E}$. Furthermore, for $x \in \mathcal{E}, a \in \mathcal{A}$ we prove that

$$\Omega_\nu(xa + xa^*) \leq 2\|a + a^*\|\Omega_\nu(x).$$

In particular, we find some bounds for Ω_ν. The main purpose of this paper is to discuss this definition and the interesting properties that Ω_ν satisfies.

2. Main results

In this section, we present some of our main results. We start with the following main properties of Ω_ν.

Theorem 2.1. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Suppose $0 \leq \nu \leq 1$. Then $\Omega_\nu(\cdot) : \mathcal{E} \to [0, +\infty)$ defines a norm on $\mathcal{E}$.*

Proof. For $x \in \mathcal{E}$, the nonnegativity follows from the fact $\Omega_\nu(x)$ is the supremum of a nonnegative valued function. Assume that $\Omega_\nu(x) = 0$ for all $x \in \mathcal{E}$. If we choose $\nu = 0$, then we have

$$\Omega_0(x) = \left\|\begin{bmatrix} 0 & e^{-i\theta} l_x \\ 0 & 0 \end{bmatrix}\right\| = \|x\|_{\mathcal{E}} = 0.$$

Thus $x = 0$. Hence, we may assume that $\nu \neq 0$. Then by Definition 1.2,

$$\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} = 0$$

for any $\theta \in \mathbb{R}$. Taking $\theta = 0$ and $\theta = \frac{\pi}{2}$, we have

$$\begin{bmatrix} 0 & (1-\nu) l_x \\ \nu r_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -(1-\nu) i l_x \\ \nu i r_x & 0 \end{bmatrix} = 0.$$

Thus

$$\begin{bmatrix} 0 & (1-\nu) l_x \\ \nu r_x & 0 \end{bmatrix} - i \begin{bmatrix} 0 & -(1-\nu) i l_x \\ \nu i r_x & 0 \end{bmatrix} = 2\nu \begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} = 0.$$

Since $\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\| = \|x\|_{\mathcal{E}}$, we get $\|x\|_{\mathcal{E}} = 0$ and therefore $x = 0$. For the triangle inequality, let $x, y \in \mathcal{E}$. Then

$$\begin{aligned}
\Omega_\nu(x+y) &= \sup_{\theta \in \mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_{x+y} \\ \nu e^{i\theta} r_{x+y} & 0 \end{bmatrix}\right\| \\
&= \sup_{\theta \in \mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_y \\ \nu e^{i\theta} r_y & 0 \end{bmatrix}\right\| \\
&\leq \sup_{\theta \in \mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\| + \sup_{\theta \in \mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_y \\ \nu e^{i\theta} r_y & 0 \end{bmatrix}\right\| \\
&= \Omega_\nu(x) + \Omega_\nu(y).
\end{aligned}$$

Let $\alpha = \mathbb{C}$. There exists $\varphi \in \mathbb{R}$ such that $\alpha = |\alpha|e^{i\varphi}$. For any $x \in \mathcal{E}, a \in \mathcal{A}$, we have

$$\begin{aligned}\Omega_\nu(\alpha x) =& \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_{\alpha x} \\ \nu e^{i\theta} r_{\alpha x} & 0 \end{bmatrix} \right\| \\ =& \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta}\overline{\alpha} l_x \\ \nu e^{i\theta}\alpha r_x & 0 \end{bmatrix} \right\| \\ =& \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i(\theta+\varphi)}|\alpha| l_x \\ \nu e^{i(\theta+\varphi)}|\alpha| r_x & 0 \end{bmatrix} \right\| \\ =&|\alpha| \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} \right\| \\ =&|\alpha|\Omega_\nu(x).\end{aligned}$$

This completes the proof. □

Since $\mathcal{L}(\mathcal{E})$ is a C^*-algebra, the operator norm $\|\cdot\|$ on $\mathcal{L}(\mathcal{E})$ is self-adjoint, in the sense that

$$\left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & \nu e^{-i\theta} l_x \\ (1-\nu) e^{i\theta} r_x & 0 \end{bmatrix} \right\|$$

for $x \in \mathcal{E}$. Then we have

Proposition 2.2. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Then*

$$\Omega_\nu(x) = \Omega_{1-\nu}(x)$$

for $x \in \mathcal{E}$.

