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# Existence results for non-instantaneous impulsive Riemann-Liouville fractional stochastic differential equations with delay

**Fatima Zahra Arioui**[a]

[a]*Laboratory of Statistics and Stochastic Processes, University of Djillali Liabes, PB 89, Sidi Bel Abbes 22000, Algeria*

**Abstract.** In this paper, we investigate a fractional stochastic differential equation with delay and non-instantaneous impulses involving the Riemann-Liouville derivative of order $\alpha \in (1/2, 1)$ with a fixed lower bound. The integral representation of the mild solution is presented, and existence results are established using Banach's fixed point theorem under appropriate assumptions. An example is provided to illustrate the main result.

## 1. Introduction

In recent years, many scientists have become interested in fractional theory because of the memory property of fractional derivatives, which represent many real-world phenomena. The applications of fractional differential equations in various fields have received great attention since the qualitative and quantitative properties of these equations, such as existence, stability, and controllability, are the most discussed.

To model phenomena using fractional differential equations, we should define physically interpretable initial conditions. In fractional differential equations with Caputo derivatives, the initial conditions are expressed in terms of the initial values of the integer-order derivatives. On the other hand, initial conditions for fractional differential equations in terms of the Riemann-Liouville derivative must be defined in terms of fractional derivatives of the unknown function, and it is not clear how to measure these values in an experiment. Heymans and Podlubny [10] proved that initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals can have physical meaning and that the corresponding quantities can be determined by measurements.

Many different real-world processes and phenomena are modeled using delay differential equations, where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times. This delay can be constant, a function of time (time-dependent delay), or a function of a variable (state-dependent delay) (see, for example [4, 9, 11] and the references therein). Moreover, many processes experience several abrupt changes in their state at discrete points, known as impulses, which can be instantaneous or non-instantaneous. It is necessary to model these processes using impulsive differential equations.

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*Email address:* ariouifatimazahra@gmail.com (Fatima Zahra Arioui)

Several papers on impulsive fractional differential equations with delay have been published. For example, some basic points in introducing non-instantaneous impulses in Caputo fractional differential equations are given in [3], along with two approaches in interpretation of the solutions. Integral representations of solutions and existence results are provided in [1] for two cases of the lower limit of the Riemann-Liouville fractional derivative in the presence of delay and impulses.

There are many publications on Riemann-Liouville fractional differential equations with delay and instantaneous impulses. However, a few papers on these equations involving non-instantaneous impulses have recently been published (see [2, 15]).

Agarwal et al. [2] investigated the following nonlinear fractional differential equations with a constant delay where initial conditions and non-instantaneous impulsive conditions are appropriately established (in integral or weighted form) depending on the type of fractional derivative (with a fixed or changeable lower limit):

$$
\begin{aligned}
&{}^{R}D_t^q x(t) = f(t, x(t), x(t-\tau)), \;\; t \in (t_k, s_k], \; k = 0, 1, \dots, m, \; 0 < q < 1, \\
&x(t) = \phi_k(t, x(s_k - 0)), \;\; t \in (s_k, t_{k+1}], \; k = 1, 2, \dots, m, \\
&{}_{t_{k+1}}I_t^{1-q} x(t)|_{t=t_{k+1}} = \phi_k(t_{k+1}, x(s_k - 0)), \\
&\qquad \text{or } \lim_{t \to t_{k+1}^+} ((t - t_{k+1})^{1-q} x(t)) = \phi_k(t_{k+1}, x(s_k - 0)), \\
&{}_{t_0}I_t^{1-q} x(t)|_{t=t_0} = \psi(0), \;\; \text{or } \lim_{t \to t_0^+} ((t - t_0)^{1-q} x(t)) = \psi(0), \\
&x(t + t_0) = \psi(t), \;\; t \in [-\tau, 0].
\end{aligned}
$$

The existence of the stochastic term (a random term) is important due to the possibility of unpredictability in the characteristics of natural systems. For this reason, stochastic differential equations (SDEs) are considered important tools in modeling and simulating real phenomena and have been applied in a variety of fields. For more references on SDEs, refer to [5, 6, 13]. During the last decades, a few papers have been presented on fractional stochastic differential equations involving Riemann-Liouville derivatives (see, for example, [8, 16]). Therefore, and based on the above model, we propose in this paper the following fractional stochastic differential equation with delay and non-instantaneous impulses:

$$
{}^{R}_{s_0}D_t^\alpha u(t) = f(t, u(t), u(t-\tau)) + g(t, u(t), u(t-\tau)) \frac{dw(t)}{dt}, \; t \in (s_j, t_{j+1}], \; j = 0, 1, \dots, m, \tag{1}
$$

$$
u(t) = \Psi_j(t, u(t_j^-)), \;\; t \in (t_j, s_j], \;\; j = 1, 2, \dots, m, \tag{2}
$$

$$
{}_{s_j}I_t^{1-\alpha} u(t)|_{t=s_j} = \int_{t_j}^{s_j} \Psi_j(s, u(t_j^-)) ds, \;\; j = 1, 2, \dots, m, \tag{3}
$$

$$
{}_0I_t^{1-\alpha} u(t)|_{t=0} = \varphi(0), \;\; \text{and} \;\; u(t) = \varphi(t), \;\; t \in [-\tau, 0]. \tag{4}
$$

where ${}^{R}_{s_0}D_t^\alpha$ denotes the Riemann-Liouville derivative of order $\frac{1}{2} < \alpha < 1$ with a fixed lower bound $s_0 = 0$, $t \in J = (s_0, b]$; the non-negative points $\{s_j\}_{j=0}^m$ and $\{t_{j+1}\}_{j=0}^m$ are given such that $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = b$. The unknown $u(\cdot)$ is a random variable that takes its values in a separable Hilbert space $\mathcal{H}$. The values $u(t_j) = u(t_j^-) = \lim_{\epsilon \to 0} u(t_j - \epsilon)$ and $u(t_j^+) = \lim_{\epsilon \to 0} u(t_j + \epsilon)$ denote the left and right hand limits of $u(t)$ at $t_j$ for $j = 1, 2, \dots, m$, respectively. The process jumps to the states $u(t_j^+) = \lim_{\epsilon \to 0} u(t_j + \epsilon)$ at the points $t_j$ where the impulsive functions $\Psi_j$ begin and continue along the interval $(t_j, s_j]$ for $j = 1, 2, \dots, m$. $f : \cup_{j=0}^m [s_j, t_{j+1}] \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $g : \cup_{j=0}^m [s_j, t_{j+1}] \times \mathcal{H} \times \mathcal{H} \to \mathcal{L}_2^0$, $\Psi_j : [t_j, s_j] \times \mathcal{H} \to \mathcal{H}$ for $j = 1, 2, ..., m$, and $\varphi : [-\tau, 0] \to \mathcal{H}$ are appropriate functions.

The paper is organized as follows: In Section 2, we introduce some materials that will be used throughout this paper. In Section 3, we present the integral representation of the solution to the system (1)-(4). Section 4 contains the main result. An example illustrating the result obtained is presented in Section 5.

## 2. Preliminaries

In this section, we will introduce some basic concepts and notations that will be used in this paper to answer the existence problem for the system (1)-(4).

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces, and $L(\mathcal{K},\mathcal{H})$ be the space of all linear and bounded operators from $\mathcal{K}$ to $\mathcal{H}$. We use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{K}$, $\mathcal{H}$ and $L(\mathcal{K},\mathcal{H})$, and we use $(\cdot,\cdot)$ to denote the inner product of $\mathcal{K}$ and $\mathcal{H}$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. An $\mathcal{H}$-valued random variable is an $\mathcal{F}$-measurable function $u(t,\cdot):\Omega\to\mathcal{H}$; in the rest of the paper, we write $u(t)$ instead of $u(t,\varpi)$ for all $\varpi\in\Omega$.

