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# On behaviours for the solution to a certain third-order stochastic integro-differential equation with time delay

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**Abstract.** In the present paper, Lyapunov functional (LF) is employed to discuss the continuability and boundedness of solutions for a third-order non-autonomous stochastic integro-differential equation (SIDE) with time delay. The third-order differential equation is ablated to a system of first-order differential equations together with its equivalent quadratic function to derive a suitable downright LF and then we study the behaviour of the solutions. A numerical example is considered to support our results. Moreover, we use the Euler-Maruyama method to get an approximate numerical solution for the considered system. The obtained result complements some recent ones in the literature.

## 1. Introduction

In the last decades, some methods have been developed to obtain information about the qualitative behaviour of solutions, stability, instability, continuability and boundedness of solutions for the delay differential equations (DDEs), see for example [7–9, 20, 21, 38].

An integro-differential equation (IDE) is an equation that involves both integrals and derivatives of an unknown function. The IDE is said to have a delay when the rate of variation in the equation state depends on past states, in this case, IDE is called a time delay IDE.

IDEs have attracted significant interest in the field of engineering and applied sciences in the last few years, which arise in several research fields, like economy, control theory, physics, chemistry, population dynamics, medicine, atomic energy, information theory, mechanics and electromagnetic theory, life science, see [12, 13, 15, 17, 19, 31, 33, 34, 41]. On the other hand, in order to capture ubiquitous noise factors in the actual situation, SIDEs emerge in anomalous diffusion [28]. Stochastic delay integro-differential equations, as the mathematical model, widely apply in biology, physics, economics and finance [10, 26].

It is worth-mentioning, that according to our observation, it can be seen some papers studied solutions of IDE and stochastic differential equations (SDEs) with or without delays, see [1–6, 14, 16, 18, 22–24, 29, 30, 32, 35, 37, 39, 42–44].

In addition, it is reasonable to mention some recent papers from the literature dealing with the qualitative behaviors of nonlinear differential equations of the third-order with delay.

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In 2019, Tunç and Ayhan [37] discussed the continuability and boundedness of solutions for a kind of nonlinear delay integro-differential equations of the third-order

$$\begin{aligned}(q(t)(p(t)x')')' + a(t)f(t,x,x')x'' + b(t)g(t,x)x' + c(t)h(x-r)\\ = \int_0^t C(t,s)x'(s)ds.\end{aligned}$$

Recently, Mahmoud and Bakhit [25] established the properties of solutions for non-autonomous third-order stochastic differential equation with a constant delay

$$\begin{aligned}&\dddot{x}(t) + a(t)f(x(t),\dot{x}(t))\ddot{x}(t) + b(t)\phi(x(t))\dot{x}(t) + c(t)\psi(x(t-r))\\ &\quad + g(t,x)\dot{\omega}(t) = p(t,x(t),\dot{x}(t),\ddot{x}(t)).\end{aligned}$$

Here, we consider a non-autonomous SIDE with variable delay of third-order as the form

$$\begin{aligned}&\dddot{x}(t) + a(t)f(t,\dot{x}(t))\ddot{x}(t) + b(t)g_1(\dot{x}(t-r(t))) + c(t)g_2(x(t-r(t)))\\ &\quad + \sigma x(t-h(t))\dot{\omega}(t) = P(t,x(t))\int_0^t G(s,\dot{x}(s))\dot{x}(s)ds,\end{aligned} \tag{1.1}$$

where, $a(t)$, $b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$, $g_1,\ g_2 : \mathbb{R} \to \mathbb{R}^+$ and $G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions. $h(t)$ is a continuous function and defined from $[0,\infty)$ to $[0,h_1]$. $\omega(t) \in \mathbb{R}^n$ is standard Brownian motion.

Essentially, our aim is to establish some sufficient conditions for the the continuability and boundedness of solutions of equation (1.1) by constructing a suitable LF.

**Remarks :**

***(i)*** In recent years, few papers have been written on the continuability and boundedness of solutions for IDEs; our study generalizes all of these papers. Moreover, most of these papers are IDEs of second-order without a stochastic term, for example [16, 29, 36, 40], but here we study SIDE for the third-order. Our results are new and improve previous results.