The inequality (2) is an essential inequality for the numerical radius. This inequality has a satisfactory version of the weighted numerical radius that we see below. It turns out that factor $\frac{1}{2}$ is a special case. Furthermore, we note that the term $\max\{\nu, 1-\nu\}$ appears in many results dealing with operator inequalities and convex functions [3, 19].

Theorem 2.3. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Suppose that $0 \le \nu \le 1$. Then we have*

$$\frac{1}{2}\|x\|_{\mathcal{E}} \le \max\{\nu, 1-\nu\}\|x\|_{\mathcal{E}} \le \Omega_\nu(x) \le \|x\|_{\mathcal{E}} \tag{3}$$

for any $x \in \mathcal{E}$. In particular, the norm $\Omega_\nu(\cdot)$ is equivalent to the norm $\|\cdot\|_{\mathcal{E}}$.

Proof. The first inequality is obvious. For the second inequality, for every $x \in \mathcal{E}$, by Definition 1.2 we have

$$\Omega_\nu(x) = \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} \right\| \ge \left\| \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} \right\|$$

for any $\theta \in \mathbb{R}$. So, by taking $\theta = 0$ and $\theta = -\frac{\pi}{2}$, we conclude that

$$\Omega_\nu(x) \ge \left\| \begin{bmatrix} 0 & (1-\nu) l_x \\ \nu r_x & 0 \end{bmatrix} \right\| \text{ and } \Omega_\nu(x) \ge \left\| \begin{bmatrix} 0 & (1-\nu) i l_x \\ -\nu i r_x & 0 \end{bmatrix} \right\|.$$

Therefore, by the triangle inequality

$$\begin{aligned}
2\Omega_\nu(x) &\geq \left\|\left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right\| + \left\|\left\|\begin{bmatrix} 0 & (1-\nu)il_x \\ -\nu ir_x & 0 \end{bmatrix}\right\|\right\| \\
&\geq \left\|\left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix} + i\begin{bmatrix} 0 & (1-\nu)il_x \\ -\nu ir_x & 0 \end{bmatrix}\right\|\right\| \\
&= 2\nu\left\|\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\|\right\| = 2\nu\|x\|_{\mathcal{E}}.
\end{aligned}$$

Replacing ν with $1-\nu$, we obtain $(1-\nu)\|x\|_{\mathcal{E}} \leq \Omega_{1-\nu}(x) = \Omega_\nu(x)$, which implies that

$$\max\{\nu, 1-\nu\}\|x\|_{\mathcal{E}} \leq \Omega_\nu(x).$$

Also, we have

$$\begin{aligned}
\Omega_\nu(x) &= \sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta}l_x \\ \nu e^{i\theta}r_x & 0 \end{bmatrix}\right\|\right\| \\
&= \sup_{\theta\in\mathbb{R}}\left\|(1-\nu)e^{-i\theta}\begin{bmatrix} 0 & l_x \\ 0 & 0 \end{bmatrix} + \nu e^{i\theta}\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\|\right\| \\
&\leq (1-\nu)\sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & l_x \\ 0 & 0 \end{bmatrix}\right\|\right\| + \nu\sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\|\right\| \\
&= (1-\nu)\left\|\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\|\right\| + \nu\left\|\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\|\right\| = \|x\|_{\mathcal{E}}.
\end{aligned}$$

This completes the proof. □

Since the value of Ω_ν depends on the parameter ν, we further investigate the properties of the function $f(\nu) = \Omega_\nu$.

Proposition 2.4. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Suppose that $0 \leq \nu \leq 1$ and $x \in \mathcal{E}$. Then the function $f(\nu) = \Omega_\nu(x)$ is a convex function on the interval $[0,1]$.*

