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Singular solutions of a fractional Dirichlet problem in a punctured domain

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Abstract. Let D be a bounded regular domain in $\mathbb{R}^n$ $(n \geq 3)$ containing 0, $0 < \alpha < 2$, and $\sigma < 1$. We take up in this article the existence and asymptotic behavior of a positive continuous solution for the following semi-linear fractional differential equation

$$(-\Delta|_D)^{\frac{\alpha}{2}} u = a(x)u^{\sigma}(x) \ \text{ in } \ D\backslash\{0\},$$

with the boundary Dirichlet conditions

$$\lim_{|x|\to 0} |x|^{n-\alpha} u(x) = 0 \text{ and } \lim_{x\to\partial D} \delta(x)^{2-\alpha} u(x) = 0,$$

where $(-\Delta|_D)^{\frac{\alpha}{2}}$ is the fractional Laplace associated to the subordinate killed Brownian motion process in D and $\delta(x) = \text{dist}\,(\,x,\ \partial D)$ denotes the Euclidean distance between x and ∂D. The function a is a positive continuous function in $D\backslash\{0\}$, which may be singular at $x = 0$ and/or at the boundary ∂D satisfying some appropriate assumption related to Karamata class. More precisely, we shall prove the existence and global asymptotic behavior of a positive continuous solution on $\overline{D}\backslash\{0\}$. We will use some potential theory arguments and Karamata regular variation theory tools.

1. Introduction

In the present work, we consider the following semilinear singular fractional problem

$$\begin{cases} (-\Delta|_D)^{\frac{\alpha}{2}} u = a(x)u^{\sigma}(x), \ x \in D\backslash\{0\}, \text{ (in the distributional sense,)} \\ u > 0 \text{ in } D\backslash\{0\}, \\ \lim\limits_{|x|\to 0} |x|^{n-\alpha} u(x) = 0, \\ \lim\limits_{x\to\partial D} \delta(x)^{2-\alpha} u(x) = 0, \end{cases} \tag{1.1}$$

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where D is a bounded $C^{1,1}-$domain in $\mathbb{R}^n$, $n \geq 3$, $0 < \alpha < 2$, $\sigma < 1$, $\delta(x) = \text{dist}\,(x,\ \partial D)$ denotes the Euclidean distance between x and ∂D and the positive continuous function a is required to satisfy some conditions related to the Karamata class $\mathcal{K}_0$ defined as follows

Definition 1.1. *[11] A function L defined on* $(0,\ \eta]$, $\eta > 0$ *belongs to the Karamata class* $\mathcal{K}_0$ *if*

$$L(t) := c\exp\left(\int_t^\eta \frac{v(s)}{s}\,ds\right),$$

where $c > 0$ *and* $v \in C([0,\ \eta])$ *with* $v(0) = 0$.

As a standard example of a function L belonging to $\mathcal{K}_0$, we quote

$$L(t) = c\prod_{k=1}^{p}(\ln_k(\frac{\psi}{t}))^{-\nu_k},$$

where $p \in \mathbb{N}^*$, $(\nu_1, \nu_2,\ \ldots\ \nu_p) \in \mathbb{R}^p$, $c > 0$, ψ is a sufficiently large positive real number and $\ln_k(x) = \ln o \ln o \ldots o \ln x$ (k times).

Recently, Fractional Dirichlet problems find a big interest by many researchers due to their real-world applications in many domains such as finance, engineering, science, economics, thermal elasticity, etc. As a motivation of our study, we give a short historical account. Both fractional Dirichlet problem and its elliptic counterpart have been widely studied, we cite works [1, 2, 6, 8, 12, 14, 16, 21]. Namely, in [1], Bachar et al., using Karamata's theory and Schauder's fixed point theorem, showed the existence of a continuous positive solution of the following problem

$$\begin{cases} (-\Delta)u = g(x)u^\sigma(x),\ x \in D\backslash\{0\}, \\ u > 0 \text{ in } D\backslash\{0\}, \\ \lim\limits_{|x|\to 0} |x|^{n-2}u(x) = 0, \\ \lim\limits_{x\to\partial D} u(x) = 0, \end{cases}$$

where $\sigma < 1$ and g is required to satisfy suitable assumptions related to the Karamata class $\mathcal{K}_0$.

In [14], Mâagli and Zribi, investigated the existence, the uniqueness, and the asymptotic behavior of a positive continuous solution of the following fractional problem

$$\begin{cases} (-\Delta|_D)^{\frac{\alpha}{2}}u = k(x)u^\sigma(x),\ x \in D, \\ u > 0 \text{ in } D, \\ \lim\limits_{x\to\partial D} \delta(x)^{2-\alpha}u(x) = 0, \end{cases}$$

where $\sigma < 1$ and k is a positive measurable function in D satisfying, for $x \in D$,

$$\frac{1}{c}\delta(x)^{-\lambda}L(\delta(x)) \leq k(x) \leq c\delta(x)^{-\lambda}L(\delta(x)),$$

with $c > 0$, $\lambda < \alpha + (2-\alpha)(1-\sigma)$ and $L \in \mathcal{K}_0$ defined on $(0,\ \eta]$, $\eta > \text{diam}(D)$. This result is already an improvement of work [8] for the case where the nonlinear term $\varphi(.,u)$ implies as a typical example $\varphi(x,u) = k(x)u^\sigma(x)$, $\sigma < 0$.

In this work, we shall prove the existence and give a global behavior of positive continuous solutions to our problems (1.1). We remark here that we are essentially inspired by the works [1] and [14]. In fact, we will extend the result of [1] to problem (1.1). We point out that Bachar et al.'s proofs in [1] carry over to some proof's here, quite nicely.

Throughout this paper, we suppose that the function a satisfies the following hypothesis.

(**H**) a is a positive continuous function in $D\backslash\{0\}$ satisfying

$$a(x) \approx |x|^{-\lambda}L_1(|x|)\delta(x)^{-\xi}L_2(\delta(x)), \tag{1.2}$$

where $\lambda < (n-2+\alpha)(1-\sigma)+\alpha\sigma$, $\xi < \alpha$ and $L_1, L_2 \in \mathcal{K}_0$ defined on $(0, \eta)$, $\eta > d$, $d =$ diam(D).
We remark that under the hypotheses (**H**) we have,

$$\int_0^\eta s^{(n-2+\alpha)(1-\sigma)+\alpha\sigma-\lambda-1} L_1(s)\, ds < \infty \text{ and } \int_0^\eta s^{\alpha-\xi-1} L_2(s)\, ds < \infty.$$

Hereinafter, for two nonnegative functions f and g defined on a set Ω, the notation $f(x) \approx g(x)$ for $x \in \Omega$ means that there exists $c > 0$ such that $\frac{1}{c} g(x) \leq f(x) \leq g(x)$, for all $x \in \Omega$.
Further, for $\sigma < 1$, we put Θ the function defined on $\overline{D}\backslash\{0\}$ by

$$\Theta(x) := |x|^{\min(0, \frac{\alpha-\lambda}{1-\sigma})} (\tilde{L}_1)^{\frac{1}{1-\sigma}}(|x|) \delta(x)^{\min(1, \frac{\alpha-\xi}{1-\sigma})} (\tilde{L}_2)^{\frac{1}{1-\sigma}}(\delta(x)), \tag{1.3}$$

where $\tilde{L}_1(t)$ and $\tilde{L}_2(t)$ are defined on $(0, \eta)$ as follows

$$\tilde{L}_1(t) := \begin{cases} 1, & \text{if } \lambda < \alpha, \\ \int_t^\eta \frac{L_1(s)}{s}\, ds, & \text{if } \lambda = \alpha, \\ L_1(t), & \text{if } \alpha < \lambda < (n-2+\alpha)(1-\sigma)+\alpha\sigma. \end{cases} \qquad \tilde{L}_2(t) := \begin{cases} 1, & \text{if } \xi < \alpha - 1 + \sigma, \\ \int_t^\eta \frac{L_2(s)}{s}\, ds, & \text{if } \xi = \alpha - 1 + \sigma, \\ L_2(t), & \text{if } \alpha - 1 + \sigma < \xi < \alpha. \end{cases}$$

Our main result is the following

Theorem 1.2. *Let $\sigma < 1$ and assume that the function a satisfies (H). Then problem (1.1) has at least one positive continuous solution $u \in \overline{D}\backslash\{0\}$ satisfying*

$$u(x) \approx \Theta(x), \text{ for } x \in \overline{D}\backslash\{0\}. \tag{1.4}$$

Our approach to prove Theorem 1.2 relies on potential theory tools associated to the operator $(-\Delta|_D)^{\frac{\alpha}{2}}$ developed in some recent papers. Also Karamata's theory and the functional class $\mathbf{K}_\alpha$ called fractional Kato class, plays a key role in our study.
The outline of this paper is organized as follows. In the next Section, we recall some potential theory tools pertaining to the fractional operator. Then we collect a basic properties of functions in $\mathbf{K}_\alpha$ and in the class $\mathcal{K}_0$. In Section 3, we state some technical lemmas and we prove the asymptotic behavior of potential functions. The last section is devoted to establish existence of solutions of problem (1.1) and will give an example to illustrate our results.
We close this section by giving the following notation. For all $s, t \in \mathbb{R}$ we denotes $s \vee t = \max(s, t)$ and $s \wedge t = \min(s, t)$. Further, We point out that $\mathcal{B}(D)$ is the set of all measurable functions on D and $\mathcal{B}^+(D)$ is the subset of non-negative measurable functions on D. Analogously, $C(D)$ is the class of all continuous functions in D and let $C_0(D)$ be the subclass of $C(D)$ consisting of functions which vanish continuously on ∂D. The letter c will denote a generic positive constant which may vary from line to line and d denotes the diameter of D.

2. Preliminaries

This section is dedicated to the presentation of the main tools that we will use throughout the paper.