Let $\{w(t)\}_{t\geq 0}$ be a $Q$-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ with the covariance operator $Q$ such that $trQ<\infty$. We assume that there exists a complete orthonormal system $\varsigma_k$ in $\mathcal{K}$, and positive real numbers $\lambda_k$ such that $Q\varsigma_k=\lambda_k\varsigma_k, k=1,2,\cdots$, and a sequence of independent Brownian motions such that $(w(t),e)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(\varsigma_k,e)\beta_k(t),\ e\in\mathcal{K},\ t\geq 0.$

Let $\mathcal{L}_2^0=\mathcal{L}_2(Q^{\frac{1}{2}}\mathcal{K},\mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ into $\mathcal{H}$ with the inner product $<\psi,\phi>_{\mathcal{L}_2^0}=tr(\psi Q\phi^*)$.

The space of all strongly measurable, square-integrable, $\mathcal{H}$-valued random variables, denoted by $\mathcal{L}_2(\Omega,\mathcal{H})$, is a Banach space equipped with the norm $\|u(\cdot)\|^2_{\mathcal{L}_2(\Omega,\mathcal{H})}=\mathbb{E}\|u(\cdot)\|^2$.

Let $C((0,b];\mathcal{L}_2(\Omega,\mathcal{H}))$ be the Banach space of all continuous $\mathcal{F}_t$-adapted measurable processes with the norm $\|u\|_C=\sup_{t\in(0,b]}(\mathbb{E}\|u(t)\|^2)^{\frac{1}{2}}$. Consider the following space

$$\begin{aligned}\mathcal{PC}_{1-\alpha}(J,\mathcal{H})=\{x:J\times\Omega\to\mathcal{H}:&\ (t-s_j)^{1-\alpha}x(t)\in C((\cup_{j=0}^{m}(s_j,t_{j+1}])\cup(\cup_{j=0}^{m-1}(t_{j+1},s_{j+1}]),\mathcal{L}_2(\Omega,\mathcal{H})),\\ &x(t_j)=x(t_j^-)=\lim_{\epsilon\to 0}x(t_j-\epsilon)<\infty\ \text{ and}\\ &x(s_j)=x(s_j^-)=\lim_{\epsilon\to 0}x(s_j-\epsilon)<\infty,\ j=1,2,\ldots,m,\\ &\mathbb{E}\left\|(t-s_j)^{1-\alpha}x(t)\right\|^2<\infty,\ \text{ for }\ t\in(s_j,s_{j+1}],\ j=0,1,\ldots,m-1,\\ &\mathbb{E}\left\|(t-s_m)^{1-\alpha}x(t)\right\|^2<\infty,\ \text{ for }\ t\in(s_m,t_{m+1}]\}.\end{aligned}$$

We introduce a norm on the space $\mathcal{PC}_{1-\alpha}(J,\mathcal{H})$ defined by:

$$\|x\|_{\mathcal{PC}_{1-\alpha}(J,\mathcal{H})}=\|x\|_J=\max_{j=0,1,\ldots,m.}\|x\|_j,$$

where $\|x\|_j=\sup_{t\in(s_j,s_{j+1}]}\left(\mathbb{E}\left\|(t-s_j)^{1-\alpha}x(t)\right\|^2\right)^{\frac{1}{2}}$ for $j=0,1,\ldots,m-1$, and $\|x\|_m=\sup_{t\in(s_m,t_{m+1}]}\left(\mathbb{E}\left\|(t-s_m)^{1-\alpha}x(t)\right\|^2\right)^{\frac{1}{2}}$.
Here $\mathcal{PC}_{1-\alpha}([-\tau,b],\mathcal{H})$ is the space of all functions $x\in\mathcal{PC}_{1-\alpha}(J,\mathcal{H})$ with $x(t)=\varphi(t)$ on $[-\tau,0]$.

Let us recall some elementary concepts related to fractional calculus (cf,. [7, 12, 14]).

**Definition 2.1.** *Let $0\leq t_0<T\leq\infty$.*

- *The Riemann-Liouville $\alpha^{th}$-order fractional integral is defined by*

$${}_{t_0}I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}u(s)ds,\ \ t\in[t_0,T],\ \alpha>0,$$

*where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}dt$.*

- *The fractional $\alpha^{th}$-order Riemann-Liouville derivative is defined by*

$$ {}_{t_0}^{R}D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\Big(\int_{t_0}^{t}(t-s)^{n-\alpha-1}u(s)ds\Big), \ \ t \in [t_0, T], \ 0 \le n-1 < \alpha < n. $$

,

**Lemma 2.2.** *For $\alpha > 0$, $k = \lfloor \alpha \rfloor + 1$ and ${}_{t_0}I_t^{k-\alpha}u(t) = u_{k-\alpha}(t)$, the following equality holds:*

$$ {}_{t_0}I_t^{\alpha}\, {}_{t_0}^{R}D_t^{\alpha}u(t) = u(t) - \sum_{i=1}^{k}\frac{u_{k-\alpha}^{(k-i)}(t_0)}{\Gamma(\alpha-i+1)}(t-t_0)^{\alpha-i}. $$

**Proposition 2.3.** *For $t > t_0$, $\gamma > 0$ and $0 < \alpha < 1$, the following are true:*

*i)* ${}_{t_0}^{R}D_t^{\alpha}(t-t_0)^{\gamma-1} = \dfrac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(t-t_0)^{\gamma-\alpha-1}$,

*ii)* ${}_{t_0}I_t^{\alpha}(t-t_0)^{\gamma-1} = \dfrac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}(t-t_0)^{\gamma+\alpha-1}$,

*iii)* ${}_{t_0}^{R}D_t^{\alpha}(t-t_0)^{\alpha-1} = 0$.

## 3. Integral representation of a mild solution

In the following, we will give the integral representation of the mild solution to the system (1)-(4).

**Lemma 3.1.** *The stochastic process $u \in \mathcal{PC}_{1-\alpha}(J, \mathcal{H})$ is a mild solution to (1)-(4) if the following integral equation is satisfied:*

$$ u(t) = \begin{cases} \dfrac{(t-s_j)^{\alpha-1}}{\Gamma(\alpha)}\displaystyle\int_{t_j}^{s_j}\Psi_j(s, u(t_j^-))ds + \frac{1}{\Gamma(\alpha)}\int_{s_j}^{t}\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds \\ \quad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{s_j}^{t}\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s) \\ \quad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{s_j}^{t}\frac{\Phi_j(s,u(s))}{(t-s)^{1-\alpha}}ds, \ \ t \in (s_j, t_{j+1}], \ j = 0, 1, \ldots, m \\ \Psi_j(t, u(t_j^-)), \ \ t \in (t_j, s_j], \ j = 1, 2, \ldots, m. \end{cases} \tag{5} $$

*where $\Psi_0 = \varphi(0)$, $\Phi_0 = 0$, and for $j = 1, \ldots, m$,*

$$ \Phi_j(t, u(t)) = \frac{\alpha}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\Big(\int_{s_k}^{t_{k+1}}(t-s)^{-\alpha-1}u(s)ds + \int_{t_{k+1}}^{s_{k+1}}(t-s)^{-\alpha-1}\Psi_{k+1}(s, u(t_{k+1}^-))ds\Big). \tag{6} $$