***(ii)*** In (1.1), if we put $\dddot{x} = (q(t)(p(t)x')')'$, $f(t,\dot{x}) = f(t,x,x')$, $g_1(\dot{x}(t-r(t))) = g(t,x)x'$, $g_2(\dot{x}(t-r(t))) = h(x-r)$, $\sigma x(t-h(t)) = 0$ and $P(t,x(t)) = 1$, we note that the equation in [37] represents a special case from the main equation (1.1) in this study.

***(iii)*** Whenever, $g_1(\dot{x}(t-r(t))) = \varphi(x(t))\dot{x}(t)$, $g_2(\dot{x}(t-r(t))) = \psi(x(t-r))$, $\sigma x(t-h(t)) = g(t,x)$ and replacing the integral term $P(t,x(t))\int_0^t G(s,\dot{x}(s))\dot{x}(s)ds$ by $p(t,x(t),\dot{x}(t),\ddot{x}(t))$, then (1.1) reduces to the studied equation in [25]. Thus, equation (1.1) generalizes the results obtained in [25]. Hence, our results include and extend all the previous results.

***(iv)*** Equation (1.1) is considered the first equation to thoroughly and systematically study the SIDE with time delay. The information just mentioned indicates the novelty and originality of the present paper.

## 2. Main Results

Let $G(t) = (G_1(t),\ldots,G_m(t))$ be an $m$-dimensional Brownian motion defined on the probability space. Consider an $n$-dimensional stochastic delay differential equation (SDDE)

$$dx(t) = N_1(t,x_t)dt + N_2(t,x_t)dG(t), \quad x_t(\theta) = x(t+\theta) \quad -r \le \theta \le 0, \ \ t \ge t_0, \tag{2.1}$$

with initial value $x(0) = x_0 \in C([-r, 0]; \mathbb{R}^n)$. Suppose that $N_1 : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $N_2 : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ satisfy the local Lipschitz condition and the linear growth condition. Therefore, for any given initial value $x(0) = x_0 \in C([-r, 0]; \mathbb{R}^n)$, it is known that equation (2.1) has a unique continuous solution on $t \geq 0$, which is denoted by $x(t; x_0)$ in this section . Suppose that $N_1(t, 0) = 0$ and $N_2(t, 0) = 0$, for all $t \geq 0$. Therefore, the SDDE admits the zero solution $x(t; 0) \equiv 0$ (see [8, 11, 26]).

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of non-negative functions $V(t, x_t)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in $t$ and twice continuously differentiable in $x$.

By Itô formula we have

$$dV(t, x_t) = \mathcal{L}V(t, x_t)dt + V_x(t, x_t)N_2(t, x_t)dG(t),$$

where

$$\begin{aligned}\mathcal{L}V(t, x_t) &= V_t(t, x_t) + V_x(t, x_t)N_1(t, x_t)\\ &\quad + \frac{1}{2}trace[N_2^T(t, x_t)V_{xx}(t, x_t)N_2(t, x_t)],\end{aligned} \tag{2.2}$$

such that $V_x = (V_{x_1}, \ldots, V_{x_n})$ and $V_{xx} = (V_{x_i x_j})_{n\times n}$.

Suppose that there exist non-negative constants $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $f_1$, $f_2$, $M$, $N$, $k$, $D_1$, $D_2$, $\gamma$, $\beta$, $L_1$, $L_2, \delta_0, L_2$ and $\delta_2$ with the negative constant $a_0$ such that the following assumptions are achieved

**($A_1$)** $a_1 \leq a(t) \leq a_2$, $b_1 \leq b(t) \leq b_2$, $c_1 \leq c(t) \leq c_2$, with $0 < m_1 \leq c(t) \leq b(t)$, $0 < m_2 \leq \dot{b}(t) \leq \dot{c}(t) \leq 0$, $\dot{a}(t) \leq a_0$, $0 < r(t) \leq \gamma$ and $0 < \dot{r}(t) \leq \beta$, $\beta \in (0, 1)$.

**($A_2$)** $g_1(y) \geq ky$, $|g_1'(y)| \leq L_1$, $g_2(x) \geq \delta_1 x$, $|g_2'(y)| \leq L_2$ and $\sup\{g_2'(x)\} = g_0$ such that $k \geq L_2$.

**($A_3$)** $|P(t, x(t))| \leq P(t) \leq M$, $|G(t, y(t))| \leq G(t) \leq N$.

**($A_4$)** $f_1 \leq f(t, y) \leq f_2$, $a_1 f_1 \geq 2$, $t \geq 0$, $y \in \mathbb{R}$.

**($A_5$)** $M\int_0^\infty G(s)ds \leq D_1$, $M\int_0^\infty G(s)ds + N\int_0^\infty P(s)ds \leq D_2$, such that, $D_1 \leq a_1 f_1 - 3$, $D_2 \leq b_1 k - a_0 f_2 - 2c_2 g_0 - 3$.