2.1. Green's function

In this paragraph, we recall sharp inequalities for the Green function $G_\alpha^D(x, y)$. The following estimates are stated in [18]

$$G_\alpha^D(x, y) \approx \frac{1}{|x-y|^{n-\alpha}} \left(1 \wedge \frac{\delta(y)\delta(x)}{|x-y|^2}\right), \quad x, y \in D. \tag{2.1}$$

For $\alpha = 2$, we refind the estimation, stated in [20], for the classical Green function

$$G^D(x,y) \approx \frac{1}{|x-y|^{n-2}}\left(1 \wedge \frac{\delta(y)\delta(x)}{|x-y|^2}\right), \; x, \; y \in D.$$

Moreover, by [8], we have

$$G^D_\alpha(x,y) \approx |x-y|^{\alpha-2} G^D(x,y). \tag{2.2}$$

Proposition 2.1. *[8, Proposition 4] Let $x, y \in D$. Then*

$$G^D_\alpha(x,y) \approx \frac{\delta(x)\delta(y)}{|x-y|^{n-\alpha}\left(|x-y|^2 + \delta(x)\delta(y)\right)}, \tag{2.3}$$

and

$$\delta(x)\delta(y) \le \; c\, G^D_\alpha(x,y).$$

Moreover, if $|x-y| \ge r$ then

$$G^D_\alpha(x,y) \le c\frac{\delta(x)\delta(y)}{r^{n+2-\alpha}}. \tag{2.4}$$

Recall that for each $f \in \mathcal{B}^+(D)$, the potential kernel Vf is defined by

$$Vf(x) = \int_D G^D_\alpha(x,y) f(y)\, dy, \; x \in D.$$

We now briefly recall known notions. Let $X = (D, F, F_t, X_t, \theta_t, P_x)$ an $n-$dimensional F_t-Brownian motion, and $T = (D, G, T_t, P_x)$ is an $\frac{\alpha}{2}-$stable subordinator starting at zero, $0 < \alpha < 2$, independent of X for every P_x. It well known that $Y_t = X_{T_t}$ is a rotationally invariant $\alpha-$stable process whose infinitesimal generator is $-(-\Delta)^{\frac{\alpha}{2}}$ and Y^D the one killed on exiting D. The subordinate killed Brownian motion Z^D_α corresponding to the fractional power $-(-\Delta|_D)^{\frac{\alpha}{2}}$ as the process obtained by subordinating the killed Brownian motion X^D at τ_D, the first exit time of X from D, via T_t. From [4, page 187 – 198], the kernel $x \to \left|x-y\right|^{\alpha-n}$ is $\alpha-$harmonic function in $R^n \setminus \{y\}$ with respect to Y and the function $x \mapsto |x|^{\alpha-n}$ is $\alpha-$superharmonic with respect to Y. By [7, Remark 2.1], the function $h(x) = |x|^{\alpha-n}$ in D, and 0 outside D is $\alpha-$superharmonic in D with respect to Y^D. Since, from [9] and [19], we know that the killed subordinate process Z^D_α has a shorter lifetime than the subordinate killed process Y^D. Then, the function $h(x)$ is $\alpha-$superharmonic with respect to Z^D_α. Moreover, it is worth to recall that for every $f \in \mathcal{B}^+(D)$, Vf is $\alpha-$superharmonic function in D with respect to Z^D_α.
Let $f \in \mathcal{B}^+(D)$. By [9], we have $Vf \neq \infty$ if and only if $\int_D \delta(y) f(y)\, dy < \infty$. Furthermore, as in the classical case, it was shown in [9, 10], that, for any $f \in \mathcal{B}^+(D)$ such that $f, Vf \in L^1_{loc}(D)$, we have in the distributional sense

$$(-\Delta|_D)^{\frac{\alpha}{2}}\, Vf = f, \text{ in } \; D \tag{2.5}$$

Remark 2.2. *We remark that the above result remains true in $D \setminus \{0\}$. More precisely, for any $f \in \mathcal{B}^+(D \setminus \{0\})$ such that $f, Vf \in L^1_{loc}(D \setminus \{0\})$, we have*

$$(-\Delta|_D)^{\frac{\alpha}{2}}\, Vf = f, \text{ in } \; D \setminus \{0\}, \text{ in the distributional sense.} \tag{2.6}$$

Indeed, it is enough to extend the function f by 0 for $x = 0$ and use (2.5).

The potential kernel V satisfies the complete maximum principle (see for intense [4, Chapter 2, Proposition 7.1]). The following Lemma is due to [8].

Lemma 2.3. *Let $h \in \mathcal{B}^+(D)$ and v be a nonnegative $\alpha-$superharmonic function. Let w be a Borel measurable function in D such that $V(h|w|) < \infty$ and $v = w + V(hw)$. Then w satisfies*

$$0 \le w \le v.$$

2.2. Kato class $\mathbf{K}_\alpha(D)$

In this paragraph, we gather some properties of functions belonging to the Kato class $\mathbf{K}_\alpha(D)$, for more details, we refer the reader to [6, 8, 14]. Let us recall the definition of $\mathbf{K}_\alpha(D)$.

Definition 2.4. *A Borel measurable function q in D belongs to the Kato class $\mathbf{K}_\alpha(D)$ if q satisfies the following condition*

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)|q(y)|\,dy\right)=0. \tag{2.7}$$

As a typical example of functions in $\mathbf{K}_\alpha(D)$, we quote $q(x)=\delta(x)^{-\lambda},\ \lambda<\alpha$.

Now, we state the following result which is a consequence of [8, Proposition 10].

Proposition 2.5.
Let q be a function in $\mathbf{K}_\alpha(D)$. Then for any $\alpha-$superharmonic function h in D with respect to Z_α^D, we have for $x_0\in\overline{D}$

$$\lim_{r\to 0}\left(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap B(x_0,r)}G_\alpha^D(x,y)|q(y)|h(y)\,dy\right)=0. \tag{2.8}$$

Remark 2.6. *If $q\in\mathbf{K}_\alpha(D)$, then the function $x\mapsto\delta(x)^{\alpha-1}q(x)$ is in $L^1(D)$, see [8, Corollary 2].*

Next, we state the following interesting proposition.

Proposition 2.7. *Let $q\in\mathbf{K}_\alpha(D)$, then the function $v(x)$ defined by*

$$v(x):=|x|^{n-\alpha}\int_D G_\alpha^D(x,y)|y|^{\alpha-n}q(y)\,dy \text{ is in } C_0(\overline{D}).$$

Proof. Let $q\in\mathbf{K}_\alpha(D)$, $\epsilon>0$ and $x_0\in\overline{D}$. Since $h(x)=\begin{cases}|x|^{\alpha-n}, & x\in D\\ 0, & x\notin D\end{cases}$ is an $\alpha-$superharmonic function in D with respect to Z_α^D, then using (2.8), there exists $r>0$ such that

$$\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B(0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\le\frac{\epsilon}{8}$$

and

$$\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B^c(0,r)\cap B(x_0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\le\frac{\epsilon}{8}.$$

We distinguish tow cases

Case 1. Let $x_0\in D$ and $x\in B(x_0,\frac{r}{2})\cap D$. Then

$$\begin{aligned}
|v(x)-v(x_0)|&\le|x|^{n-\alpha}\int_D|G_\alpha^D(x,y)|y|^{\alpha-n}|q(y)||\,dy+|x_0|^{n-\alpha}\int_D|G_D^\alpha(x_0,y)|y|^{\alpha-n}|q(y)||\,dy\\
&\le 2\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B(0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\\
&+2\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B^c(0,r)\cap B(x_0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\\
&+\int_{D\cap B^c(0,r)\cap B^c(x_0,r)}\Big||x|^{n-\alpha}G_\alpha^D(x,y)-|x_0|^{n-\alpha}G_\alpha^D(x,y)\Big||y|^{\alpha-n}|q(y)|\,dy\\
&\le\frac{\epsilon}{2}+\int_{D\cap B^c(0,r)\cap B^c(x_0,r)}\Big||x|^{n-\alpha}G_\alpha^D(x,y)-|x_0|^{n-\alpha}G_\alpha^D(x,y)\Big||y|^{\alpha-n}|q(y)|dy.
\end{aligned}$$

Let $y \in D_0 = D \cap B^c(0,r) \cap B^c(x_0,r)$, then

$$|x-y| \geq |y-x_0| - |x-x_0| \geq \frac{r}{2} \text{ and } |y| \geq r,$$

so, using (2.4), we obtain

$$\begin{aligned}|x|^{n-\alpha}G_\alpha^D(x,y)|y|^{\alpha-n} &\leq c\frac{d\delta(y)^{\alpha-1}\delta(y)^{2-\alpha}d^{n-\alpha}r^{\alpha-n}}{(r)^{n-\alpha+2}}\\ &\leq cd^{n-2\alpha+3}r^{2\alpha-2n-2}\delta(y)^{\alpha-1}\\ &\leq c\delta(y)^{\alpha-1}.\end{aligned}$$

By Remark 2.6, it follows

$$|x|^{n-\alpha}G_\alpha^D(x,y)|y|^{\alpha-n}|q(y)| \leq c\,\delta(y)^{\alpha-1}|q(y)| \in L^1(D_0).$$

Since $|x|^{n-\alpha}G_\alpha^D(x,y)$ is continuous on $(B(x_0,\ 2r)\cap D)\times D_0$, we get, by Lebesgue's dominated convergence theorem,

$$\lim_{x\to x_0}\int_{D_0}\Big||x|^{n-\alpha}G_\alpha^D(x,y) - |x_0|^{n-\alpha}G_\alpha^D(x,y)\Big||y|^{\alpha-n}|q(y)|\,dy = 0.$$

It follows that there exists $r_1 > 0$ with $r_1 < \frac{r}{2}$ such that for $x \in B(x_0,r_1)\cap D$,

$$\int_{D_0}\Big||x|^{n-\alpha}G_\alpha^D(x,y) - |x_0|^{n-\alpha}G_\alpha^D(x,y)\Big||y|^{\alpha-n}|q(y)|\,dy \leq \frac{\epsilon}{2}.$$

Hence for $x \in B(x_0,r_1)\cap D$, we have

$$\Big|v(x) - v(x_0)\Big| \leq \epsilon.$$

This implies that

$$\lim_{x\to x_0} v(x) = v(x_0).$$

Case 2. Let $x_0 \in \partial D$ and $x \in B(x_0,2r)\cap D$. Then we have

$$\begin{aligned}|v(x)| \leq &\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B(0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\\ &+\sup_{z\in D}|z|^{n-\alpha}\int_{D\cap B^c(0,r)\cap B(x_0,r)}G_\alpha^D(z,y)|y|^{\alpha-n}|q(y)|\,dy\\ &+\int_{D_0}|x|^{n-\alpha}G_\alpha^D(x,y)|y|^{\alpha-n}|q(y)|\,dy.\end{aligned}$$

Since $\lim_{x\to x_0}|x|^{n-\alpha}G_\alpha^D(x,y)|y|^{\alpha-n} = 0$, for all $y \in D_0$, we deduce by similar arguments as above that

$$\lim_{x\to x_0} v(x) = v(x_0).$$

So, we conclude that $v \in C_0(\overline{D})$.