*Proof.*

Let $t \in (s_0, t_1]$. From Lemma 2.2, the general integral equation of the system (1)-(4) can be expresses as

$$ u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s). \tag{7} $$

Let $t \in (s_1, t_2]$, then

$$\begin{aligned}
{}^R_0D^\alpha_t u(t) &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dt}\int_0^{t_1}(t-s)^{-\alpha}u(s)ds + \frac{d}{dt}\int_{t_1}^{s_1}(t-s)^{-\alpha}u(s)ds + \frac{d}{dt}\int_{s_1}^{t}(t-s)^{-\alpha}u(s)ds\right)\\
&= -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha-1}u(s)ds - \frac{\alpha}{\Gamma(1-\alpha)}\int_{t_1}^{s_1}(t-s)^{-\alpha-1}\Psi_1(s,u(t_1^-))ds\\
&\quad + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{s_1}^{t}(t-s)^{-\alpha}u(s)ds\\
&= -\Phi_1(t,u(t)) +^R_{s_1} D^\alpha_t u(t),
\end{aligned}$$

where $\Phi_1(t,u(t)) = \dfrac{\alpha}{\Gamma(1-\alpha)}\Big(\int\limits_0^{t_1}(t-s)^{-\alpha-1}u(s)ds + \int\limits_{t_1}^{s_1}(t-s)^{-\alpha-1}\Psi_1(s,u(t_1^-))ds\Big)$.
From (1) and (3), we obtain the following initial problem for $t \in (s_1, t_2]$

$$\begin{aligned}
&{}^R_{s_1}D^\alpha_t u(t) = f(t,u(t),u(t-\tau)) + g(t,u(t),u(t-\tau))\frac{dw(t)}{dt} + \Phi_1(t,u(t)),\\
&{}_{s_1}I_t^{1-\alpha}u(t)|_{t=t_1} = \int_{t_1}^{s_1}\Psi_1(s,u(t_1^-))ds.
\end{aligned}$$

From Lemma 2.2, we get

$$\begin{aligned}
u(t) &= \frac{(t-s_1)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{s_1}\Psi_1(s,u(t_1^-))ds + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{t}\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{t}\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s) + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{t}\frac{\Phi_1(s,u(s))}{(t-s)^{1-\alpha}}ds.
\end{aligned}$$

Continuing this process for $j = 2, 3, \ldots, m$, we prove the integral representation (5).

Now we show that the mild solution satisfies the system (1)-(4). From (5), we have

$$\begin{aligned}
u(t) &= \frac{(t-s_j)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_j}^{s_j}\Psi_j(s,u(t_j^-))ds + \frac{1}{\Gamma(\alpha)}\int\limits_{s_j}^{t}\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int\limits_{s_j}^{t}\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s) + \frac{1}{\Gamma(\alpha)}\int\limits_{s_j}^{t}\frac{\Phi_j(s,u(s))}{(t-s)^{1-\alpha}}ds
\end{aligned}$$

By the Definition 2.1, we have for every $t \in (s_j, t_{j+1}]$, $j = 0, 1, \ldots, m$.

$$\begin{aligned}
{}^R_0D^\alpha_t u(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^{t}(t-s)^{-\alpha}u(s)ds\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\sum_{k=0}^{j-1}\left(\int_{s_k}^{t_{k+1}}(t-s)^{-\alpha}u(s)ds + \int_{t_{k+1}}^{s_{k+1}}(t-s)^{-\alpha}u(s)ds\right) + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{s_j}^{t}(t-s)^{-\alpha}u(s)ds\\
&= -\frac{\alpha}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{s_k}^{t_{k+1}}(t-s)^{-\alpha-1}u(s)ds - \frac{\alpha}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{t_{k+1}}^{s_{k+1}}(t-s)^{-\alpha}\Psi_{k+1}(t,u(t_{k+1}^-))ds\\
&\quad + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{s_j}^{t}(t-s)^{-\alpha}\Big(\frac{(s-s_j)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_j}^{s_j}\Psi_j(\tilde{s},u(t_j^-))d\tilde{s} + \frac{1}{\Gamma(\alpha)}\int\limits_{s_j}^{s}\frac{f(\tilde{s},u(\tilde{s}),u(\tilde{s}-\tau))}{(s-\tilde{s})^{1-\alpha}}d\tilde{s}
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{s_j}^{s}\frac{g(\tilde{s},u(\tilde{s}),u(\tilde{s}-\tau))}{(s-\tilde{s})^{1-\alpha}}dw(\tilde{s})+\frac{1}{\Gamma(\alpha)}\int_{s_j}^{s}\frac{\Phi_j(\tilde{s},u(\tilde{s}))}{(s-\tilde{s})^{1-\alpha}}d\tilde{s}\Big)ds.$$

Then, from (6) and Proposition 2.3, we have

$$\begin{aligned}
{}_0^RD_t^{\alpha}u(t) &= -\Phi_j(t,u(t))+\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\Big({}_{s_j}I_t^{1-\alpha}(t-s_j)^{\alpha-1}\int_{t_j}^{s_j}\Psi_j(\tilde{s},u(t_j^-))d\tilde{s}\Big)\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{s_j}^{t}(t-s)^{-\alpha}\Big({}_{s_j}I_s^{\alpha}f(s,u(s),u(s-\tau))+{}_{s_j}I_s^{\alpha}g(s,u(s),u(s-\tau))\frac{dw(s)}{ds}+{}_{s_j}I_s^{\alpha}\Phi_j(s,u(s))\Big)ds\\
&= -\Phi_j(t,u(t))+{}_{s_j}^{R}D_t^{\alpha}\Big({}_{s_j}I_t^{\alpha}f(t,u(t),u(t-\tau))+{}_{s_j}I_t^{\alpha}g(t,u(t),u(t-\tau))\frac{dw(t)}{dt}+{}_{s_j}I_t^{\alpha}\Phi_j(t,u(t))\Big)\\
&= f(t,u(t),u(t-\tau))+g(t,u(t),u(t-\tau))\frac{dw(t)}{dt}. \qquad (8)
\end{aligned}$$

We can see that the mild solution (5) is adequate for the system (1)-(4). Based on this, we define a mild solution for (1)-(4). □

## 4. Existence results

Let us introduce the following hypotheses:

(H1) The function $f:\cup_{j=0}^{m}[s_j,t_{j+1}]\times\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ satisfies the following condition:

There exist two constants $L_f, M_f>0$, such that for all $t\in[s_j,t_{j+1}]$, $j=0,1,\dots,m$ and $x_1,x_2,y_1,y_2\in\mathcal{H}$, we have

$$\mathbb{E}\left\|f(t,x_1,y_1)-f(t,x_2,y_2)\right\|^2\le L_f\mathbb{E}\|x_1-x_2\|^2+M_f\mathbb{E}\|y_1-y_2\|^2.$$

(H2) The function $g:\cup_{j=0}^{m}[s_j,t_{j+1}]\times\mathcal{H}\times\mathcal{H}\to\mathcal{L}_2^0$ satisfies the following condition:

There exist two constants $L_g, M_g>0$, such that for all $t\in[s_j,t_{j+1}]$, $j=0,1,\dots,m$ and $x_1,x_2,y_1,y_2\in\mathcal{H}$, we have

$$\mathbb{E}\left\|g(t,x_1,y_1)-g(t,x_2,y_2)\right\|^2\le L_g\mathbb{E}\|x_1-x_2\|^2+M_g\mathbb{E}\|y_1-y_2\|^2.$$

(H3) For $j=1,2,\dots,m$, the impulsive functions $\Psi_j:[t_j,s_j]\times\mathcal{H}\to\mathcal{H}$ are continuous and satisfy the following condition:

$$\mathbb{E}\left\|\Psi_j(t,x)-\Psi_j(t,y)\right\|^2\le\lambda_{\Psi_j}\mathbb{E}\|x-y\|^2,\ \ \forall x,y\in\mathcal{H},\ \text{ with }\ \lambda_{\Psi_j}\in(0,1].$$

(H4) For $j=1,2,\dots,m$, the impulsive functions $\Psi_j:[t_j,s_j]\times\mathcal{H}\to\mathcal{H}$ are of order $(s_j-t)^{\alpha+1}$, i.e.,

$$\Psi_j(t,x)=(s_j-t)^{\alpha+1}\psi_j(t,x),\ \ \forall x\in\mathcal{H},$$

where the functions $\psi_j$ satisfy $\mathbb{E}\left\|\psi_j(t,x)-\psi_j(t,y)\right\|^2\le\lambda_{\psi_j}\mathbb{E}\left\|x-y\right\|^2$, $\forall x,y\in\mathcal{H}$, with $\lambda_{\psi_j}\in(0,1]$.