Proof. Let $x \in \mathcal{E}$ and let $0 \leq \mu, \nu, t \leq 1$, we have

$$\begin{aligned}
& f(t\nu + (1-t)\mu) = \Omega_{t\nu+(1-t)\mu}(x) \\
=& \sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & (1-t\nu-\mu+t\mu)e^{-i\theta}l_x \\ (t\nu+(1-t)\mu)e^{i\theta}r_x & 0 \end{bmatrix}\right\|\right\| \\
=& \sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & t(1-\nu)e^{-i\theta}l_x \\ t\nu e^{i\theta}r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & (1-t)(1-\mu)e^{-i\theta}l_x \\ (1-t)\mu e^{i\theta}r_x & 0 \end{bmatrix}\right\|\right\| \\
\leq& t\sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta}l_x \\ \nu e^{i\theta}r_x & 0 \end{bmatrix}\right\|\right\| + (1-t)\sup_{\theta\in\mathbb{R}}\left\|\left\|\begin{bmatrix} 0 & (1-\mu)e^{-i\theta}l_x \\ \mu e^{i\theta}r_x & 0 \end{bmatrix}\right\|\right\| \\
=& t\Omega_\nu(x) + (1-t)\Omega_\mu(x) \\
=& tf(\nu) + (1-t)f(\mu).
\end{aligned}$$

Therefore, f is convex on $[0,1]$. □

In the following result, we present more elaborated formulas for Ω_ν .

Proposition 2.5. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Suppose that $0 \le \nu \le 1$ and $x \in \mathcal{E}$. Then*
(i) $\Omega_\nu(x) = \sup_{\alpha^2+\beta^2=1} \left\| \alpha \mathfrak{R}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) + \beta \mathfrak{I}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\|$.
(ii) $\Omega_\nu(x) = \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)(e^{i\theta}-ie^{i\varphi})^* l_x \\ \nu(e^{i\theta}-ie^{i\varphi})r_x & 0 \end{bmatrix}\right\|$.

Proof. (i) For any $x \in \mathcal{E}$, we have

$$\begin{aligned}
\Omega_\nu(x) &= \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\| \\
&= \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)(\cos\theta - i\sin\theta) l_x \\ \nu(\cos\theta + i\sin\theta) r_x & 0 \end{bmatrix}\right\| \\
&= \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)\cos\theta l_x \\ \nu\cos\theta r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -(1-\nu)i\sin\theta l_x \\ \nu i\sin\theta r_x & 0 \end{bmatrix}\right\| \\
&= \sup_{\theta\in\mathbb{R}} \left\|\cos\theta \begin{bmatrix} 0 & (1-\nu) l_x \\ \nu r_x & 0 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & (1-\nu)i l_x \\ -\nu i r_x & 0 \end{bmatrix}\right\| \\
&= \sup_{\alpha^2+\beta^2=1} \left\|\alpha \begin{bmatrix} 0 & (1-\nu) l_x \\ \nu r_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & (1-\nu)i l_x \\ -\nu i r_x & 0 \end{bmatrix}\right\| \\
&= \sup_{\alpha^2+\beta^2=1} \left\| \alpha \mathfrak{R}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) + \beta \mathfrak{I}_\nu\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\|.
\end{aligned}$$

(ii) For any $x \in \mathcal{E}$, we have

$$\begin{aligned}
\Omega_\nu(x) &= \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & (1-\nu)e^{-i(\theta+\frac{\pi}{2})} i l_x \\ -\nu e^{i(\theta+\frac{\pi}{2})} i r_x & 0 \end{bmatrix}\right\| \\
&\le \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & (1-\nu)e^{-i\varphi} i l_x \\ -\nu e^{i\varphi} i r_x & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)(e^{i\theta}-ie^{i\varphi})^* l_x \\ \nu(e^{i\theta}-ie^{i\varphi}) r_x & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu) l_{(e^{i\theta}-ie^{i\varphi})x} \\ \nu r_{(e^{i\theta}-ie^{i\varphi})x} & 0 \end{bmatrix}\right\| \\
&\le \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \Omega_\nu((e^{i\theta}-ie^{i\varphi})x) = \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} |e^{i\theta}-ie^{i\varphi}|\Omega_\nu(x) \\
&= \frac{\Omega_\nu(x)}{2}\sup_{\theta,\varphi\in\mathbb{R}} \sqrt{2-2\sin(\theta-\varphi)} = \Omega_\nu(x),
\end{aligned}$$

and thus

$$\Omega_\nu(x) = \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\|\begin{bmatrix} 0 & (1-\nu)(e^{i\theta}-ie^{i\varphi})^* l_x \\ \nu(e^{i\theta}-ie^{i\varphi}) r_x & 0 \end{bmatrix}\right\|.$$

□

3. Some upper and below bounds for $\Omega_\nu(\cdot)$

So far, we have given the basic inequalities for the weighted numerical radius. In this section, we intend to present more detailed inequalities.