This completes the proof. □

2.3. Karamata class $\mathcal{K}_0$

In this paragraph, we recall some useful properties related to the Karamata class $\mathcal{K}_0$. For more details we refer readers to [5, 17].

Lemma 2.8. *Let $L_1,\ L_2 \in \mathcal{K}_0$, $p \in \mathbb{R}$ and $\varepsilon > 0$. Then we have*

(*i*) $L_1L_2 \in \mathcal{K}_0$ *and* $L_1^p \in \mathcal{K}_0$.

(*ii*) $\lim_{t\to 0^+} t^\varepsilon L_1 = 0$ *and* $\lim_{t\to 0^+} t^{-\varepsilon}L_1 = \infty$.

(iii) $\lim_{t\to 0^+} \frac{L_1(t)}{\int_t^\eta \frac{L_1(s)}{s}\,ds} = 0$ *and* $t \mapsto \int_t^\eta \frac{L_1(s)}{s}\,ds \in \mathcal{K}_0$.

(iv) *If* $\int_0^\eta \frac{L_1(s)}{s}\,ds$ *converges, then* $\lim_{t\to 0^+} \frac{L_1(t)}{\int_0^t \frac{L_1(s)}{s}\,ds} = 0$ *and* $t \mapsto \int_0^t \frac{L_1(s)}{s}\,ds \in \mathcal{K}_0$.

Applying Karamata's theorem (see [15, 17]), we get the following

Lemma 2.9. *Let* $\nu \in \mathbb{R}$ *and* $L \in \mathcal{K}_0$ *. Then the following assertions hold*

(i) *If* $\nu > -1$, *then* $\int_0^\eta s^\nu L(s)\,ds$ *converges and* $\int_0^t s^\nu L(s)\,ds \approx \frac{t^{1+\nu}L(t)}{1+\nu}$ *as* $t \to 0^+$.

(ii) *If* $\nu < -1$, *then* $\int_0^\eta s^\nu L(s)\,ds$ *diverges and* $\int_t^\eta s^\nu L(s)\,ds \approx -\frac{t^{1+\nu}L(t)}{1+\nu}$ *as* $t \to 0^+$.

We end this section by the following result due to Mâagli et al. [13].

Lemma 2.10. *Let* $L \in \mathcal{K}_0$ *defined on* $(0,\eta]$, $\eta > 1$, *and* $a, b \in (0,\ 1)$, $c \geq 1$ *such that* $\frac{1}{c}b \leq a \leq cb$. *Then, there exists* $m \geq 0$ *such that*

$$c^{-m}L(b) \leq L(a) \leq c^m L(b).$$

3. Key tools

In what follows, we recall some important lemmas and prove properties that play an important role in the proof of our result.

Lemma 3.1. *[8, Lemma 1] Let* $x \in D$, $D_1 = \{y \in D,\ \delta(x)\delta(y) \geq |x-y|^2\}$ *and* $D_2 = D_1^c$. *Then*

(1) *If* $y \in D_1$, *then*

$$\left(\frac{3-\sqrt{5}}{2}\right)\delta(x) \leq \delta(y) \leq \left(\frac{3+\sqrt{5}}{2}\right)\delta(x).$$

and

$$|x-y| \leq \frac{1+\sqrt{5}}{2}(\delta(x) \wedge \delta(y)).$$

(2) *If* $y \in D_2$, *then*

$$(\delta(x) \vee \ \delta(y)) \leq \frac{\sqrt{5}+1}{2}|x-y|.$$

In particular, we have

$$B\left(x, \frac{\sqrt{5}-1}{2}\delta(x)\right) \subset D_1 \subset B\left(x, \frac{\sqrt{5}+1}{2}\delta(x)\right). \tag{3.1}$$

In the same way as for the proof of the above result, we prove the following lemma.

Lemma 3.2. *Let* $x \in D$ *and* $D_x := \{y \in D\ ;\ |x-y|^2 \leq |x||y|\}$. *Then*

(i) *If* $y \in D_x$ *then*

$$\frac{3-\sqrt{5}}{2}|x| \leq |y| \leq \frac{3+\sqrt{5}}{2}|x|,$$

and

$$|x-y| \leq \frac{\sqrt{5}+1}{2}\left(|x| \wedge |y|\right).$$

Furthermore

$$B\left(x, \frac{\sqrt{5}-1}{2}|x|\right) \subset D_x \subset B\left(x, \frac{\sqrt{5}+1}{2}|x|\right). \tag{3.2}$$

(*ii*) *If* $y \in D_x^c$ *then*

$$|x| \vee |y| \leq \frac{\sqrt{5}+1}{2}|x-y|.$$

Proof. (*i*) Let $y \in D_x$. Since $|y| \leq |x-y| + |x|$, then

$$|y| \leq \sqrt{|x||y|} + |x|.$$

So

$$\sqrt{|y|}^2 - \sqrt{|x||y|} - |x| \leq 0,$$

that is

$$\left(\sqrt{|y|} + \frac{\sqrt{5}-1}{2}\sqrt{|x|}\right)\left(\sqrt{|y|} - \frac{\sqrt{5}+1}{2}\sqrt{|x|}\right) \leq 0.$$

It follows that

$$|y| \leq \frac{3+\sqrt{5}}{2}|x|.$$

Thus, interchange the role of x and y, we obtain $|x| \leq \frac{3+\sqrt{5}}{2}|y|$. Which implies that

$$\frac{3-\sqrt{5}}{2}|x| \leq |y| \leq \frac{3+\sqrt{5}}{2}|x|.$$

In particular, we have

$$|x-y| \leq \frac{1+\sqrt{5}}{2}|x| \wedge |y|.$$

According to the above, we deduce that

$$B\left(x, \frac{\sqrt{5}-1}{2}|x|\right) \subset D_x \subset B\left(x, \frac{\sqrt{5}+1}{2}|x|\right).$$

(*ii*) We may assume that $(|x| \vee |y|) = |y|$. Then the inequalities $|y| \leq |x-y| + |x|$ and $|x||y| \leq |x-y|^2$ imply that

$$|y|^2 \leq |x-y||y| + |x-y|^2,$$

hence

$$|y|^2 - |x-y||y| - |x-y|^2 \leq 0,$$

that is

$$\left(|y| + \frac{\sqrt{5}-1}{2}|x-y|\right)\left(|y| - \frac{\sqrt{5}+1}{2}|x-y|\right) \leq 0.$$

With an exchange of role between x and y we obtain

$$|x| \vee |y| \leq \frac{\sqrt{5}+1}{2}|x-y|.$$

This ends the proof. □

Remark 3.3. *Let* $r > 0$ *such that* $B(x, 3r) \subset D$. *Then for every* $x \in D$ *with* $0 < |x| < r$ *we have*

$$\int_{B(0,2r)\cap D_x} \frac{dy}{|x-y|^{n-\alpha}} \approx |x|^{\alpha}. \tag{3.3}$$

Indeed, by (3.2) *we have*

$$\int_{B(0,2r)\cap B(x,c_1|x|)} \frac{dy}{|x-y|^{n-\alpha}} \leq \int_{B(0,2r)\cap D_x} \frac{dy}{|x-y|^{n-\alpha}} \leq \int_{B(0,2r)\cap B(x,c_2|x|)} \frac{dy}{|x-y|^{n-\alpha}},$$

where $c_1 = \frac{\sqrt{5}-1}{2}$ *and* $c_2 = \frac{\sqrt{5}+1}{2}$. *Using a change of spherical variable, we obtain* (3.3).

Lemma 3.4. *Let $r > 0$ such that $B(0, 3r) \subset D$. Then, we have*

(*i*) *If $x \in B(0, r)$ and $y \in B(0, 2r)$, then*

$$G_\alpha^D(x, y) \approx \frac{1}{|x-y|^{n-\alpha}}.$$

(*ii*) *If $x \in B(0, r)$ and $y \in D \backslash B(0, 2r)$, then*

$$G_\alpha^D(x, y) \approx \delta(y).$$

(*iii*) *If $x \in D \backslash B(0, 2r)$ and $y \in B(0, 2r)$, then*

$$G_\alpha^D(x, y) \approx \delta(x).$$

Proof. (*i*) Since the following function

$$(x, y) \longmapsto \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right)$$

is bounded and greater than a positive constant on $B(0, r) \times B(0, 2r)$. So, by (2.1), we obtain

$$G_\alpha^D(x, y) \approx \frac{1}{|x-y|^{n-\alpha}}.$$

(*ii*) Let $y \in D \backslash B(0, 2r)$ the function

$$g : x \mapsto \frac{\delta(x)}{|x-y|^{n-\alpha}\left(|x-y|^2 + \delta(x)\delta(y)\right)},$$

is continuous in $B(0, r)$. Hence g is bounded and greater than a positive constant on $B(0, r)$. It follows that

$$G_\alpha^D(x, y) \approx \delta(y).$$

(*iii*) Let $x \in D \backslash B(0, 2r)$, by the same manner as in (*ii*), we obtain

$$G_\alpha^D(x, y) \approx \delta(x).$$

Thus the lemma is proved. □

An important step in our proof of the Proposition 3.7 uses the following result due to Bachar et al. [1].