**($A_6$)** $0 < h(t) \leq h_1$, $|\dot{h}(t)| \leq \frac{1}{2}$, such that $2\sigma^2 \leq 2c_1\delta_1 - a_1 f_1 - b_1 k - D_1 - 2$.

**Theorem 2.1.** *Suppose that all the assumptions* ($A_1$)–($A_6$) *are satisfied. Then all solutions of system* (1.1) *are continuable and bounded provided that*

$$\begin{aligned}\gamma < \min\Bigg[&\left\{\frac{(2c_1\delta_1 - a_1 f_1 - b_1 k - 2 - 2D_1 - 2\sigma^2)}{2(b_2 L_1 + c_2 L_2)}\right\},\\ &\left\{\frac{(1-\beta)(b_1 k - a_0 f_2 - 2c_2 g_0 - 3 - D_2)}{2(b_2 L_1(1-\beta) + c_2 L_2(4-\beta))}\right\},\\ &\left\{\frac{(1-\beta)(a_1 f_1 - 3 - D_1)}{2\left\{b_2 L_1(4-\beta) + c_2 L_2(1-\beta)\right\}}\right\}\Bigg].\end{aligned}$$

**Proof.** We can rewrite (1.1) as the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= P(t, x)\int_0^t G(s, y(s))y(s)ds - a(t)f(t, y)z - b(t)g_1(y) - \sigma x(t - h(t))\dot{\omega}(t)\\ &\quad + b(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds - c(t)g_2(x) + c(t)\int_{t-r(t)}^t g_2'(x(s))y(s)ds.\end{aligned} \tag{2.3}$$

The LF $V(t, x_t, y_t, z_t)$ around the system (2.3) can be defined as

$$\begin{aligned}V(t,x_t,y_t,z_t) =&c(t)\int_0^x g_2(\xi)d\xi + b(t)\int_0^y g_1(\zeta)d\zeta + c(t)g_2(x)y + a(t)\int_0^y f(t,\zeta)\zeta d\zeta\\&+ yz + \frac{z^2}{4} + (x+\frac{z}{2})^2 + \lambda_1\int_{-r(t)}^0\int_{t+s}^t y^2(u)duds + \sigma^2\int_{t-h(t)}^t x^2(s)ds\\&+ \lambda_2\int_{-r(t)}^0\int_{t+s}^t z^2(u)duds + \frac{3}{2}\int_0^t\int_t^\infty |P(\eta,x(\eta))|G(s)y^2(s)d\eta ds.\end{aligned} \tag{2.4}$$

Define the functions $V_1$ and $V_2$, as the following

$$V_1 = c(t)\int_0^x g_2(\xi)d\xi + b(t)\int_0^y g_1(\zeta)d\zeta + c(t)g_2(x)y,$$

and

$$V_2 = a(t)\int_0^y f(t,\zeta)\zeta d\zeta + yz + \frac{z^2}{4} + (x+\frac{z}{2})^2,$$

such that

$$\begin{aligned}V =&V_1 + V_2 + \lambda_1\int_{-r(t)}^0\int_{t+s}^t y^2(u)duds + \sigma^2\int_{t-h(t)}^t x^2(s)ds\\&+ \lambda_2\int_{-r(t)}^0\int_{t+s}^t z^2(u)duds + \frac{3}{2}\int_0^t\int_t^\infty |P(\eta,x(\eta))|G(s)y^2(s)d\eta ds.\end{aligned} \tag{2.5}$$

First, for the function $V_1$, since $g_1(y) \geq ky$, we get

$$\begin{aligned}V_1 &\geq c(t)\int_0^x g_2(\xi)d\xi + \frac{1}{2}kb(t)\Big(y + \frac{c(t)g_2(x)}{kb(t)}\Big)^2 - \frac{1}{2kb(t)}c^2(t)g_2^2(x)\\&\geq c(t)\int_0^x \Big(1 - \frac{c(t)}{2kb(t)}g_2'(\xi)\Big)g_2(\xi)d\xi.\end{aligned}$$