Theorem 3.1. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and let $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. For $0 \le \nu \le 1, x \in \mathcal{E}$, the following inequality holds:*

$$\frac{(1+|2\nu-1|)(1+\nu)}{4}\|x\|_{\mathcal{E}} + \frac{1+|2\nu-1|}{8}(\Delta+\Delta') + \frac{(1+|2\nu-1|)}{4}|\Gamma-\Gamma'| \le \Omega_\nu(x),$$

where $\Gamma = \max\left\{\|x\|_{\mathcal{E}}, \left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right\}$, $\Gamma' = \max\left\{\|x\|_{\mathcal{E}}, \left\|\begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right\}$,
$\Delta = \left|\|x\|_{\mathcal{E}} - \left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right|$ *and* $\Delta' = \left|\|x\|_{\mathcal{E}} - \left\|\begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right|$.

Proof. Since $\Omega_\nu(x) = \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta}l_x \\ \nu e^{i\theta}r_x & 0 \end{bmatrix}\right\|$, by taking $\theta = 0$ and $\theta = \frac{\pi}{2}$, we have

$$\Omega_\nu(x) \ge \left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\| \text{ and } \Omega_\nu(x) \ge \left\|\begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|. \tag{4}$$

So, by (3) and (4), we have $\Omega_\nu(x) \ge \frac{1+|2\nu-1|}{2}\max\{\Gamma,\Gamma'\}$. Therefore,

$$\begin{aligned}
\Omega_\nu(x) &\ge \frac{(1+|2\nu-1|)(\Gamma+\Gamma')}{4} + \frac{(1+|2\nu-1|)(|\Gamma-\Gamma'|)}{4} \\
&= \frac{1+|2\nu-1|}{4}\left(\frac{1}{2}\left(\|x\|_{\mathcal{E}} + \left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right) + \frac{1}{2}\Delta\right) \\
&+ \frac{1+|2\nu-1|}{4}\left(\frac{1}{2}\left(\|x\|_{\mathcal{E}} + \left\|\begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right) + \frac{1}{2}\Delta'\right) \\
&+ \frac{(1+|2\nu-1|)(|\Gamma-\Gamma'|)}{4} \\
&= \frac{1+|2\nu-1|}{8}\left(\left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right) \\
&+ \frac{1+|2\nu-1|}{4}\|x\|_{\mathcal{E}} + \frac{1+|2\nu-1|}{8}(\Delta+\Delta') + \frac{(1+|2\nu-1|)(|\Gamma-\Gamma'|)}{4} \\
&\ge \frac{1+|2\nu-1|}{8}\left(\left\|\begin{bmatrix} 0 & (1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -(1-\nu)l_x \\ \nu r_x & 0 \end{bmatrix}\right\|\right) \\
&+ \frac{1+|2\nu-1|}{4}\|x\|_{\mathcal{E}} + \frac{1+|2\nu-1|}{8}(\Delta+\Delta') + \frac{(1+|2\nu-1|)(|\Gamma-\Gamma'|)}{4} \\
&= \frac{(1+|2\nu-1|)(1+\nu)}{4}\|x\|_{\mathcal{E}} + \frac{1+|2\nu-1|}{8}(\Delta+\Delta') + \frac{(1+|2\nu-1|)}{4}|\Gamma-\Gamma'|.
\end{aligned}$$

Thus

$$\frac{(1+|2\nu-1|)(1+\nu)}{4}\|x\|_{\mathcal{E}} + \frac{1+|2\nu-1|}{8}(\Delta+\Delta') + \frac{(1+|2\nu-1|)}{4}|\Gamma-\Gamma'| \le \Omega_\nu(x).$$