Lemma 3.5. *Let $\beta \le 2$ and $L \in \mathcal{K}_0$ defined on $(0, \eta)$, $\eta \ge d$, such that $\int_0^\eta t^{1-\beta} L(t)\, dt < \infty$. Then*

$$0 < \int_D \delta(x)^{1-\beta} L(\delta(x))\, dx < \infty,$$

that is $\int_D \delta(x)^{1-\beta} L(\delta(x))\, dx \approx 1$.

Proposition 3.6. *Let L_i be a function in $\mathcal{K}_0$ and λ_i be a real number, for $i \in \{1, 2\}$. Then the following assertions are equivalent*

(*i*) $\varphi : x \mapsto |x|^{-\lambda_1} L_1(|x|) \delta(x)^{-\lambda_2} L_2(\delta(x)) \in \mathbf{K}_\alpha(D)$.

(*ii*) $\int_0^\eta t^{\alpha-1-\lambda_i} L_i(t)\, dt < \infty$ *for $i \in \{1, 2\}$.*

(*iii*) $\lambda_i < \alpha$ *or* $\lambda_i = \alpha$ *with* $\int_0^\eta \frac{L_i(t)}{t}\, dt < \infty$.

Proof. By Lemma 2.9, we have obviously $(ii) \Longleftrightarrow (iii)$. Let us prove $(i) \Longleftrightarrow (ii)$.
Suppose (i). Let $r > 0$ such that $B(0, 3r) \subset D$. By [14, Theorem 2], we have $V\varphi(0) < \infty$. This implies that

$$\int_{B(0,2r)} G_\alpha^D(0, y)\varphi(y)dy < \infty.$$

It follows from Lemma 3.4 (i), that

$$\int_{B(0,2r)} |y|^{\alpha-n-\lambda_1} L_1(|y|)dy < \infty.$$

So

$$\int_0^{2r} t^{\alpha-1-\lambda_1} L_1(t)dt < \infty,$$

and then

$$\int_0^{\eta} t^{\alpha-1-\lambda_1} L_1(t)dt < \infty.$$

On the other hand, by Remark 2.6, we have $\int_D \delta(y)^{\alpha-1}\varphi(y)dy < \infty$. In particular,

$$\int_{D\setminus B(0,2r)} \delta(y)^{\alpha-1}\varphi(y)dy \approx \int_{D\setminus B(0,2r)} \delta(y)^{\alpha-1-\lambda_2} L_2(\delta(y))dy < \infty.$$

Since

$$\int_D \delta(y)^{\alpha-1-\lambda_2} L_2(\delta(y))dy \approx \int_{D\setminus B(0,2r)} \delta(y)^{\alpha-1-\lambda_2} L_2(\delta(y))dy < \infty,$$

we obtain

$$\int_0^{\eta} t^{\alpha-1-\lambda_2} L_2(t)dt < \infty.$$

Conversely, assume (ii). Let $r > 0$ such that $B(0, 3r) \subset D$ and let $r_1 \in (0, r)$. Then there exist $c > 0$ such that

$$\begin{aligned}
\int_{D\cap B(x,r_1)} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)\varphi(y)dy &\leq c \int_{D\cap B(x,r_1)\cap B(0,2r)} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)\varphi(y)dy \\
&\quad + c \int_{D\cap B(x,r_1)\cap B^c(0,2r)} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)\varphi(y)dy \\
&= A(x) + B(x).
\end{aligned}$$

Using Lemma 3.4 (i), we obtain

$$\begin{aligned}
A(x) &\leq c \int_{D_x\cap B(x,r_1)\cap B(0,2r)} \frac{|y|^{-\lambda_1}}{|x-y|^{n-\alpha}} L_1(|y|)dy \\
&\quad + c \int_{D_x^c\cap B(x,r_1)\cap B(0,2r)} \frac{|y|^{-\lambda_1}}{|x-y|^{n-\alpha}} L_1(|y|)dy \\
&= A_1(x) + A_2(x),
\end{aligned}$$

where D_x is the set defined in Lemma 3.2. Note that, if $y \in B(x, r_1) \cap B(0, 2r)$ then $|x| < 3r$. Now, for $y \in D_x$, by Lemma 3.2, (i) and Lemma 2.10, we have

$$A_1(x) \leq c|x|^{-\lambda_1} L_1(|x|) \int_{D_x\cap B(x,r_1)\cap B(0,2r)} \frac{dy}{|x-y|^{n-\alpha}},$$

using (3.3), we obtain

$$A_1(x) \leq c|x|^{\alpha-\lambda_1} L_1(|x|).$$

Since $\int_0^\eta t^{\alpha-1-\lambda_1}L_1(t)dt < \infty$, then by (iii) we have $\lambda_1 \leq \alpha$.
If $\lambda_1 < \alpha$ then, by Lemma 2.8 (ii) and using the fact that $\alpha-\lambda_1 > 0$, there exist $c > 0$ such that $|x|^{\alpha-\lambda_1}L_1(|x|) \leq c$. That is

$$A_1(x) \leq c.$$

If $\lambda_1 = \alpha$ and $\int_0^\eta \frac{L_1(t)}{t}dt < \infty$, then, by Lemma 3.2 (i) and using the fact that the function $t \mapsto t^{-\alpha}L_1(t)$ is nonincreasing in $(0, w]$ for some $w > 0$, we obtain

$$A_1(x) \leq c \int_{D_x\cap B(x,r_1)} \frac{L_1(\frac{\sqrt{5}+1}{2}|x-y|)}{|x-y|^n}dy \leq c\int_0^{\frac{\sqrt{5}+1}{2}r} \frac{L_1(r)}{r}dr \leq c.$$

For $y \in D_x^c$, by Lemma 3.2, (ii), we have

$$\begin{aligned} A_2(x) &\leq c \int_{D_x^c\cap B(x,r_1)\cap B(0,2r)} \frac{|y|^{-\lambda_1}L_1(|y|)}{(|x| \vee |y|)^{n-\alpha}}dy \\ &\leq c \int_{(|x|-r_1)^+}^{(|x|+r_1)\wedge 2r} \frac{t^{n-1-\lambda_1}L_1(t)}{(|x| \vee t)^{n-\alpha}}dt. \end{aligned}$$

Since the function $g(z) := \int_0^z t^{\alpha-1-\lambda_1}L_1(t)dt$ is uniformly continuous on $[0, 3r]$, we obtain

$$\lim_{r_1\to 0} \sup_{|x|\leq 3r}[g((|x| + r_1) \wedge 2r) - g((|x| - r_1)^+)] = 0.$$

So, we deduce that

$$\lim_{r_1\to 0} sup_{x\in D}A(x) = 0. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned} B(x) &\leq c \int_{D\cap B(x,r_1)\cap B^c(0,2r)} \frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\delta(y)^{-\lambda_2}L_2(\delta(y))dy \\ &\leq c \int_{D\cap B(x,r_1)} \frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\delta(y)^{-\lambda_2}L_2(\delta(y))dy. \end{aligned}$$

Now, since $\int_0^\eta t^{\alpha-1-\lambda_2}L_2(t)dt < \infty$ then, by [14, Proposition 7], we have

$$\delta(y)^{-\lambda_2}L_2(\delta(y)) \in \mathbf{K}_\alpha(D),$$

that is

$$\lim_{r_1\to 0} \sup_{x\in D} \int_{D\cap B(x,r_1)} \frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\delta(y)^{-\lambda_2}L_2(\delta(y))dy = 0.$$

So

$$\lim_{r_1\to 0} \sup_{x\in D} B(x) = 0. \tag{3.5}$$

Combining (3.5) and (3.4), we conclude that the function

$$\varphi : \ x \mapsto |x|^{-\lambda_1}L_1(|x|)\delta(x)^{-\lambda_2}L_2(\delta(y)) \text{ is in } \mathbf{K}_\alpha(D).$$

This completes the proof. □

Now, we give an interesting result that plays a key role in the proof of Theorem 1.2.

Proposition 3.7. *Let $\gamma < n-2+\alpha$, $\mu < \alpha$ and $M,\ N \in \mathcal{K}_0$. Let*

$$b(x) = |x|^{-\gamma} M(|x|)\delta(x)^{-\mu} N(\delta(x)),\ x \in D\backslash\{0\}.$$

Then, for $x \in D\backslash\{0\}$,

$$Vb(x) \approx |x|^{\min(0,\alpha-\gamma)} \tilde{M}(|x|)\delta(x)^{\min(1,\alpha-\mu)} \tilde{N}(\delta(x)),$$

where $\tilde{M}$ and $\tilde{N}$ are defined on $(0,\eta)$ by

$$\tilde{M}(t) := \begin{cases} 1, & \text{if } \gamma < \alpha, \\ \int_t^\eta \frac{M(s)}{s}\, ds, & \text{if } \gamma = \alpha, \\ M(t), & \text{if } \alpha < \gamma < n-2+\alpha, \end{cases} \quad \text{and} \quad \tilde{N}(t) := \begin{cases} 1, & \text{if } \mu < \alpha-1, \\ \int_t^\eta \frac{N(s)}{s}\, ds, & \text{if } \mu = \alpha-1, \\ N(s), & \text{if } \alpha-1 < \mu < \alpha. \end{cases}$$

Proof. First we remark that

$$\int_0^\eta t^{n-3+\alpha-\gamma} M(t)\, dt < \infty \text{ and } \int_0^\eta t^{\alpha-1-\mu} N(t)\, dt < \infty, \text{ for } \eta > d. \tag{3.6}$$

Let $0 < r < \frac{3-\sqrt{5}}{2}$ such that $B(0,\ 3r) \subset D$. For $x \in D\backslash\{0\}$, we have

$$Vb(x) = \int_{B(0,\ 2r)} G_\alpha^D(x,y) b(y)\, dy + \int_{D\backslash B(0,\ 2r)} G_\alpha^D(x,y) b(y)\, dy.$$

We distinguish three cases.