**Theorem 4.1.** *Assume that conditions (H1)-(H4) hold. Then (1)-(4) has a unique solution $u\in\mathcal{PC}_{1-\alpha}([-\tau,b],\mathcal{H})$ if the following condition holds:*

$$\begin{aligned}
\eta \;=\; \max_{j=1,\dots,m}\Big(&\frac{4\lambda_{\Psi_j}(s_j-t_j)^2(t_j-s_{j-1})^{2(\alpha-1)}}{\Gamma^2(\alpha)}+\frac{4L_f\Gamma(2\alpha-1)(t_{j+1}-s_j)^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)}+\frac{4M_f(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)^2}{\Gamma^2(\alpha+1)}\\
&+\frac{4L_g\beta(2\alpha-1,2\alpha-1)(t_{j+1}-s_j)^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{4M_g(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)}{(2\alpha-1)\Gamma^2(\alpha)}
\end{aligned}$$

$$+\frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\Big(\sum_{k=0}^{j-1}\frac{(t_{j+1}-s_j)^2}{(2\alpha-1)(s_j-t_{k+1})^2}+\sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{j+1}-s_j)^2(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_j-s_{k+1})^{2\alpha}}+\frac{\lambda_{\psi_j}(s_j-t_j)^2(t_{j+1}-s_j)^2}{(t_j-s_{j-1})^{2(1-\alpha)}}\Big)\Big)<1. \tag{9}$$

*Proof.* Define the operator $N$ for any $u\in\mathcal{PC}_{1-\alpha}([-\tau,b],\mathcal{H})$ by

$$(Nu)(t)=\begin{cases}\varphi(t), & \text{for } t\in[-\tau,0]\\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\varphi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds\\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s), & \text{for } t\in(0,t_1]\\ \Psi_j(t,u(t_j^-)), & \text{for } t\in(t_j,s_j],\ j=1,2,\dots,m\\ \frac{(t-s_j)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_j}^{s_j}\Psi_j(s,u(t_j^-))ds+\frac{1}{\Gamma(\alpha)}\int_{s_j}^t\frac{f(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}ds\\ \quad+\frac{1}{\Gamma(\alpha)}\int_{s_j}^t\frac{g(s,u(s),u(s-\tau))}{(t-s)^{1-\alpha}}dw(s)\\ \quad+\frac{1}{\Gamma(\alpha)}\int_{s_j}^t\frac{\Phi_j(s,u(s))}{(t-s)^{1-\alpha}}ds, & \text{for } t\in(s_j,t_{j+1}],\ j=1,2,\dots,m.\end{cases}$$

It is evident that $N$ maps from $\mathcal{PC}_{1-\alpha}([-\tau,b],\mathcal{H})$ into itself. Now we prove that $N$ is a contraction mapping. Let $u,v\in\mathcal{PC}_{1-\alpha}([-\tau,b],\mathcal{H})$, then

**Case 1**: For $t\in(s_0,t_1]$ with $t_1\le\tau+s_0$, we have

$$(Nu)(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\varphi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s,u(s),\varphi(s-\tau))}{(t-s)^{1-\alpha}}ds+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s,u(s),\varphi(s-\tau))}{(t-s)^{1-\alpha}}dw(s).$$

Therefore,

$$\begin{aligned}&\mathbb{E}\Big\|t^{1-\alpha}\Big((Nu)(t)-(Nv)(t)\Big)\Big\|^2\\ &\le 2\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{(f(s,u(s),\varphi(s-\tau))-f(s,v(s),\varphi(s-\tau)))}{(t-s)^{1-\alpha}}ds\Big\|^2\\ &\quad+2\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{(g(s,u(s),\varphi(s-\tau))-g(s,v(s),\varphi(s-\tau)))}{(t-s)^{1-\alpha}}dw(s)\Big\|^2\\ &=I_1'+I_2'.\end{aligned}\tag{10}$$

Using Hölder's inequality, (H1), and Proposition 2.3, we get

$$\begin{aligned}I_1'&=2\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{(f(s,u(s),\varphi(s-\tau))-f(s,v(s),\varphi(s-\tau)))}{(t-s)^{1-\alpha}}ds\Big\|^2\\ &\le\frac{2t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|f(s,u(s),\varphi(s-\tau))-f(s,v(s),\varphi(s-\tau))\|^2ds\\ &\le\frac{2t^{2-\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha-1}L_f\mathbb{E}\|u(s)-v(s)\|^2ds\\ &\le\frac{2L_ft^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{2(\alpha-1)}\mathbb{E}\|s^{1-\alpha}(u(s)-v(s))\|^2ds\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2L_f t^{2-\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha)}\|u-v\|_0^2 \int_0^t (t-s)^{\alpha-1} s^{2(\alpha-1)} ds \\
&\leq \frac{2L_f t^{2-\alpha}}{\Gamma(\alpha+1)} \times \frac{\Gamma(2\alpha-1)t^{3\alpha-2}}{\Gamma(3\alpha-1)}\|u-v\|_0^2 \\
&\leq \frac{2L_f \Gamma(2\alpha-1) t_1^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)}\|u-v\|_0^2. \qquad (11)
\end{aligned}$$

From Proposition 2.3 and (H2), we get

$$\begin{aligned}
I_2' &= 2\mathbb{E}\Big\|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{(g(s,u(s),\varphi(s-\tau)) - g(s,v(s),\varphi(s-\tau)))}{(t-s)^{1-\alpha}} dw(s)\Big\|^2 \\
&\leq \frac{2t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\Big\|g(s,u(s),\varphi(s-\tau)) - g(s,v(s),\varphi(s-\tau))\Big\|^2 ds \\
&\leq \frac{2t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)} L_g \mathbb{E}\|u(s)-v(s)\|^2 ds \\
&\leq \frac{2L_g t^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)} s^{2(\alpha-1)} \mathbb{E}\|s^{1-\alpha}(u(s)-v(s))\|^2 ds \\
&\leq \frac{2L_g t^{2(1-\alpha)}}{\Gamma^2(\alpha)} \times \frac{\Gamma(2\alpha-1)\Gamma(2\alpha-1)t^{4\alpha-3}}{\Gamma(4\alpha-2)}\|u-v\|_0^2 \\
&\leq \frac{2L_g \beta(2\alpha-1,2\alpha-1) t_1^{2\alpha-1}}{\Gamma^2(\alpha)}\|u-v\|_0^2. \qquad (12)
\end{aligned}$$