Since $0 < m_1 \leq c(t) \leq b(t)$ and from $(A_2)$, we conclude

$$V_1 \geq \frac{1}{2}m_1(1-\frac{L_2}{k})\delta_1 x^2.$$

Since, $k \geq L_2$, therefore there exists a positive constant $\delta_2$, such that

$$\delta_2 = 1 - \frac{L_2}{k} \geq 0.$$

It follows that

$$V_1 \geq \frac{1}{2}\delta_1\delta_2 x^2. \tag{2.6}$$

Second, for the function $V_2$, in view of the assumptions $(A_1)$ and $(A_4)$, we find

$$V_2 \geq (\frac{z}{2}+y)^2 + (x+\frac{z}{2})^2 + (\frac{1}{2}a_1f_1 - 1)y^2. \tag{2.7}$$

Let $a_1f_1 \geq 2$, then for some positive constants $\delta_3, \delta_4, \delta_5$, we get

$$V_2 \geq \delta_3x^2 + \delta_4y^2 + \delta_5z^2.$$

Now, since $\lambda_1 \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(u)duds$, $\sigma^2 \int_{t-h(t)}^{t} x^2(s)ds$, $\lambda_2 \int_{-r(t)}^{0}\int_{t+s}^{t} z^2(u)duds$ *and* $\frac{3}{2}\int_0^t\int_t^\infty |P(\eta, x(\eta))|G(s)y^2(s)d\eta ds$ are positive, then from (2.5), we get

$$V \geq V_1 + V_2.$$

Therefore, from (2.6) and (2.7), we find

$$V(t, x_t, y_t, z_t) \geq \frac{1}{2}\delta_1\delta_2 x^2 + \delta_3 x^2 + \delta_4 y^2 + \delta_5 z^2.$$

Hence, for positive constant $\delta_6$, we conclude

$$V(t, x_t, y_t, z_t) \geq \delta_6(x^2 + y^2 + z^2). \tag{2.8}$$

This implies that $V(t, x_t, y_t, z_t) \geq 0$.

Now, we compute the stochastic time derivative of the LF $V(t, x_t, y_t, z_t)$ by using Itô formula (2.2), then we find

$$\begin{aligned}
\mathcal{L}V =& \dot{a}(t)\int_0^y f(t,\zeta)\zeta d\zeta + \dot{c}(t)\int_0^x g_2(\xi)d\xi + \dot{b}(t)\int_0^y g_1(\zeta)d\zeta + \dot{c}(t)g_2(x)y \\
& - a(t)f(t,y)z^2 - b(t)g_1(y)y - a(t)f(t,y)xz - b(t)g_1(y)x - c(t)g_2(x)x \\
& + 2xy + yz + z^2 + c(t)g_2'(x)y^2 + (x+y+z)\Big\{b(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds \\
& + c(t)\int_{t-r(t)}^t g_2'(x(s))y(s)ds + P(t,x)\int_0^t G(s,y(s))y(s)ds\Big\} \\
& + \lambda_1 r(t)y^2 - \lambda_1(1-\dot{r}(t))\int_{t-r(t)}^t y^2(u)du + \sigma^2 x^2(t) \\
& - \sigma^2 x^2(t-h(t))(1-\dot{h}(t)) + \frac{1}{2}\sigma^2 x^2(t-h(t)) + \lambda_2 r(t)z^2 \\
& - \lambda_2(1-\dot{r}(t))\int_{t-r(t)}^t z^2(u)du + \frac{3}{2}\frac{d}{dt}\int_0^t\int_t^\infty |P(\eta,x(\eta))|G(s)y^2(s)d\eta ds.
\end{aligned} \tag{2.9}$$

We know that

$$\begin{aligned}
&\frac{d}{dt}\int_0^t\int_t^\infty |P(\eta,x(\eta))|G(s)y^2(s)d\eta ds \\
&\quad = -|P(t,x(t))|\int_0^t G(s)y^2(s)ds + G(t)y^2(t)\int_t^\infty |P(\eta,x(\eta))|d\eta.
\end{aligned} \tag{2.10}$$

We now check that

$$F(t,x,y) = \dot{c}(t)\int_0^x g_2(\xi)d\xi + \dot{b}(t)\int_0^y g_1(\zeta)d\zeta + \dot{c}(t)g_2(x)y \leq 0.$$

We can write the above inequality as the following

$$F(t,x,y) = \dot{c}(t)\Big\{\int_0^x g_2(\xi)d\xi + \frac{\dot{b}(t)}{\dot{c}(t)}\int_0^y g_1(\zeta)d\zeta + g_2(x)y\Big\}.$$