Case 1 : If $0 < |x| < r$.
By Lemma 3.4 (*i*) and (*ii*), we have

$$Vb(x) \approx \int_{B(0,2r)} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy + \int_{D\backslash B(0,\ 2r)} \delta(y)^{1-\mu} N(\delta(y))\, dy.$$

Using (3.6) and Lemma 3.5, we obtain

$$\int_{D\backslash B(0,\ 2r)} \delta(y)^{1-\mu} N(\delta(y))\, dy \approx 1.$$

Hence

$$Vb(x) \approx \int_{B(0,2r)} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy + 1. \tag{3.7}$$

Therefore

$$\begin{aligned} Vb(x) &\approx \int_{B(0,2r)\cap D_x} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy + \int_{B(0,2r)\cap D_x^c} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy + 1 \\ &= I_1(x) + I_2(x), \end{aligned}$$

where D_x is the set defined in Lemma 3.2 and

$$I_1(x) = \int_{B(0,2r)\cap D_x} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy,$$

$$I_2(x) = \int_{B(0,2r)\cap D_x^c} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy + 1.$$

Let us estimate $I_1(x)$. By Lemma 2.10 and Lemma 3.2 (i) we have

$$|x| \approx |y| \text{ and } M(|x|) \approx M(|y|).$$

Then

$$I_1(x) \approx |x|^{-\gamma} M(|x|) \int_{B(0,2r)\cap D_x} \frac{dy}{|x-y|^{n-\alpha}}.$$

by Remark 3.3, we obtain

$$I_1(x) \approx |x|^{\alpha-\gamma} M(|x|). \tag{3.8}$$

Now let us estimate $I_2(x)$. For $y \in D_x^c$, we have $\frac{|y|^{2-\alpha}}{|x-y|^{2-\alpha}} \leq c$. Then

$$\begin{aligned} J(x) = \int_{B(0,2r)\cap D_x^c} \frac{|y|^{-\gamma} M(|y|)}{|x-y|^{n-\alpha}}\, dy &= \int_{B(0,2r)\cap D_x^c} \frac{|y|^{2-\alpha}}{|x-y|^{2-\alpha}} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{|x-y|^{n-2}}\, dy \\ &\leq c \int_{B(0,2r)\cap D_x^c} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{|x-y|^{n-2}}\, dy. \end{aligned}$$

By Lemma 3.2 (ii), we have

$$|x-y|^{2-n} \leq c(|x| \vee |y|)^{2-n},$$

this implies that

$$\begin{aligned} J(x) &\leq c \int_{B(0,2r)\cap D_x^c} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{(|x| \vee |y|)^{n-2}}\, dy \\ &\leq c \int_{B(0,2r)} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{(|x| \vee |y|)^{n-2}}\, dy \\ &\leq c \int_{B(0,2r)\cap(|x|\leq|y|)} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{(|x| \vee |y|)^{n-2}}\, dy + c \int_{B(0,2r)\cap(|x|\geq|y|)} \frac{|y|^{\alpha-2-\gamma} M(|y|)}{(|x| \vee |y|)^{n-2}}\, dy \\ &\leq c|x|^{2-n} \int_{B(0,2r)\cap(|y|\leq|x|)} |y|^{\alpha-2-\gamma} M(|y|)\, dy + c \int_{B(0,2r)\cap(|y|\geq|x|)} |y|^{\alpha-\gamma-n} M(|y|)\, dy. \end{aligned}$$

Since the functions to be integrated are radial, then

$$J(x) \leq c|x|^{2-n} \int_0^{|x|} t^{n-2+\alpha-\gamma-1} M(t)\, dt + c \int_{|x|}^{2r} t^{\alpha-\gamma-1} M(t)\, dt = I_{21}(x) + I_{22}(x), \tag{3.9}$$

where

$$I_{21}(x) = c|x|^{2-n} \int_0^{|x|} t^{n-3+\alpha-\gamma} M(t)\, dt,$$

$$I_{22}(x) = c \int_{|x|}^{2r} t^{\alpha-\gamma-1} M(t)\, dt.$$

By (3.8) and (3.9), we conclude that

$$Vb(x) \leq I_1(x) + I_{21}(x) + I_{22}(x) + 1.$$

Now, we have three sub-cases

(a) If $\alpha < \gamma < n-2+\alpha$ then, by Lemma 2.9 (ii) we have

$$I_{21}(x) \approx |x|^{\alpha-\gamma} M(|x|).$$

Using Lemma 2.9 (ii) and the fact that $\int_{2r}^{\eta} t^{\alpha-\gamma-1} M(t)\, dt \approx 1$, we obtain

$$\begin{aligned} I_{22}(x) + 1 &\approx \int_{|x|}^{2r} t^{\alpha-\gamma-1} M(t)\, dt + \int_{2r}^{\eta} t^{\alpha-\gamma-1} M(t)\, dt \\ &= \int_{|x|}^{\eta} t^{\alpha-\gamma-1} M(t)\, dt \\ &\approx |x|^{\alpha-\gamma} M(|x|). \end{aligned}$$

This and (3.8) implies that

$$Vb(x) \approx |x|^{\alpha-\gamma} M(|x|). \tag{3.10}$$

(b) If $\gamma = \alpha$, then

$$I_{21}(x) = c|x|^{2-n} \int_0^{|x|} t^{n-3} M(t)\, dt,$$

by Lemma 2.9 (i), we obtain

$$I_{21}(x) \approx M(|x|).$$

Since $0 < \int_{2r}^{\eta} \frac{M(t)}{t}\, dt < \infty$, we have

$$I_{22}(x) + 1 \approx \int_{2r}^{\eta} \frac{M(t)}{t}\, dt + \int_{|x|}^{2r} \frac{M(t)}{t}\, dt = \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt.$$

Then, by (3.8), we obtain

$$\begin{aligned} Vb(x) &\le c\left(2M(|x|) + \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt\right) \\ &\le c \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt \left(1 + \frac{2M(|x|)}{\int_{|x|}^{\eta} \frac{M(t)}{t}\, dt}\right), \end{aligned}$$

so, by Lemma 2.8 (iii), we obtain

$$Vb(x) \le c \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt. \tag{3.11}$$

On the other hand, by (3.7) and using the fact that $|x-y| \le 2\max(|x|, |y|)$, we get

$$\begin{aligned} Vb(x) &\ge c \int_{B(0,2r)} \frac{|y|^{-\alpha} M(|y|)}{\max(|x|, |y|)^{n-\alpha}} dy + 1 \\ &\ge c|x|^{\alpha-n} \int_0^{|x|} t^{n-1-\alpha} M(t) dt + c \int_{|x|}^{2r} \frac{M(t)}{t} dt + 1. \end{aligned}$$

By Lemma 2.9 (i) and since $\int_{2r}^{\eta} \frac{M(t)}{t} dt \approx 1$, we obtain

$$Vb(x) \ge c \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt. \tag{3.12}$$

So, using (3.11) and (3.12), we obtain

$$Vb(x) \approx \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt. \tag{3.13}$$

(c) If $\gamma < \alpha$ then, by Lemma 2.9 (i)

$$I_{21}(x) \approx |x|^{\alpha-\gamma} M(|x|).$$

Since $\alpha - \gamma - 1 > -1$ then, by Lemma 2.9 (i), $\int_0^\eta t^{\alpha-\gamma-1} M(t)dt$ converges and

$$I_{22}(x) + 1 = 1 + c\int_{|x|}^{2r} t^{\alpha-\gamma-1} M(t)dt \leq 1 + c\int_0^\eta t^{\alpha-\gamma-1} M(t)dt \leq C. \tag{3.14}$$

Lemma 2.8 (ii), (3.8) and (3.14) yield to

$$Vb(x) \leq c. \tag{3.15}$$

On the other hand, since

$$Vb(x) = I_1(x) + I_2(x),$$

then

$$Vb(x) \geq I_2(x) = J(x) + 1 \geq 1.$$

So

$$Vb(x) \geq c. \tag{3.16}$$

Using (3.15) and (3.16), we obtain

$$Vb(x) \approx 1. \tag{3.17}$$

Finally, by (3.10), (3.13) and (3.17), we deduce that

$$Vb(x) \approx \begin{cases} 1 & \text{if } \gamma < \alpha, \\ \int_{|x|}^{\eta} \frac{M(t)}{t}\, dt & \text{if } \gamma = \alpha, \\ |x|^{\alpha-\gamma} M(|x|) & \text{if } \alpha < \gamma < n-2+\alpha. \end{cases} \tag{3.18}$$

We conclude that, for $0 < |x| < r$,

$$Vb(x) \approx |x|^{\min(0,\ \alpha-\gamma)} \tilde{M}(|x|). \tag{3.19}$$

Case 2 : If $x \in D\backslash B(0,\ 3r)$. Using Lemma 3.4, we obtain

$$Vb(x) \approx \delta(x)\int_{B(0,2r)} |y|^{-\gamma} M(|y|)\, dy + \int_{D\backslash B(0,2r)} G_\alpha^D(x,y)\delta(y)^{-\mu} N(\delta(y))\, dy.$$

Since

$$\begin{aligned}\int_{B(0,2r)} |y|^{-\gamma} M(|y|)\, dy &= \int_0^{2r} t^{n-1-\gamma} M(t)\, dt \\ &\leq \int_0^{\eta} t^{n-1-\gamma} M(t)\, dt \\ &\leq \eta^{2-\alpha}\int_0^{\eta} t^{n-3+\alpha-\gamma} M(t)\, dt \\ &< \infty.\end{aligned}$$

Then

$$\int_{B(0,2r)} |y|^{-\gamma} M(|y|)\, dy \approx 1.$$

Hence

$$Vb(x) \approx \delta(x) + \int_{D\setminus B(0,2r)} G_\alpha^D(x,y)\delta(y)^{-\mu}N(\delta(y))\,dy.$$