□

Theorem 3.2. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $\mathcal{L}(\mathcal{E})$ be the linking algebra of $\mathcal{E}$. Let $0 \le \nu \le 1$, $a \in \mathcal{A}$ and $x \in \mathcal{E}$. Then*

$$\Omega_\nu(xa + xa^*) \le 2\|a + a^*\|\Omega_\nu(x).$$

Proof. For any $b \in \mathcal{A}$ and $y \in \mathcal{E}$, we have

$$r_{xa}(b) = (xa)b = x(T_a(b)) = r_x T_a(b)$$

and

$$l_{xa}(y) = \langle xa, y\rangle_{\mathcal{E}} = a^*\langle x, y\rangle_{\mathcal{E}} = a^*(l_x(y)) = T_{a^*} l_x(y).$$

Hence $r_{xa} = r_x T_a$ and $l_{xa} = T_{a^*} l_x$. Therefore, let $\theta \in \mathbb{R}$,

$$\begin{aligned}
&\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_{xa+xa^*} \\ \nu e^{i\theta} r_{xa+xa^*} & 0 \end{bmatrix}\right\|\right\| \\
=&\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta}(T_{a^*} l_x + T_a l_x) \\ \nu e^{i\theta}(r_x T_a + r_x T_{a^*}) & 0 \end{bmatrix}\right\|\right\| \\
=&\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} T_{a+a^*} l_x \\ \nu e^{i\theta} r_x T_{a+a^*} & 0 \end{bmatrix}\right\|\right\| \\
=&\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\|\right\| \\
\leq&\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\|\right\|\left\|\left\|\begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix}\right\|\right\| \\
&+\left\|\left\|\begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix}\right\|\right\|\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_x \\ \nu e^{i\theta} r_x & 0 \end{bmatrix}\right\|\right\| \\
\leq& 2\Omega_\nu(x)\left\|\left\|\begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix}\right\|\right\| = 2\|a + a^*\|\Omega_\nu(x).
\end{aligned}$$

so

$$\left\|\left\|\begin{bmatrix} 0 & (1-\nu)e^{-i\theta} l_{xa+xa^*} \\ \nu e^{i\theta} r_{xa+xa^*} & 0 \end{bmatrix}\right\|\right\| \leq 2\|a + a^*\|\Omega_\nu(x).$$

Taking the supremum over $\theta \in \mathbb{R}$ in the above inequality, we deduce that

$$\Omega_\nu(xa + xa^*) \leq 2\|a + a^*\|\Omega_\nu(x).$$

□

As a direct consequence of Theorem 3.2, we obtain the following result.

Corollary 3.3. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $0 \leq \nu \leq 1$. Let $a \in \mathcal{A}$ and $x \in \mathcal{E}$. If $xa = xa^*$, then*

$$\Omega_\nu(xa) \leq \|a + a^*\|\Omega_\nu(x).$$

Remark 3.4. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $0 \leq \nu \leq 1$. Let $a \in \mathcal{A}$ and $x \in \mathcal{E}$. Replace a by ia in Theorem 3.2, we obtain*

$$\Omega_\nu(xa - xa^*) \leq 2\|a - a^*\|\Omega_\nu(x).$$

Thus

$$\Omega_\nu(xa \pm xa^*) \leq 2\|a \pm a^*\|\Omega_\nu(x).$$

Using Proposition 2.5, we obtain the following upper bound for the weighted numerical radius of elements in a Hilbert $\mathcal{A}$-module.

Theorem 3.5. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $0 \leq \nu \leq 1$. Then*

$$\Omega_\nu(x) \leq \frac{\sqrt{2 + 4\nu(\nu - 1)}}{2} \inf_{\varphi\in\mathbb{R}}\left(\left\|\left\|\begin{bmatrix} 0 & e^{-i\varphi} l_x \\ e^{i\varphi} r_x & 0 \end{bmatrix}\right\|\right\|^2 + \left\|\left\|\begin{bmatrix} 0 & ie^{-i\varphi} l_x \\ -ie^{i\varphi} r_x & 0 \end{bmatrix}\right\|\right\|^2\right)^{\frac{1}{2}}$$

for every $x \in \mathcal{E}$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$. Then clearly

$$\left\| \alpha \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} + i(2\nu - 1)\left(\alpha \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right) \right\|$$
$$= \left\| c \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + d \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|,$$

where $c = \alpha - \beta(2\nu - 1)i$ and $d = \beta + \alpha(2\nu - 1)i$. Using triangle and Cauchy-Schwarz inequalities, respectively, we have

$$\left\| c \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + d \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|$$
$$\leq |c| \left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\| + |d| \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|$$
$$\leq (|c|^2 + |d|^2)^{\frac{1}{2}} \left(\left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.$$