From the condition $|g_1'(y)| \leq L_1$ and using the mean-value theorem, we get

$$F(t,x,y) \leq \dot{c}(t)\Big\{\int_0^x g_2(\xi)d\xi + \frac{L_1\dot{b}(t)}{2\dot{c}(t)}\Big(y + \frac{\dot{c}(t)}{L_1\dot{b}(t)}g_2(x)\Big)^2 - \frac{\dot{c}(t)}{2L_1\dot{b}(t)}g_2^2(x)\Big\}.$$

Since $\dot{b}(t) \le \dot{c}(t) \le 0$; by the hypostasis $(A_1)$, we can see that

$$F(t,x,y) \le \dot{c}(t)\Big\{\int_0^x \Big(1 - g_2'(\xi)\Big) g_2(\xi)d\xi + \frac{L_1}{2}\Big(y + \frac{\dot{c}(t)}{L_1\dot{b}(t)} g_2(x)\Big)^2\Big\}.$$

Since $1 - g_2'(x) \le 1 + |g_2'(x)| \le 1 + L_2 \ge 0$ and $\dot{c}(t) \le 0$, therefore, we conclude that $F(t,x,y) \le 0$.

Now, by substituting from (2.10) in (2.9), using the condition $\dot{h}(t) \le \frac{1}{2}$ and considering $F(t,x,y) \le 0$, we get

$$\begin{aligned}\mathcal{L}V =& \dot{a}(t)\int_0^y f(t,\zeta)\zeta d\zeta - a(t)f(t,y)z^2 - b(t)g_1(y)y - a(t)f(t,y)xz\\ &- b(t)g_1(y)x - c(t)g_2(x)x + 2xy + yz + z^2 + c(t)\dot{g}_2(x)y^2 + \sigma^2x^2(t)\\ &+ (x+y+z)\Big\{b(t)\int_{t-r(t)}^t \dot{g}_1(y(s))z(s)ds + c(t)\int_{t-r(t)}^t \dot{g}_2(x(s))y(s)ds\\ &+ P(t,x)\int_0^t G(s,y(s))y(s)ds\Big\} + \lambda_1 r(t)y^2 + \lambda_2 r(t)z^2\\ &- \lambda_1(1-\dot{r}(t))\int_{t-r(t)}^t y^2(u)du - \lambda_2(1-\dot{r}(t))\int_{t-r(t)}^t z^2(u)du\\ &- \frac{3}{2}|P(t,x(t))|\int_0^t G(s)y^2(s)ds + \frac{3}{2}G(t)y^2(t)\int_t^\infty |P(\eta,x(\eta))|d\eta.\end{aligned}$$

Since $|G(s,y(s)| \le G(s)$ and $P(t,x) \le M$ and using the fact $2mn \le m^2 + n^2$, we have

$$\begin{aligned}zP(t,x(t))\int_0^t G(s,y(s))y(s)ds &\le |z||P(t,x)|\int_0^t G(s)|y(s)|ds\\ &\le \frac{1}{2}|P(t,x)|\int_0^t G(s)(y^2(s)+z^2(t))ds\\ &\le \frac{1}{2}Mz^2\int_0^\infty G(s)ds + \frac{1}{2}|P(t,x)|\int_0^t G(s)y^2(s)ds.\end{aligned} \tag{2.11}$$

Also, we obtain

$$\begin{aligned}yP(t,x(t))\int_0^t G(s,y(s))y(s)ds &\le |y||P(t,x)|\int_0^t G(s)|y(s)|ds\\ &\le \frac{1}{2}|P(t,x)|\int_0^t G(s)(y^2(s)+y^2(t))ds\\ &\le \frac{1}{2}My^2\int_0^\infty G(s)ds + \frac{1}{2}|P(t,x)|\int_0^t G(s)y^2(s)ds.\end{aligned} \tag{2.12}$$

Similarly, we find

$$xP(t,x(t))\int_0^t G(s,y(s))y(s)ds \le \frac{1}{2}Mx^2\int_0^\infty G(s)ds + \frac{1}{2}|P(t,x)|\int_0^t G(s)y^2(s)ds. \tag{2.13}$$