By substituting (11) and (12) into (10), we get for $t \in (0, t_1]$,

$$\begin{aligned}
&\mathbb{E}\Big\|t^{1-\alpha}\Big((Nu)(t) - (Nv)(t)\Big)\Big\|^2 \\
&\leq \Big(\frac{2L_f \Gamma(2\alpha-1) t_1^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)} + \frac{2L_g \beta(2\alpha-1,2\alpha-1) t_1^{2\alpha-1}}{\Gamma^2(\alpha)}\Big)\|u-v\|_0^2 \\
&\leq \eta\|u-v\|_0^2. \qquad (13)
\end{aligned}$$

**Case 2**: For $t \in (s_j, t_{j+1}]$, $j = 1, 2, \ldots, m$.
Let $s_j > \tau + s_0$. For any $s \in [s_j, t]$, we have

$$\begin{aligned}
&\mathbb{E}\Big\|(t-s_j)^{1-\alpha}\Big((Nu)(t) - (Nv)(t)\Big)\Big\|^2 \\
&\leq 4\mathbb{E}\Big\|\frac{1}{\Gamma(\alpha)}\int_{t_j}^{s_j}\Big(\Psi_j(s,u(t_j^-)) - \Psi_j(s,v(t_j^-))\Big)ds\Big\|^2 \\
&\quad + 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^t \frac{(f(s,u(s),u(s-\tau)) - f(s,v(s),v(s-\tau)))}{(t-s)^{1-\alpha}} ds\Big\|^2 \\
&\quad + 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^t \frac{(g(s,u(s),u(s-\tau)) - g(s,v(s),v(s-\tau)))}{(t-s)^{1-\alpha}} dw(s)\Big\|^2 \\
&\quad + 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^t \frac{\Big(\Phi_j(s,u(s)) - \Phi_j(s,v(s))\Big)}{(t-s)^{1-\alpha}} ds\Big\|^2 \\
&= I_1 + I_2 + I_3 + I_4. \qquad (14)
\end{aligned}$$

From (H3), we have

$$I_1 = 4\mathbb{E}\Big\|\frac{1}{\Gamma(\alpha)}\int_{t_j}^{s_j}\Big(\Psi_j(s,u(t_j^-)) - \Psi_j(s,v(t_j^-))\Big)ds\Big\|^2$$

$$
\begin{aligned}
&\leq \frac{4(s_j-t_j)}{\Gamma^2(\alpha)}\int_{t_j}^{s_j}\mathbb{E}\Big\|\Psi_j(s,u(t_j^-))-\Psi_j(s,v(t_j^-))\Big\|^2ds\\
&\leq \frac{4(s_j-t_j)^2}{\Gamma^2(\alpha)}\lambda_{\Psi_j}\mathbb{E}\|u(t_j^-)-v(t_j^-)\|^2\\
&\leq \frac{4(s_j-t_j)^2\lambda_{\Psi_j}}{\Gamma^2(\alpha)}(t_j-s_{j-1})^{2(\alpha-1)}\mathbb{E}\Big\|(t_j-s_{j-1})^{1-\alpha}(u(t_j^-)-v(t_j^-))\Big\|^2\\
&\leq \frac{4(s_j-t_j)^2\lambda_{\Psi_j}}{\Gamma^2(\alpha)}(t_j-s_{j-1})^{2(\alpha-1)}\|u-v\|_{j-1}^2. \qquad (15)
\end{aligned}
$$

First, we calculate $I_2$ and $I_3$. Since we previously assumed that $s_j > \tau + s_0$, then we have for any $s \in [s_j, t]$ with $j = 1, 2, \ldots, m$, $s_0 < s_j - \tau \leq s - \tau$. Let $k$ be a natural number such that $s - \tau \in (s_k, t - \tau]$ for $s \in [s_j, t]$ and $t \in (s_j, t_{j+1}]$.
Using Hölder's inequality, (H1), and Proposition 2.3, we obtain

$$
\begin{aligned}
I_2 &= 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^{t}\frac{(f(s,u(s),u(s-\tau))-f(s,v(s),v(s-\tau)))}{(t-s)^{1-\alpha}}ds\Big\|^2\\
&\leq \frac{4(t-s_j)^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}ds\int_{s_j}^{t}(t-s)^{\alpha-1}\mathbb{E}\|f(s,u(s),u(s-\tau))-f(s,v(s),v(s-\tau))\|^2ds\\
&\leq \frac{4(t-s_j)^{2-\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{s_j}^{t}(t-s)^{\alpha-1}\Big(L_f\mathbb{E}\|u(s)-v(s)\|^2+M_f\mathbb{E}\|u(s-\tau)-v(s-\tau)\|^2\Big)ds\\
&\leq \frac{4(t-s_j)^{2-\alpha}L_f}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{s_j}^{t}(t-s)^{\alpha-1}(s-s_j)^{2(\alpha-1)}\mathbb{E}\|(s-s_j)^{1-\alpha}(u(s)-v(s))\|^2ds\\
&\quad+\frac{4(t-s_j)^{2-\alpha}M_f}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{s_j}^{t}(t-s)^{\alpha-1}(s-\tau-s_k)^{2(\alpha-1)}\mathbb{E}\|(s-\tau-s_k)^{1-\alpha}(u(s-\tau)-v(s-\tau))\|^2ds\\
&\leq \frac{4(t-s_j)^{2-\alpha}L_f}{\Gamma(\alpha)\Gamma(\alpha+1)}\|u-v\|_j^2\int_{s_j}^{t}(t-s)^{\alpha-1}(s-s_j)^{2(\alpha-1)}ds\\
&\quad+\frac{4(t-s_j)^{2-\alpha}M_f}{\Gamma(\alpha)\Gamma(\alpha+1)}\|u-v\|_k^2\int_{s_j}^{t}(t-s)^{\alpha-1}(s-\tau-s_k)^{2(\alpha-1)}ds\\
&\leq \Big(\frac{4(t-s_j)^{2-\alpha}L_f}{\Gamma(\alpha)\Gamma(\alpha+1)}\times\frac{\Gamma(\alpha)\Gamma(2\alpha-1)(t-s_j)^{3\alpha-2}}{\Gamma(3\alpha-1)}+\frac{4(t-s_j)^{2-\alpha}M_f}{\Gamma(\alpha)\Gamma(\alpha+1)}(s_j-\tau-s_0)^{2(\alpha-1)}\int_{s_j}^{t}(t-s)^{\alpha-1}ds\Big)\|u-v\|_J^2\\
&\leq \Big(\frac{4L_f\Gamma(2\alpha-1)(t-s_j)^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)}+\frac{4M_f(t-s_j)^2(s_j-\tau)^{2(\alpha-1)}}{\Gamma^2(\alpha+1)}\Big)\|u-v\|_J^2\\
&\leq \Big(\frac{4L_f\Gamma(2\alpha-1)(t_{j+1}-s_j)^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)}+\frac{4M_f(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)^2}{\Gamma^2(\alpha+1)}\Big)\|u-v\|_J^2. \qquad (16)
\end{aligned}
$$