The function $y \mapsto \delta(y)^{-\mu}N(\delta(y))$ is continuous in $B(0,2r)$. This implies that

$$\delta(y)^{-\mu}N(\delta(y)) \approx 1.$$

Thus, using Lemma 3.4 (*iii*), we obtain

$$\begin{aligned} Vb(x) &\approx \int_{B(0,2r)} G_\alpha^D(x,y)\delta(y)^{-\mu}N(\delta(y))\,dy + \int_{D\setminus B(0,2r)} G_\alpha^D(x,y)\delta(y)^{-\mu}N(\delta(y))\,dy \\ &= \int_D G_\alpha^D(x,y)\delta(y)^{-\mu}N(\delta(y))\,dy. \end{aligned}$$

Since $\mu < \alpha$, then by [14, Proposition 8] we obtain, for $x \in D\setminus B(0,3r)$,

$$Vb(x) \approx \begin{cases} \delta(x) & \text{if } \mu < \alpha - 1, \\ \delta(x)\int_{\delta(x)}^{\eta} \frac{N(t)}{t}\,dt & \text{if } \mu = \alpha - 1, \\ \delta(x)^{\alpha-\mu}N(\delta(x)) & \text{if } \alpha - 1 < \mu < \alpha. \end{cases} \tag{3.20}$$

This implies that

$$Vb(x) \approx \delta(x)^{\min(1,\alpha-\mu)}\tilde{N}(\delta(x)), \text{ for } x \in D\setminus B(0,3r). \tag{3.21}$$

Case 3 If $r \le |x| \le 3r$. The function $x \mapsto |x|^{\min(0,\alpha-\gamma)}\tilde{M}(|x|)\delta(x)^{\min(1,\alpha-\mu)}\tilde{N}(\delta(x))$ is positive and continuous on $D\setminus\{0\}$. Let $b(x) = |x|^{\alpha-n}q(x)$, where $q(x) = |x|^{n-\alpha-\gamma}M(|x|)\delta(x)^{-\mu}N(\delta(x))$. Then, by (3.6) and Proposition 3.6, we have

$$q \in \mathbf{K}_\alpha(D).$$

Thus, by Proposition 2.7, we deduce that the function Vb is positive and continuous on $D\setminus\{0\}$. So

$$Vb(x) \approx |x|^{\min(0,\alpha-\gamma)}\tilde{M}(|x|)\delta(x)^{\min(1,\alpha-\mu)}\tilde{N}(\delta(x)), \tag{3.22}$$

for all x in the compact J defined by $J := \{x \in D,\ r \le |x| \le 3r\}$.

Combining (3.19), (3.21) and (3.22), we obtain for $x \in D\setminus\{0\}$,

$$Vb(x) \approx |x|^{\min(0,\alpha-\gamma)}\tilde{M}(|x|)\delta(x)^{\min(1,\alpha-\mu)}\tilde{N}(\delta(x)).$$

This ends the proof. □

Proposition 3.8. *Let a be a function satisfying* (H). *Then*

$$V(a(x)\Theta^\sigma(x)) \approx \Theta(x),\ x \in D\setminus\{0\},$$

where Θ *is defined by* (1.3)

Proof. Let a be a function satisfying (H). By (1.2) and (1.3), we obtain

$$a(x)\Theta(x)^\sigma \approx |x|^{-(\lambda-\min(0,\frac{\alpha-\lambda}{1-\sigma})\sigma)}(L_1\tilde{L}_1^{\frac{\sigma}{1-\sigma}})(|x|)\delta(x)^{-(\xi-\min(1,\frac{\alpha-\xi}{1-\sigma})\sigma)}(L_2\tilde{L}_2^{\frac{\sigma}{1-\sigma}})(\delta(x))$$

Let

$$\gamma = \lambda - \min(0, \frac{\alpha-\lambda}{1-\sigma})\sigma \text{ and } \mu = \xi - \min(1, \frac{\alpha-\xi}{1-\sigma})\sigma.$$

Since $\xi < \alpha$ and $\lambda < (n-2+\alpha)(1-\sigma)+\alpha\sigma$, then we have $\mu < \alpha$ and $\gamma < n-2+\alpha$.
Using Lemma 2.8 (*i*), (*iii*), (*iv*) and Proposition 3.6, we obtain

$$L_1(t)(\tilde{L}_1(t))^{\frac{\sigma}{1-\sigma}} \in \mathcal{K}_0 \text{ and } L_2(t)(\tilde{L}_2(t))^{\frac{\sigma}{1-\sigma}} \in \mathcal{K}_0.$$

By Proposition 3.7, with $M(t) = L_1(t)(\tilde{L}_1(t))^{\frac{\sigma}{1-\sigma}}$ and $N(t) = L_2(t)(\tilde{L}_2(t))^{\frac{\sigma}{1-\sigma}}$, we deduce that

$$V(a(x)\Theta^\sigma(x)) \approx |x|^{\min(0,\alpha-\gamma)}\tilde{M}(|x|)\delta(x)^{\min(1,\alpha-\mu)}\tilde{N}(\delta(x)).$$

Now, since

$$\min(0,\ \alpha-\gamma) = \min\Big(0,\ \alpha-\lambda+\min\Big(0,\frac{\alpha-\lambda}{1-\sigma}\Big)\sigma\Big) = \min\Big(0,\ \frac{\alpha-\lambda}{1-\sigma}\Big),$$
$$\min(1,\alpha-\mu) = \min\Big(1,\ \alpha-\xi+\min\Big(1,\frac{\alpha-\xi}{1-\sigma}\Big)\sigma\Big) = \min\Big(1,\ \frac{\alpha-\xi}{1-\sigma}\Big),$$

and, by an elementary calculation, we have

$$\tilde{M}(t) \approx \tilde{L}_1(t))^{\frac{1}{1-\sigma}} \text{ and } \tilde{N}(t) \approx \tilde{L}_2(t))^{\frac{1}{1-\sigma}}.$$

Then, we deduce for $x \in D\backslash\{0\}$

$$V(a(x)\Theta^\sigma(x)) \approx \Theta(x).$$

This ends the proof. □

For $\omega > 0$, we denote by (P_ω) the following problem

$$(P_\omega)\begin{cases}(-\Delta|_D)^{\frac{\alpha}{2}}u = a(x)u^\sigma(x),\ x\in D\backslash\{0\}, \text{ (in the distributional sense)},\\ u>0 \text{ in } D\backslash\{0\},\\ \lim\limits_{|x|\to 0}|x|^{n-\alpha}u(x)=\omega,\\ \lim\limits_{x\to\partial D}\delta(x)^{2-\alpha}u(x)=0,\end{cases}$$

Proposition 3.9. *Let $\sigma < 0$ and assume that hypothesis $(H)$is satisfied. Then for esach $\omega > 0$, problem P_ω has at least one positive solution $u_\omega \in C(\overline{D}\backslash\{0\})$ satisfying for $x \in \overline{D}\backslash\{0\}$*

$$u_\omega = \omega|x|^{n-\alpha} + V(a(x)u_\omega^\sigma)(x). \tag{3.23}$$

Proof. By hypothesis (H), we have

$$q(x) = |x|^{(\alpha-n)(\sigma-1)}a(x) \approx |x|^{-(\lambda-(\alpha-n)(\sigma-1))}L_1(|x|)\delta(x)^{-\xi}L_2(\delta(x)).$$

Since, $\sigma < 0$ and $0 < \alpha < 2$ then

$$\lambda-(\alpha-n)(\sigma-1) < (1-\sigma)\alpha+\alpha\sigma = \alpha.$$

In addition, $\xi < \alpha$. So ,by Proposition 3.6, we get that $q(x) \in \mathbf{K}_\alpha(D)$.
Using Proposition 2.7, we deduce that the function

$$x \mapsto |x|^{n-\alpha}\int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}\,dy \text{ is in } C_0(\overline{D}). \tag{3.24}$$

Let Σ be the closed convex set given by

$$\Sigma = \{v \in C(\overline{D})\ :\ \omega \le v \le \beta,\ \beta = \omega + \omega^\sigma\|h\|_\infty\}.$$

We define the operator T on Σ by

$$Tv(x) = \omega + |x|^{n-\alpha} \int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}v^\sigma(y)\,dy.$$

We shall prove that T has a fixed point in Σ. First, let $x_0 \in \overline{D}$, since $\sigma < 0$ then for all $v \in \Sigma$ we have $v^\sigma \le \omega^\sigma$. Thus

$$|Tv(x) - Tv(x_0)| \le \omega^\sigma \int_D \left| |x|^{n-\alpha}G_\alpha^D(x,y) - |x_0|^{n-\alpha}G_\alpha^D(x_0,y)\right| |y|^{\alpha-n}q(y)\,dy,\ x \in \overline{D}$$

where $q(x) = |x|^{(\alpha-n)(\sigma-1)}a(x) \in \mathbf{K}_\alpha(D)$.
By same arguments as in the proof of Proposition 2.7, we obtain for all $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\text{if } x \in \overline{D} \text{ and } |x - x_0| < \delta \text{ then } \omega^\sigma \int_D \left| |x|^{n-\alpha}G_\alpha^D(x,y) - |x_0|^{n-\alpha}G_\alpha^D(x_0,y)\right| |y|^{\alpha-n}q(y)\,dy \le \epsilon.$$

So, for all $\epsilon > 0$ there exists $\delta > 0$, such that

$$\text{if } x \in \overline{D} \text{ and } |x - x_0| < \delta \text{ then } |Tv(x) - Tv(x_0)| \le \epsilon, \quad \text{for all } v \in \Sigma.$$