In view of the assumptions $(A_1) - (A_4)$, the above inequalities (2.11), (2.12) and (2.13), we get

$$\begin{aligned}\mathcal{L}V(t,x_t,y_t,z_t) \leq& \frac{1}{2}a_0f_2y^2 - a_1f_1z^2 - b_1ky^2 - a_1f_1xz - b_1kxy - c_1\delta_1x^2\\&+ 2xy + yz + c_2g_0y^2 + z^2 + \sigma^2x^2 + \lambda_1\gamma y^2 + \lambda_2\gamma z^2 + \frac{1}{2}Mx^2\int_0^\infty G(s)ds\\&+ (x+y+z)\Big\{b(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds + c(t)\int_{t-r(t)}^t g_2'(x(s))y(s)ds\Big\}\\&+ \frac{1}{2}My^2\int_0^\infty G(s)ds + \frac{1}{2}Mz^2\int_0^\infty G(s)ds - \lambda_1(1-\beta)\int_{t-r(t)}^t y^2(u)du\\&- \lambda_2(1-\beta)\int_{t-r(t)}^t z^2(u)du + \frac{3}{2}Ny^2(t)\int_t^\infty |P(\eta,x(\eta))|d\eta.\end{aligned}$$

Applying the inequality $|mn| \leq \frac{1}{2}(m^2+n^2)$, we obtain

$$\begin{aligned}\mathcal{L}V(t,x_t,y_t,z_t) \leq& -\frac{1}{2}\Big\{2c_1\delta_1 - a_1f_1 - b_1k - 2 - M\int_0^\infty G(s)ds - 2\sigma^2 - (b_2L_1 + c_2L_2)\gamma\Big\}\\&- \frac{1}{2}\Big\{b_1k - a_0f_2 - 2c_2g_0 - 3 - M\int_0^\infty G(s)ds - N\int_0^\infty P(s)ds\\&- (b_2L_1 + c_2L_2 + 2\lambda_1)\gamma\Big\}y^2 - \frac{1}{2}\Big\{a_1f_1 - 3 - M\int_0^\infty G(s)ds\\&- (b_2L_1 + c_2L_2 + 2\lambda_2)\gamma\Big\}z^2 + \Big\{\frac{3}{2}c_2L_2 - \lambda_1(1-\beta)\Big\}\int_{t-r(t)}^t y^2(s)ds\\&+ \Big\{\frac{3}{2}b_2L_1 - \lambda_2(1-\beta)\Big\}\int_{t-r(t)}^t z^2(s)ds.\end{aligned}$$

Choosing $\lambda_1 = \frac{3c_2L_2}{2(1-\beta)}$, $\lambda_2 = \frac{3b_2L_1}{2(1-\beta)}$ and from $(A_5)$, we get

$$\begin{aligned}\mathcal{L}V(t,x_t,y_t,z_t) \leq& -\frac{1}{2}\Big\{2c_1\delta_1 - a_1f_1 - b_1k - 2 - D_1 - 2\sigma^2 - (b_2L_1 + c_2L_2)\gamma\Big\}\\&- \frac{1}{2}\Big\{b_1k - a_0f_2 - 2c_2g_0 - 3 - D_2 - \frac{b_2L_1(1-\beta) + c_2L_2(4-\beta)}{(1-\beta)}\gamma\Big\}y^2\\&- \frac{1}{2}\Big\{a_1f_1 - 3 - D_1 - \frac{b_2L_1(4-\beta) + c_2L_2(1-\beta)}{(1-\beta)}\gamma\Big\}z^2.\end{aligned}$$

Suppose that $(x(t), y(t), z(t))$ is a solution of system (2.3) with initial condition $(x_0, y_0, z_0)$. Since $V(t)$ is a positive definite and decreasing functional on the trajectories of system (2.3) also we have

$$\mathcal{L}V(t,x_t,y_t,z_t) \leq 0,$$

then, we can say that $V(t)$ is bounded on $[t_0, T]$. Now, integrating the above inequality from $t_0$ to $T$, we obtain

$$V(T, x(T), y(T), z(T)) \leq V(t_0, x(t_0), y(t_0), z(t_0)) = V_0.$$

Thus, it follows from (2.8) that

$$x^2(T), y^2(T), z^2(T) \leq \frac{V_0}{\delta_6}.$$

Therefore, we conclude that $|x(t)|, |y(t), |z(t)|$ are bounded on $t \to T^-$. Hence, we can conclude that $T < \infty$ is impossible and we must have $T = \infty$.