From Proposition 2.3 and (H2), we have

$$
\begin{aligned}
I_3 &= 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^{t}\frac{(g(s,u(s),u(s-\tau))-g(s,v(s),v(s-\tau)))}{(t-s)^{1-\alpha}}dw(s)\Big\|^2\\
&\leq \frac{4(t-s_j)^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{2(\alpha-1)}\mathbb{E}\|g(s,u(s),u(s-\tau))-g(s,v(s),v(s-\tau))\|^2ds\\
&\leq \frac{4(t-s_j)^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{2(\alpha-1)}\Big(L_g\mathbb{E}\|u(s)-v(s)\|^2+M_g\mathbb{E}\|u(s-\tau)-v(s-\tau)\|^2\Big)ds
\end{aligned}
$$

$$\leq \frac{4(t-s_j)^{2(1-\alpha)}L_g}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{2(\alpha-1)}(s-s_j)^{2(\alpha-1)}\mathbb{E}\|(s-s_j)^{1-\alpha}(u(s)-v(s))\|^2ds$$
$$+\frac{4(t-s_j)^{2(1-\alpha)}M_g}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{2(\alpha-1)}(s-\tau-s_k)^{2(\alpha-1)}\mathbb{E}\Big\|(s-\tau-s_k)^{1-\alpha}(u(s-\tau)-v(s-\tau)\Big\|^2ds$$
$$\leq \frac{4(t-s_j)^{2(1-\alpha)}L_g}{\Gamma^2(\alpha)}\|u-v\|_j^2\int_{s_j}^{t}(t-s)^{2(\alpha-1)}(s-s_j)^{2(\alpha-1)}ds$$
$$+\frac{4(t-s_j)^{2(1-\alpha)}M_g}{\Gamma^2(\alpha)}\|u-v\|_k^2\int_{s_j}^{t}(t-s)^{2(\alpha-1)}(s-\tau-s_k)^{2(\alpha-1)}ds$$
$$\leq \Big(\frac{4(t-s_j)^{2(1-\alpha)}L_g}{\Gamma^2(\alpha)}\times\frac{\Gamma(2\alpha-1)\Gamma(2\alpha-1)(t-s_j)^{4\alpha-3}}{\Gamma(4\alpha-2)}$$
$$+\frac{4(t-s_j)^{2(1-\alpha)}M_g}{\Gamma^2(\alpha)}(s_j-\tau-s_0)^{2(\alpha-1)}\int_{s_j}^{t}(t-s)^{2(\alpha-1)}ds\Big)\|u-v\|_J^2$$
$$\leq \Big(\frac{4L_g\beta(2\alpha-1,2\alpha-1)(t_{j+1}-s_j)^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{4M_g(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)}{(2\alpha-1)\Gamma^2(\alpha)}\Big)\|u-v\|_J^2. \tag{17}$$

Second, we calculate $I_4$. From (6) and the Hölder inequality, we obtain for $t\in(s_j,t_{j+1}]$, $j=1,2,\ldots,m$,

$$I_4 = 4\mathbb{E}\Big\|\frac{(t-s_j)^{1-\alpha}}{\Gamma(\alpha)}\int_{s_j}^{t}\frac{\big(\Phi_j(s,u(s))-\Phi_j(s,v(s))\big)}{(t-s)^{1-\alpha}}ds\Big\|^2$$
$$\leq \frac{4(t-s_j)^{2(1-\alpha)}}{\Gamma^2(\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}ds\int_{s_j}^{t}(t-s)^{\alpha-1}\mathbb{E}\Big\|\Phi_j(s,u(s))-\Phi_j(s,v(s))\Big\|^2ds$$
$$\leq \frac{4(t-s_j)^{2-\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{s_j}^{t}(t-s)^{\alpha-1}\mathbb{E}\Big\|\Phi_j(s,u(s))-\Phi_j(s,v(s))\Big\|^2ds$$
$$\leq \frac{4(t-s_j)^{2-\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{s_j}^{t}(t-s)^{\alpha-1}\mathbb{E}\Big\|\frac{\alpha}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\Big(\int_{s_k}^{t_{k+1}}\frac{\big(u(z)-v(z)\big)}{(s-z)^{\alpha+1}}dz$$
$$+\int_{t_{k+1}}^{s_{k+1}}\frac{\big(\Psi_{k+1}(z,u(t_{k+1}^-))-\Psi_{k+1}(z,v(t_{k+1}^-))\big)}{(s-z)^{\alpha+1}}dz\Big)\Big\|^2ds$$
$$\leq \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}\sum_{k=0}^{j-1}\Big(\mathbb{E}\Big\|\int_{s_k}^{t_{k+1}}\frac{u(z)-v(z)}{(s-z)^{\alpha+1}}dz\Big\|^2$$
$$+\mathbb{E}\Big\|\int_{t_{k+1}}^{s_{k+1}}\frac{\Psi_{k+1}(z,u(t_{k+1}^-))-\Psi_{k+1}(z,v(t_{k+1}^-))}{(s-z)^{\alpha+1}}dz\Big\|^2\Big)ds$$
$$= I_{4,1}+I_{4,2}.$$

We have to calculate $I_{4,1}$ and $I_{4,2}$. For $s\in[s_j,t]$ and $t\in(s_j,t_{j+1}]$, $j=1,2,\ldots,m$, we use Hölder's inequality and $s-t_{k+1}<s-s_k$, we get

$$\sum_{k=0}^{j-1}\mathbb{E}\Big\|\int_{s_k}^{t_{k+1}}\frac{u(z)-v(z)}{(s-z)^{\alpha+1}}dz\Big\|^2$$
$$\leq \sum_{k=0}^{j-1}\int_{s_k}^{t_{k+1}}\frac{1}{(s-z)^{2(\alpha+1)}}dz\int_{s_k}^{t_{k+1}}\mathbb{E}\|u(z)-v(z)\|^2dz$$

$$\leq \sum_{k=0}^{j-1} \frac{(t_{k+1}-s_k)}{(s-t_{k+1})^2(s-s_k)^{2\alpha}} \int_{s_k}^{t_{k+1}} (z-s_k)^{2(\alpha-1)}\mathbb{E}\Big\|(z-s_k)^{1-\alpha}(u(z)-v(z))\Big\|^2 dz$$

$$\leq \sum_{k=0}^{j-1} \frac{(t_{k+1}-s_k)^{2\alpha}}{(2\alpha-1)(s-t_{k+1})^2(s-s_k)^{2\alpha}}\|u-v\|_k^2. \tag{18}$$

From (18), and by applying $\dfrac{t_{k+1}-s_k}{s-s_k} < 1$ and $s_j - t_{k+1} \leq s - t_{k+1}$, we obtain

$$\begin{aligned}
I_{4,1} &= \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \int_{s_j}^{t} (t-s)^{\alpha-1} \sum_{k=0}^{j-1} \mathbb{E}\Big\| \int_{s_k}^{t_{k+1}} \frac{u(z)-v(z)}{(s-z)^{\alpha+1}} dz \Big\|^2 ds \\
&\leq \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \int_{s_j}^{t} (t-s)^{\alpha-1} \sum_{k=0}^{j-1} \frac{(t_{k+1}-s_k)^{2\alpha}}{(2\alpha-1)(s-t_{k+1})^2(s-s_k)^{2\alpha}}\|u-v\|_k^2 ds \\
&\leq \frac{8\alpha j(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \sum_{k=0}^{j-1} \frac{1}{(2\alpha-1)}\|u-v\|_k^2 \int_{s_j}^{t} \frac{(t-s)^{\alpha-1}}{(s-t_{k+1})^2} ds \\
&\leq \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \sum_{k=0}^{j-1} \frac{1}{(2\alpha-1)}\|u-v\|_k^2 \frac{1}{(s_j-t_{k+1})^2} \int_{s_j}^{t} (t-s)^{\alpha-1} ds \\
&\leq \frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \sum_{k=0}^{j-1} \frac{(t_{j+1}-s_j)^2}{(2\alpha-1)(s_j-t_{k+1})^2}\|u-v\|_k^2.
\end{aligned} \tag{19}$$