A simple calculation implies $|c|^2 + |d|^2 = 2 + 4\nu(\nu - 1)$. Hence, we have

$$\left\| \alpha \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} + i(2\nu - 1)\left(\alpha \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right) \right\|$$
$$= 2 \left\| \alpha \begin{bmatrix} 0 & (1 - \nu)l_x \\ \nu r_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & (1 - \nu)il_x \\ -\nu ir_x & 0 \end{bmatrix} \right\|$$
$$= 2 \left\| \alpha \mathfrak{R}_\nu \left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} \right) + \beta \mathfrak{I}_\nu \left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} \right) \right\|$$
$$\leq (2 + 4\nu(\nu - 1))^{\frac{1}{2}} \left(\left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.$$

By Proposition 2.5(1), taking the supremum over all $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$, then

$$\Omega_\nu(x) \leq \frac{\sqrt{2 + 4\nu(\nu - 1)}}{2} \left(\left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.$$

Replacing x by $e^{i\varphi}x$, we get the desired result. □

In the next theorem, we obtain another upper bound for the weighted numerical radius of elements in a Hilbert $\mathcal{A}$-module.

Theorem 3.6. *Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $0 \leq \nu \leq 1$. Then*

$$\Omega_\nu(x) \leq \frac{1}{2} \inf_{\varphi \in \mathbb{R}} \left(\left\| \begin{bmatrix} 0 & e^{-i\varphi}l_x \\ e^{i\varphi}r_x & 0 \end{bmatrix} + i(2\nu - 1) \begin{bmatrix} 0 & ie^{-i\varphi}l_x \\ -ie^{i\varphi}r_x & 0 \end{bmatrix} \right\|^2 \right.$$
$$\left. + \left\| \begin{bmatrix} 0 & ie^{-i\varphi}l_x \\ -ie^{i\varphi}r_x & 0 \end{bmatrix} - i(2\nu - 1) \begin{bmatrix} 0 & e^{-i\varphi}l_x \\ e^{i\varphi}r_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}$$

for every $x \in \mathcal{E}$.

Proof. For $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}
&\left\| \alpha \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} + i(2\nu-1)\left(\alpha \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right) \right\| \\
=&2 \left\| \alpha \mathfrak{R}_\nu \left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} \right) + \beta \mathfrak{I}_\nu \left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} \right) \right\| \\
=&\left\| \alpha \left(\begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + i(2\nu-1) \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right) + \beta \left(\begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - i(2\nu-1) \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right) \right\| \\
\leq&|\alpha| \left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + i(2\nu-1) \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\| + |\beta| \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - i(2\nu-1) \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\| \\
\leq&(\alpha^2+\beta^2)^{\frac{1}{2}} \left(\left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + i(2\nu-1) \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - i(2\nu-1) \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum over $\alpha^2 + \beta^2 = 1$, then by Proposition 2.5(1),

$$\Omega_\nu(x) \leq \frac{1}{2} \left(\left\| \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + i(2\nu-1) \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 0 & il_x \\ -ir_x & 0 \end{bmatrix} - i(2\nu-1) \begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.$$

Now, replacing x by $e^{i\varphi}x$, we can obtain

$$\begin{aligned}
\Omega_\nu(x) \leq \frac{1}{2} \inf_{\varphi \in \mathbb{R}} &\left(\left\| \begin{bmatrix} 0 & e^{-i\varphi} l_x \\ e^{i\varphi} r_x & 0 \end{bmatrix} + i(2\nu-1) \begin{bmatrix} 0 & ie^{-i\varphi} l_x \\ -ie^{i\varphi} r_x & 0 \end{bmatrix} \right\|^2 \right. \\
&\left. + \left\| \begin{bmatrix} 0 & ie^{-i\varphi} l_x \\ -ie^{i\varphi} r_x & 0 \end{bmatrix} - i(2\nu-1) \begin{bmatrix} 0 & e^{-i\varphi} l_x \\ e^{i\varphi} r_x & 0 \end{bmatrix} \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

□

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