**Example 2.1.** *Here, as an application, we give the following numerical example*

$$\begin{aligned}
&\dddot{x}(t) + \Big(260 - 5t + \cos t\Big)\Big(2 + 100\sin^2 y + \frac{\sin^2 y}{1+t^2}\Big)\ddot{x}(t) \\
&\quad + \Big(2 + \frac{1}{1+t}\Big)\Big(\sin(y(t - r(t)) + 15y(t - r(t))\Big) \\
&\quad + \Big(10 + \frac{1}{1+t^5}\Big)\Big(\frac{x}{1 + x^2(t - r(t))} + 10x\Big) \\
&\quad + \frac{1}{2}\sin(x(t - \frac{1}{2}e^{-t}))\dot{\omega}(t) = \frac{e^{-t}}{1+x^2}\int_0^t \frac{e^{-s}}{1+\dot{x}^2(s)}\dot{x}(s)ds.
\end{aligned} \tag{2.14}$$

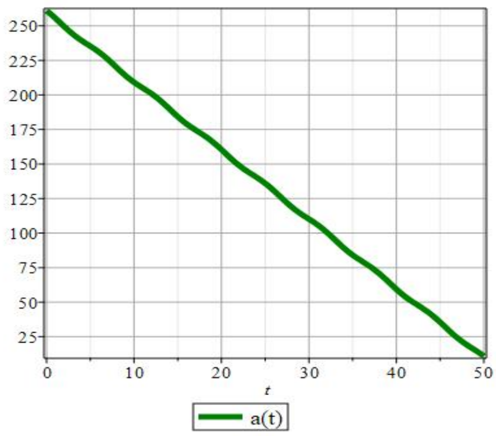
*The following system can be implied from the equation above as the following*

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= \frac{e^{-t}}{1+x^2}\int_0^t \frac{e^{-s}}{1+y^2(s)}y(s)ds - \Big(260 - 5t + \cos t\Big)\Big(2 + 100\sin^2 y + \frac{\sin^2 y}{1+t^2}\Big)z \\
&\quad - \Big(2 + \frac{1}{1+t}\Big)(\sin y + 15y) - \frac{1}{2}\sin(x(t - \frac{1}{2}e^{-t}))\dot{\omega}(t) \\
&\quad + \Big(2 + \frac{1}{1+t}\Big)\int_{t-r(t)}^t (\cos(y(s)) + 15)ds - \Big(10 + \frac{1}{1+t^5}\Big)\Big(\frac{x}{1+x^2} + 10x\Big) \\
&\quad + \Big(10 + \frac{1}{1+t^5}\Big)\int_{t-r(t)}^t \Big(\frac{1 - x^2(s)}{(1 + x^2(s))^2} + 10\Big)ds.
\end{aligned} \tag{2.15}$$

*If we compare the above system with* (2.3) *and by using the conditions of Theorem 2.1, we can obtain the following estimates:*

$$a(t) = 260 - 5t + \cos t, \;\; 10.6 \le a(t) \le 261, \;\; \textit{then} \;\; a_1 = 10.6, \; a_2 = 261,$$

$$a'(t) = -5 - \sin t \le -4 = a_0,$$



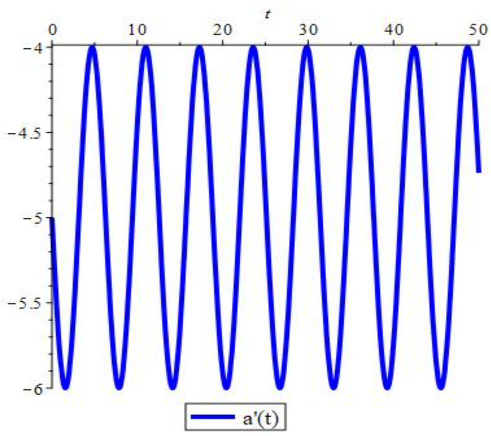




Figure 1: Path of $a(t), a'(t)$

.