This implies that the family $T\Sigma$ is equicontinuous in each point of $\overline{D}$. In particular, for all $v \in \Sigma$, $Tv \in C(\overline{D})$ and as a result $T\Sigma \subset \Sigma$. Therefore, since the family $\{Tv(x),\ v \in \Sigma\}$ is uniformly bounded in $\overline{D}$, then by Arzela-Ascoli theorem (see [3]), the set $T(\Sigma)$ is relatively compact in $C(\overline{D})$. Next, let us prove the continuity of T in Σ. We consider a sequence $(v_k)_k$ in Σ which converges uniformly to a function v in Σ. Then we have

$$|Tv_k(x) - Tv(x)| \le |x|^{n-\alpha} \int_D G_\alpha^D(x,y)|y|^{(\alpha-n)\sigma}a(y)|v_k^\sigma(y) - v^\sigma(y)|\,dy.$$

By (3.24), the dominated convergence theorem and using the fact that $|v_k^\sigma(y) - v^\sigma(y)| \le 2\omega^\sigma$, we deduce that

$$\forall x \in \overline{D},\ Tv_k(x) \to Tv(x) \text{ as } k \to \infty.$$

Moreover, since $T(\Sigma)$ is relatively compact in $C(\overline{D})$, we get

$$\|Tv_k - Tv\|_\infty \to 0 \text{ as } k \to \infty.$$

So T is a compact mapping of Σ to it self. Applying, now, the Schauder fixed point theorem, there exists $v_\omega \in \Sigma$ such that for each $x \in \overline{D}$

$$v_\omega = \omega + |x|^{n-\alpha} \int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}v_\omega^\sigma(y)\,dy. \tag{3.25}$$

From (3.24), (3.25) and since $v_\omega^\sigma \le \omega^\sigma$, we obtain

$$\lim_{x\to\partial D} v_\omega(x) = \omega. \tag{3.26}$$

Let $r > 0$ such that $B(0,3r) \subset D$ and let $x \in B(0,r)\backslash\{0\}$. Since $\omega \le v_\omega \le \beta$, by (3.25) and hypothesis (H) and similar argument as in the proof of Proposition 3.7, we obtain

$$\begin{aligned}
v_\omega(x) - \omega &\approx |x|^{n-\alpha} \int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}\,dy \\
&\approx |x|^{n-\alpha}\left(\int_{B(0,2r)} \frac{|y|^{-(\lambda-(\alpha-n)\sigma)}L_1(|y|)}{|x-y|^{n-\alpha}}\,dy + 1\right) \\
&\le c|x|^{n-\alpha}\left(|x|^{\alpha-\lambda+(\alpha-n)\sigma}L_1(|x|) + \int_{B(0,2r)\cap D_x^c} \frac{|y|^{\alpha-2-\lambda+(\alpha-n)\sigma}L_1(|y|)}{(|x| \vee |y|)^{n-2}}\,dy + 1\right) \\
&\le c|x|^{n-\alpha}\left(|x|^{\alpha+(\alpha-n)\sigma-\lambda}L_1(|x|) + \int_0^{2r} \frac{t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1}L_1(t)}{(|x| \vee t)^{n-2}}\,dt + 1\right).
\end{aligned}$$

Since $\lambda < (n-2+\alpha)(1-\sigma)+\alpha\sigma$ and $n+(\alpha-n)\sigma-\lambda > (2-\alpha)(1-\sigma) > 0$, by Lemma 2.8, (*ii*), we get

$$\lim_{|x|\to 0} |x|^{n+(\alpha-n)\sigma-\lambda} L_1(|x|) = 0.$$

For $n \geq 3$, $x \in B(0,r)\backslash\{0\}$ and $t \in (0,2r)$ we have

$$\begin{aligned} |x|^{n-\alpha} \frac{t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1}}{(|x| \vee t)^{n-2}} L_1(t) &\leq |x|^{2-\alpha} t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1} L_1(t) \\ &\leq r^{2-\alpha} t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1} L_1(t) \\ &= \Psi(t). \end{aligned}$$

Since

$$\int_0^\eta t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1} L_1(t)\, dt \leq \eta^{(\alpha-2)\sigma} \int_0^\eta t^{(n-2+\alpha)(1-\sigma)+\alpha\sigma-\lambda-1} L_1(t)\, dt < \infty,$$

this implies that

$$\Psi(t) \in L^1((0,\eta)),$$

and

$$\lim_{|x|\to 0} |x|^{n-\alpha} \frac{t^{n+(\alpha-n)\sigma+(\alpha-2)-\lambda-1}}{(|x| \vee t)^{n-2}} L_1(t) = 0.$$

By the dominate convergence theorem, we deduce that

$$\lim_{|x|\to 0} v_\omega(x) = \omega. \tag{3.27}$$

Let $u_\omega(x) = |x|^{\alpha-n} v_\omega(x)$, for $x \in D\backslash\{0\}$. Then $u_\omega \in C(\overline{D}\backslash\{0\})$, and we have

$$u_\omega(x) = \omega|x|^{\alpha-n} + \int_D G_\alpha^D(x,y) a(y) u_\omega^\sigma(y)\, dy, \tag{3.28}$$

so

$$\omega|x|^{\alpha-n} \leq u_\omega(x) \leq \beta|x|^{\alpha-n}. \tag{3.29}$$

Now, since the function $y \mapsto a(y)u_\omega^\sigma(y) \in L^1_{loc}(D\backslash\{0\})$ and we have

$$x \mapsto \int_D G_\alpha^D(x,y) a(y) u_\omega^\sigma(y)\, dy \in L^1_{loc}(D\backslash\{0\})$$

thus, by (2.6), applying $(-\Delta|_D)^{\frac{\alpha}{2}}$ on both sides of equation (3.28), we conclude that u_ω satisfies

$$(-\Delta|_D)^{\frac{\alpha}{2}} u_\omega = a(x)u_\omega^\sigma(x),\ x \in D\backslash\{0\},\ \text{(in the distributional sense)},$$

in addition, from (3.27) and (3.29), we get

$$\lim_{|x|\to 0} |x|^{n-\alpha} u_\omega(x) = \omega,\ \lim_{x\to\partial D} \delta(x)^{2-\alpha} u_\omega(x) = 0.$$

This ends the proof. □

Corollary 3.10. *Let a be a function satisfying (H), $\sigma < 0$ and $u_{\omega_i} \in C(\overline{D}\backslash\{0\})$ is a solution of P_{ω_i} $i = 1,\ 2$. Then, for $0 < \omega_1 \leq \omega_2$, we have*

$$0 \leq u_{\omega_2}(x) - u_{\omega_1}(x) \leq (\omega_2 - \omega_1)|x|^{\alpha-n},\ x \in \overline{D}\backslash\{0\}. \tag{3.30}$$

Proof. Let h be the function defined on $\overline{D}\backslash\{0\}$ by

$$h(x) = \begin{cases} a(x)\frac{u^{\sigma}_{\omega_2}(x)-u^{\sigma}_{\omega_1}(x)}{u_{\omega_1}(x)-u_{\omega_2}(x)}, & \text{if } u_{\omega_2}(x) \neq u_{\omega_1}(x), \\ 0, & \text{if } u_{\omega_2}(x) = u_{\omega_1}(x). \end{cases}$$

Since $\sigma < 0$, then $h \in \mathcal{B}^+(D\backslash\{0\})$ and we have

$$u_{\omega_2} - u_{\omega_1} + V(h(u_{\omega_2} - u_{\omega_1})) = (\omega_2 - \omega_1)|x|^{\alpha-n},\ x \in \overline{D}\backslash\{0\}.$$

Furthermore, by (3.23), (3.24) and (3.29), we conclude that, for $x \in \overline{D}\backslash\{0\}$

$$\begin{aligned} V(h|u_{\omega_2} - u_{\omega_1}|)(x) &\leq (\omega_1^{\sigma} + \omega_2^{\sigma}) \int_D G^D_{\alpha}(x,y)a(y)|y|^{(\alpha-2)\sigma}\, dy \\ &< \infty. \end{aligned}$$

Hence the result (3.30) holds by Lemma 2.3. □

Proposition 3.11. *Let $\sigma < 0$ and assume that hypothesis (H) is satisfied. Then problem P has at least one positive solution $u \in C(\overline{D}\backslash\{0\})$ such that, for $x \in \overline{D}\backslash\{0\}$*

$$u = V(a(x)u^{\sigma})(x). \tag{3.31}$$

Proof. Let $(\omega_k)_k$ be a sequence of positive real numbers decreasing to zero and denote by u_k the positive continuous solution of problem P_{ω_k}. By Corollary 3.10, the sequence $(u_k)_k$ decreases to a function u and since the sequence $(u_k - \omega_k|x|^{\alpha-n})_k$ increases to u. Then, by (3.23) and (3.29), we have for each $x \in \overline{D}\backslash\{0\}$ and $\sigma < 0$,

$$\begin{aligned} u(x) &\geq u_k(x) - \omega_k|x|^{\alpha-n} \\ &= \int_D G^D_{\alpha}(x,y)a(y)u_k^{\sigma}(y)\, dy \\ &\geq \beta_k^{\sigma} \int_D G^D_{\alpha}(x,y)a(y)\, dy > 0, \end{aligned}$$

where $\beta_k = \omega_k + \omega_k^{\sigma}\|h\|_{\infty}$. Therefore, by the monotone convergence theorem, we obtain

$$u(x) = \int_D G^D_{\alpha}(x,y)a(y)u^{\sigma}(y)\, dy.$$

Let us prove that u is a positive continuous solution of (1.1). Since for each $x \in \overline{D}\backslash\{0\}$, we have

$$u(x) = \inf_k u_k(x) = \sup_k (u_k - \omega_k|x|^{\alpha-n}),$$

then u is upper and lower semi-continuous function on $\overline{D}\backslash\{0\}$ and so $u \in C(\overline{D}\backslash\{0\})$.
Hence the function $y \mapsto a(y)u^{\sigma}(y)$ is in $L^1_{loc}(D\backslash\{0\})$ and we have

$$y \mapsto \int_D G^D_{\alpha}(x,y)a(y)u^{\sigma}(y)\, dy \in L^1_{loc}(D\backslash\{0\}).$$

Using (2.6), we conclude that

$$(-\Delta|_D)^{\frac{\alpha}{2}} u = a(x)u^{\sigma}(x)\ x \in D\backslash\{0\},\ (\text{ in the distributional sense }).$$

Now, since $0 < u(x) \leq u_k(x)$ for each $x \in D\backslash\{0\}$, and u_k is a solution of problem (P_{ω_k}), then

$$\lim_{|x|\to 0} |x|^{n-\alpha}u(x) = 0.$$

By (3.26) and using the fact that $0 < |x| \leq d$ as $x \to \partial D$, we have

$$\lim_{x\to\partial D} \delta(x)^{2-\alpha}u(x) = \lim_{x\to\partial D} \delta(x)^{2-\alpha}|x|^{\alpha-n}v_{\omega}(x) = 0.$$

Consequently, u is a solution of problem (1.1). This completes the proof. □

4. Proof of Theorem 1.2

Let a be a function satisfying (H) and let Θ be the function defined in (1.3), by proposition 3.8, there exists $M \geq 1$ such that

$$\frac{1}{M}\Theta(x) \leq V(a(x)\Theta(x)^{\sigma}) \leq M\Theta(x) \text{ for all } x \in D\backslash\{0\}, \tag{4.1}$$

We divide the proof of Theorem 1.2 into two cases according to the sign of σ.