Using Hölder's inequality and (H3), we get

$$\begin{aligned}
&\sum_{k=0}^{j-1} \mathbb{E}\Big\| \int_{t_{k+1}}^{s_{k+1}} \frac{\Psi_{k+1}(z,u(t_{k+1}^-)) - \Psi_{k+1}(z,v(t_{k+1}^-))}{(s-z)^{\alpha+1}} dz \Big\|^2 \\
&\leq \sum_{k=0}^{j-1} \int_{t_{k+1}}^{s_{k+1}} \frac{1}{(s-z)^{\alpha+1}} dz \int_{t_{k+1}}^{s_{k+1}} \frac{\lambda_{\Psi_{k+1}}}{(s-z)^{\alpha+1}} \mathbb{E}\|u(t_{k+1}^-) - v(t_{k+1}^-)\|^2 dz \\
&\leq \sum_{k=0}^{j-1} \frac{1}{\alpha}\Big(\frac{1}{(s-s_{k+1})^\alpha} - \frac{1}{(s-t_{k+1})^\alpha}\Big) \int_{t_{k+1}}^{s_{k+1}} \frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{(s-z)^{\alpha+1}} \mathbb{E}\Big\|(t_{k+1}-s_k)^{1-\alpha}(u(t_{k+1}^-) - v(t_{k+1}^-))\Big\|^2 dz \\
&\leq \sum_{k=0}^{j-1} \frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha(s-s_{k+1})^\alpha}\|u-v\|_k^2 \int_{t_{k+1}}^{s_{k+1}} \frac{1}{(s-z)^{\alpha+1}} dz \\
&\leq \sum_{k=0}^{j-1} \frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s-s_{k+1})^{2\alpha}}\|u-v\|_k^2.
\end{aligned} \tag{20}$$

For $k = 0, 1, \ldots, j-2$ with $j = 2, 3, \ldots, m$, using (20) and applying $s_j - s_{k+1} \leq s - s_{k+1}$, we get

$$\begin{aligned}
I_{4,2} &= \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \int_{s_j}^{t} (t-s)^{\alpha-1} \sum_{k=0}^{j-2} \mathbb{E}\Big\| \int_{t_{k+1}}^{s_{k+1}} \frac{\Psi_{k+1}(z,u(t_{k+1}^-)) - \Psi_{k+1}(z,v(t_{k+1}^-))}{(s-z)^{\alpha+1}} dz \Big\|^2 ds \\
&\leq \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \int_{s_j}^{t} (t-s)^{\alpha-1} \sum_{k=0}^{j-2} \frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s-s_{k+1})^{2\alpha}}\|u-v\|_k^2 ds \\
&\leq \frac{8j(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} \sum_{k=0}^{j-2} \frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha}\|u-v\|_k^2 \int_{s_j}^{t} \frac{(t-s)^{\alpha-1}}{(s-s_{k+1})^{2\alpha}} ds
\end{aligned}$$

$$\leq \frac{8j(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha(s_j-s_{k+1})^{2\alpha}}\|u-v\|_k^2\int_{s_j}^{t}(t-s)^{\alpha-1}ds$$
$$\leq \frac{8j(t_{j+1}-s_j)^{2}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_j-s_{k+1})^{2\alpha}}\|u-v\|_k^2. \tag{21}$$

For $k = j-1$ with $j = 1, 2, \ldots, m$, using the Hölder inequality, (H4), and applying $\frac{s_j - z}{s - z} \leq 1$, we get

$$I_{4,2} = \frac{8j\alpha(t-s_j)^{2-\alpha}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}\mathbb{E}\Big\|\int_{t_j}^{s_j}\frac{\Psi_j(z,u(t_j^-))-\Psi_j(z,v(t_j^-))}{(s-z)^{\alpha+1}}dz\Big\|^2 ds$$
$$\leq \frac{8j\alpha(t-s_j)^{2-\alpha}(s_j-t_j)}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}\int_{t_j}^{s_j}\frac{\mathbb{E}\|\Psi_j(z,u(t_j^-))-\Psi_j(z,v(t_j^-))\|^2}{(s-z)^{2(\alpha+1)}}dzds$$
$$\leq \frac{8j\alpha(t-s_j)^{2-\alpha}(s_j-t_j)}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}\int_{t_j}^{s_j}\frac{\mathbb{E}\|(s_j-z)^{\alpha+1}(\psi_j(z,u(t_j^-))-\psi_j(z,v(t_j^-)))\|^2}{(s-z)^{2(\alpha+1)}}dzds$$
$$\leq \frac{8j\alpha(t-s_j)^{2-\alpha}(s_j-t_j)}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\int_{s_j}^{t}(t-s)^{\alpha-1}\int_{t_j}^{s_j}\frac{(s_j-z)^{2(\alpha+1)}}{(s-z)^{2(\alpha+1)}}\mathbb{E}\|\psi_j(z,u(t_j^-))-\psi_j(z,v(t_j^-))\|^2dzds$$
$$\leq \frac{8j\alpha\lambda_{\psi_j}(t-s_j)^{2-\alpha}(s_j-t_j)(t_j-s_{j-1})^{2(\alpha-1)}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\|u-v\|_{j-1}^2\int_{s_j}^{t}(t-s)^{\alpha-1}\int_{t_j}^{s_j}\frac{(s_j-z)^{2(\alpha+1)}}{(s-z)^{2(\alpha+1)}}dzds$$
$$\leq \frac{8j\lambda_{\psi_j}(s_j-t_j)^2(t_{j+1}-s_j)^2}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)(t_j-s_{j-1})^{2(1-\alpha)}}\|u-v\|_{j-1}^2. \tag{22}$$

For $k = 0, 1, \ldots, j-1$ with $j = 1, 2, \ldots, m$, we derive the following result from (21) and (22),

$$I_{4,2} \leq \frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\Big(\sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{j+1}-s_j)^2(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_j-s_{k+1})^{2\alpha}} + \frac{\lambda_{\psi_j}(s_j-t_j)^2(t_{j+1}-s_j)^2}{(t_j-s_{j-1})^{2(1-\alpha)}}\Big)\|u-v\|_J^2. \tag{23}$$

As a result of (19) and (23), we have

$$I_4 \leq \frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\Big(\sum_{k=0}^{j-1}\frac{(t_{j+1}-s_j)^2}{(2\alpha-1)(s_j-t_{k+1})^2} + \sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{j+1}-s_j)^2(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_j-s_{k+1})^{2\alpha}} + \frac{\lambda_{\psi_j}(s_j-t_j)^2(t_{j+1}-s_j)^2}{(t_j-s_{j-1})^{2(1-\alpha)}}\Big)\|u-v\|_J^2. \tag{24}$$

Substituting (15)-(17) and (24) into (14), we get for $t \in (s_j, t_{j+1}]$, $j = 1, 2, \ldots, m$

$$\mathbb{E}\Big\|(t-s_j)^{1-\alpha}\Big((Nu)(t)-(Nv)(t)\Big)\Big\|^2$$
$$\leq \Big[\frac{4\lambda_{\Psi_j}(s_j-t_j)^2(t_j-s_{j-1})^{2(\alpha-1)}}{\Gamma^2(\alpha)} + \frac{4L_f\Gamma(2\alpha-1)(t_{j+1}-s_j)^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)} + \frac{4M_f(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)^2}{\Gamma^2(\alpha+1)}$$
$$+ \frac{4L_g\beta(2\alpha-1,2\alpha-1)(t_{j+1}-s_j)^{2\alpha-1}}{\Gamma^2(\alpha)} + \frac{4M_g(s_j-\tau)^{2(\alpha-1)}(t_{j+1}-s_j)}{(2\alpha-1)\Gamma^2(\alpha)}$$
$$+ \frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\Big(\sum_{k=0}^{j-1}\frac{(t_{j+1}-s_j)^2}{(2\alpha-1)(s_j-t_{k+1})^2}$$