Case 1. If $\sigma < 0$.

By Proposition 3.11, problem (3.7) has a positive continuous solution u satisfying (3.31). Let us prove that u satisfies (1.4). Assume that $p(x) = a(x)\Theta(x)^{\sigma}$, then, by (4.1), we obtain

$$M^{\sigma}(Vp)^{\sigma}(x) \leq \Theta^{\sigma}(x) \leq M^{-\sigma}(Vp)^{\sigma}(x). \tag{4.2}$$

Put c be a positive constant, then we have

$$cVp(x) = V((cVp)^{\sigma})(x) + V(ca\Theta^{\sigma} - c^{\sigma}(Vp)^{\sigma}))(x). \tag{4.3}$$

Define the function f by

$$f(x) = ca(x)[\Theta(x)^{\sigma} - c^{\sigma}(Vp)^{\sigma}(x)] \in \mathcal{B}^{+}(D\backslash\{0\}),$$

then, by (4.3), for $c = M^{-\frac{\sigma}{1-\sigma}}$,

$$f(x) = ca(x)(\Theta(x)^{\sigma} - M^{\sigma}(Vp)^{\sigma}(x)),\ x \in D\backslash\{0\}.$$

Using (3.31) and (4.3), we obtain

$$cVp - u + V(a(u^{\sigma} - (cVp)^{\sigma}) = cVp - u + V(a(u^{\sigma}) - (cVp)^{\sigma}) = Vf.$$

Let g be a function defined on $D\backslash\{0\}$ by

$$g(x) = \begin{cases} a(x)\frac{u^{\sigma}(x)-(cVp)^{\sigma}(x)}{cVp(x)-u(x)}, & \text{if } u(x) \neq cVp(x), \\ 0, & \text{if } u(x) = cVp(x). \end{cases}$$

Since $\sigma < 0$, then $g \in \mathcal{B}^{+}(D\backslash\{0\})$ and

$$g(x)(cVp - u)(x) = a(x)(u^{\sigma} - (cVp)^{\sigma})(x), \text{ for all } x \in D\backslash\{0\}. \tag{4.4}$$

Hence

$$cVp - u + V(g(cVp - u)) = Vf.$$

By (3.31), (4.1), (4.2) and (4.4), we obtain

$$\begin{aligned} V(g|cVp - u|) &\leq V(au^{\sigma}) + V(a(cVp)^{\sigma}) \\ &\leq u + cVp \\ &\leq u + cM\Theta \\ &< \infty. \end{aligned}$$

So, by Lemma 2.3, we have

$$u \leq cVp.$$

In the same way as above, we get that

$$\frac{1}{c}Vp \leq u.$$

Thus, by (4.1), u satisfies (1.4).

Case 2. $0 \leq \sigma < 1$

Let $g(x) = |x|^{n-\alpha}\Theta(x),\ x \in D$. By (4.1), we have, for $p(x) = a(x)\Theta(x)^{\sigma}$,

$$\frac{1}{M}g(x) \leq |x|^{n-\alpha}Vp(x) \leq Mg(x). \tag{4.5}$$

Put $c = M^{\frac{1}{1-\sigma}}$, where M is the constant defined in (4.1).

Let $B = \{v \in C_0(\overline{D}),\ \frac{1}{c}g \leq v \leq cg\}$, be the closed convex non empty set and let T be the operator defined on B, for all $v \in B$, by

$$Tv(x) := |x|^{n-\alpha}\int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}v^{\sigma}(y)\,dy,\ \ x \in D.$$

Further, by (4.5), we obtain

$$\frac{1}{c}g \leq Tv \leq cg$$

Since, for all $v \in B$, we have

$$|v^{\sigma}(y)| \leq c^{\sigma}\|g^{\sigma}\|_{\infty},\ y \in D,$$

by the same arguments as in the proof of Proposition 3.9, we deduce that

$$Tv \in C_0(\overline{D}),\ \text{for all}\ \ v \in B.$$

Then

$$T(B) \subset B.$$

Consider the sequence of function $(v_k)_k$ defined by

$$v_0 = \frac{1}{c}g \text{ and } v_{k+1} = Tv_k,\ \ k \in \mathbb{N}.$$

From the monotonicity of T and using the fact that $T(B) \subset B$, we deduce that

$$\frac{1}{c}g = v_0 \leq v_1 \leq v_2 \leq ... \leq v_k \leq v_{k+1} \leq cg.$$

So, thanks to the convergence monotone theorem, the sequence $(v_k)_k$ convergence to a function v satisfying for each $x \in \overline{D}$

$$\frac{1}{c}g(x) \leq v(x) \leq cg(x),$$

and

$$v(x) = |x|^{n-\alpha}\int_D G_\alpha^D(x,y)a(y)|y|^{(\alpha-n)\sigma}v^{\sigma}(y)\,dy.$$

Since v is bounded, we prove by similar arguments as in the proof of Proposition 3.9 that $v \in C_0(\overline{D})$. Put $u(x) = |x|^{\alpha-n}v(x)$. Then $u \in C_0(\overline{D}\backslash\{0\})$ and satisfies the following equation

$$u(x) = V(au^{\sigma})(x),\ x \in \overline{D}\backslash\{0\}. \tag{4.6}$$

Now, using the fact that $y \mapsto a(y)u^{\sigma}(y)$ is in $L^1_{loc}(D\backslash\{0\})$ and by (4.6) the function $x \mapsto V(au^{\sigma})$ is also in $L^1_{loc}(D\backslash\{0\})$. Then, by (2.6), we deduce that u is a solution of problem (1.1). This ends the proof.

We end this section with the following example that illustrates our result.

Example 4.1. *Let $\sigma < 1$ and a be a nonnegative function in $C(D\backslash\{0\})$ such that, for $x \in D\backslash\{0\}$,*

$$a(x) \approx |x|^{-\lambda} \left(\ln\left(\frac{4d}{|x|}\right)\right)^{-\beta_1} \delta(x)^{-\xi} \left(\ln\left(\frac{4d}{\delta(x)}\right)\right)^{-\beta_2},$$

where $\lambda < (n-2+\alpha)(1-\sigma) + \alpha\sigma$, $\xi < \alpha$, $\beta_1 > 1$, $\beta_2 > 1$ and $d := \mathrm{diam}(D)$. Then, by Theorem 1.2, problem (1.1) has at least one positive continuous solution u satisfying the following.

If $\alpha < \lambda < (n-2+\alpha)(1-\sigma) + \alpha\sigma$ and $\alpha - 1 + \sigma < \xi < \alpha$, then

$$u(x) \approx |x|^{\frac{\alpha-\lambda}{1-\sigma}} \left(\ln\left(\frac{4d}{|x|}\right)\right)^{\frac{-\beta_1}{1-\sigma}} \delta(x)^{\frac{\alpha-\xi}{1-\sigma}} \left(\ln\left(\frac{4d}{\delta(x)}\right)\right)^{\frac{-\beta_2}{1-\sigma}}.$$

If $\lambda \leq \alpha$ and $\xi \leq \alpha - 1 + \sigma$, then

$$u(x) \approx |x|\delta(x).$$

Indeed, let $L_1(t) = \left(\ln\left(\frac{4d}{t}\right)\right)^{-\beta_1}$ and $L_2(t) = \left(\ln\left(\frac{4d}{t}\right)\right)^{-\beta_2}$, $0 < t < d$.
If $\lambda < \alpha$ and $\xi < \alpha - 1 + \sigma$, we have $\tilde{L}_1(t) = 1$ and $\tilde{L}_2(t) = 1$. Then

$$u(x) \approx |x|\delta(x).$$

If $\lambda = \alpha$ and $\xi = \alpha - 1 + \sigma$, we have $\tilde{L}_1(t) \approx 1$ and $\tilde{L}_2(t) \approx 1$. Then

$$u(x) \approx |x|\delta(x).$$

If $\alpha < \lambda < (n-2+\alpha)(1-\sigma) + \alpha\sigma$ and $\alpha - 1 + \sigma < \xi < \alpha$ we have $\tilde{L}_1(t) = \left(\ln\left(\frac{4d}{t}\right)\right)^{-\beta_1}$ and $\tilde{L}_2(t) = \left(\ln\left(\frac{4d}{t}\right)\right)^{-\beta_2}$. Then

$$u(x) \approx |x|^{\frac{\alpha-\lambda}{1-\sigma}} \left(\ln\left(\frac{4d}{|x|}\right)\right)^{\frac{-\beta_1}{1-\sigma}} \delta(x)^{\frac{\alpha-\xi}{1-\sigma}} \left(\ln\left(\frac{4d}{\delta(x)}\right)\right)^{\frac{-\beta_2}{1-\sigma}}.$$

"My manuscript has no associate data"

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