$$+\sum_{k=0}^{j-2}\frac{\lambda_{\Psi_{k+1}}(t_{j+1}-s_j)^2(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_j-s_{k+1})^{2\alpha}}+\frac{\lambda_{\psi_j}(s_j-t_j)^2(t_{j+1}-s_j)^2}{(t_j-s_{j-1})^{2(1-\alpha)}}\Big)\Big]\|u-v\|_J^2$$

$$\leq \eta\|u-v\|_J^2. \tag{25}$$

**Case 3**: For $t \in (t_j, s_j],\ j = 1, 2, \ldots, m,$

$$\mathbb{E}\Big\|(t-s_{j-1})^{1-\alpha}\Big((Nu)(t)-(Nv)(t)\Big)\Big\|^2$$

$$= \mathbb{E}\Big\|(t-s_{j-1})^{1-\alpha}\Big((\Psi_j(t,N(u(t_j^-)))-\Psi_j(t,N(v(t_j^-)))\Big)\Big\|^2$$

$$\leq \lambda_{\Psi_j}\mathbb{E}\Big\|(t-s_{j-1})^{1-\alpha}\Big(N(u(t_j^-)))-N(v(t_j^-))\Big)\Big\|^2$$

$$\leq \lambda_{\Psi_j}\Big[\frac{4\lambda_{\Psi_{j-1}}(s_{j-1}-t_{j-1})^2(t_{j-1}-s_{j-2})^{2(\alpha-1)}}{\Gamma^2(\alpha)}+\frac{4L_f\Gamma(2\alpha-1)(t_j-s_{j-1})^{2\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha-1)}+\frac{4M_f(s_{j-1}-\tau)^{2(\alpha-1)}(t_j-s_{j-1})^2}{\Gamma^2(\alpha+1)}$$

$$+\frac{4L_g\beta(2\alpha-1,2\alpha-1)(t_j-s_{j-1})^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{4M_g(s_{j-1}-\tau)^{2(\alpha-1)}(t_j-s_{j-1})}{(2\alpha-1)\Gamma^2(\alpha)}$$

$$+\frac{8j}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)}\Big(\sum_{k=0}^{j-2}\frac{(t_j-s_{j-1})^2}{(2\alpha-1)(s_{j-1}-t_{k+1})^2}$$

$$+\sum_{k=0}^{j-3}\frac{\lambda_{\Psi_{k+1}}(t_j-s_{j-1})^2(t_{k+1}-s_k)^{2(\alpha-1)}}{\alpha^2(s_{j-1}-s_{k+1})^{2\alpha}}+\frac{\lambda_{\psi_{j-1}}(s_{j-1}-t_{j-1})^2(t_j-s_{j-1})^2}{(t_{j-1}-s_{j-2})^{2(1-\alpha)}}\Big)\Big]\|u-v\|_J^2$$

$$\leq \eta\|u-v\|_J^2. \tag{26}$$

Therefore, from (13), (25), and (26), we obtain

$$\|Nu-Nv\|_J \leq \sqrt{\eta}\|u-v\|_J.$$

From condition (9), the operator $N$ satisfies the condition of the contraction mapping principle, then $N$ has a fixed point. We get the uniqueness in the same way by proving that $\|u-v\|_J \leq \sqrt{\eta}\|u-v\|_J$ for $u, v \in \mathcal{PC}_{1-\alpha}([-\tau, b], \mathcal{H})$, and we conclude that $u(t) = v(t)$. Hence $N$ has a unique fixed point $u \in \mathcal{PC}_{1-\alpha}([-\tau, b], \mathcal{H})$. □

## 5. An example

Let $s_j = 2j,\ t_{j+1} = 2j+1,\ j = 0, 1, 2, 3,\ b = 7.$
Consider the following problem

$$ {}_{s_0}^{R}D_t^{0.9}u(t) = \frac{e^{-t}|u(t)|}{10(1+e^{-t})(1+|u(t)|)}+\frac{|u(t-1)|}{10(1+|u(t-1)|)}$$

$$+\frac{1}{\sqrt{2}}\left(\frac{0.1|u(t)|}{10+|u(t)|}+\frac{0.2|u(t-1)|}{10+|u(t-1)|}\right)\frac{dw(t)}{dt},\ t \in (0,1]\cup(2,3]\cup(4,5]\cup(6,7], \tag{27}$$

$$u(t) = \frac{(s_j-t)^{1+\alpha}}{\sqrt{10}}e^{-(s_j-t)^{1+\alpha}}u(t_j), \qquad t \in (1,2]\cup(3,4]\cup(5,6], \tag{28}$$

$$ {}_{s_j}I_t^{0.1}u(t)|_{t=s_j} = \int_{t_j}^{s_j}\frac{(s_j-s)^{1+\alpha}}{\sqrt{10}}e^{-(s_j-s)^{1+\alpha}}u(t_j)ds,\ \ j = 1,2,3 \tag{29}$$

$$ {}_0I_t^{0.1}u(t)|_{t=0} = \varphi(0),\ \text{ and }\ u(t) = \varphi(t),\ \ t \in [-1,0]. \tag{30}$$

We have
$f(t,u(t),u(t-1)) = \frac{e^{-t}|u(t)|}{10(1+e^{-t})(1+|u(t)|)}+\frac{|u(t-1)|}{10(1+|u(t-1)|)}, g(t,u(t),u(t-1)) = \frac{1}{\sqrt{2}}\left(\frac{0.1|u(t)|}{10+|u(t)|}+\frac{0.2|u(t-1)|}{10+|u(t-1)|}\right),$

$\Psi_j(t, u(t_j^-)) = \frac{(s_j-t)^{1+\alpha}}{\sqrt{10}} e^{-(s_j-t)^{1+\alpha}} u(t_j), \tau = 1, q = 0.9, L_f = 0.02, M_f = 0.02, L_g = 0.01, M_g = 0.04, \lambda_{\Psi_j} = 0.025, \lambda_{\psi_j} = 0.1, J = (0, 7]$.

$$\begin{aligned}\eta \quad = \quad & \max_{j=1,2,3}\Big(\frac{4\times 0.025(s_j-t_j)^2(t_j-s_{j-1})^{-0.2}}{\Gamma^2(0.9)} + \frac{4\times 0.02\Gamma(0.8)(t_{j+1}-s_j)^{1.8}}{\Gamma(1.9)\Gamma(1.7)} + \frac{4\times 0.02(s_j-1)^{-0.2}(t_{j+1}-s_j)^2}{\Gamma^2(1.9)} \\ & + \frac{4\times 0.01\times\beta(0.8,0.8)(t_{j+1}-s_j)^{0.8}}{\Gamma^2(0.9)} + \frac{4\times 0.04(s_j-1)^{-0.2}(t_{j+1}-s_j)}{0.8\Gamma^2(0.9)} \\ & + \frac{8j}{\Gamma^2(0.9)\Gamma^2(0.1)}\Big(\sum_{k=0}^{j-1}\frac{(t_{j+1}-s_j)^2}{0.8(s_j-t_{k+1})^2} \\ & + \sum_{k=0}^{j-2}\frac{0.025(t_{j+1}-s_j)^2(t_{k+1}-s_k)^{-0.2}}{0.81(s_j-s_{k+1})^{1.8}} + \frac{0.1(s_j-t_j)^2(t_{j+1}-s_j)^2}{(t_j-s_{j-1})^{0.2}}\Big)\Big) \simeq 0.85 < 1.\end{aligned}$$

According to Theorem 4.1, the system (27)-(30) has a unique solution $u \in \mathcal{PC}_{0.9}([-1, 7], \mathcal{H})$.

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