*Figure 1, shows the path of* $a(t), a'(t)$, *on the interval* $t \in [0, 50]$.

$$b(t) = 2 + \frac{1}{1+t}, \;\; 2 \le b(t) \le 3, \;\; \textit{therefore}, \;\; b_1 = 2, b_2 = 3,$$

$$b'(t) = \frac{-1}{(1+t)^2} \le 0,$$

$$c(t) = 10 + \frac{1}{1+t^5}, \;\; 10 \le c(t) \le 11 \;\; \textit{so}, \;\; c_1 = 10, c_2 = 11,$$

$$c'(t) = \frac{-5t^4}{(1+t^5)^2} \le 0,$$

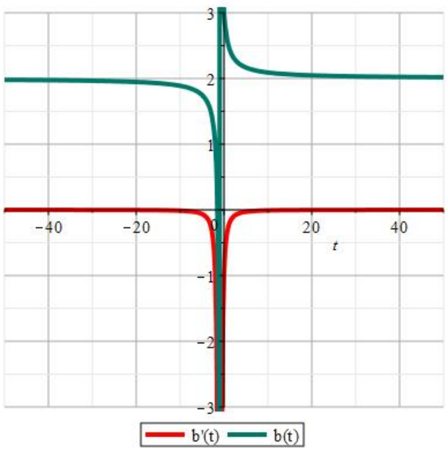



Figure 2: Path of $b(t), b'(t)$.

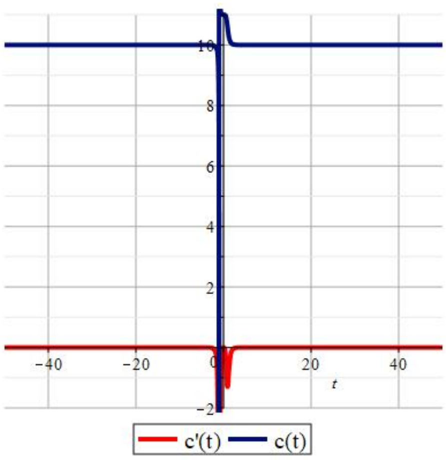



Figure 3: Path of $c(t), c'(t)$

.

*we can see that Figures 2 and 3, illustrate the behaviour of* $b(t), b'(t), c(t)$ *and* $c'(t)$*, through the interval* $t \in [-50, 50]$.

*The function*

$$g_1(y) = \sin y + 15y, \quad \frac{g_1(y)}{y} = \frac{\sin y}{y} + 15,$$

*since* $|\frac{\sin y}{y}| \leq 1$*, we have*

$$-1 \leq \frac{\sin y}{y} \leq 1, \quad then \quad \frac{g_1(y)}{y} = \frac{\sin y}{y} + 15 \geq 14 = k.$$

*It follows that*

$$g_1'(y) = \cos y + 15, \quad |g_1'(y)| \leq 16 = L_1.$$

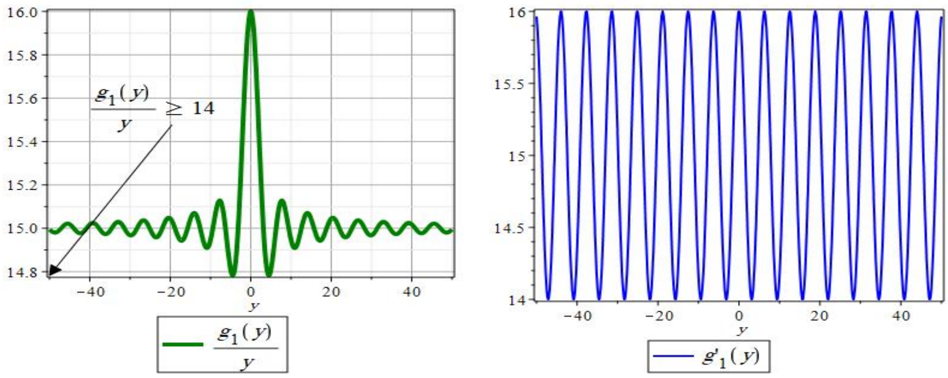



Figure 4: Trajectory of $\frac{g_1(y)}{y}, g_1'(y)$

.

*The behavior of the functions* $\frac{g_1(y)}{y}$*,* $g_1'(y)$ *are shown in Figure 4 on the interval* $y \in [-50, 50]$.

*Also, we get*

$$g_2(x) = \frac{x}{1+x^2} + 10x, \quad \frac{g_2(x)}{x} \geq 10 = \delta_1.$$

*Hence, the derivative of the function* $g_2(x)$ *with respect to* $x$*, is*

$$g_2'(x) = \frac{1-x^2}{(1+x^2)^2} + 10, \quad |g_2'(x)| \leq 11 = L_2, \quad \sup\{g_2'(x)\} = 11 = g_0.$$

*Then, we have*

$$k = 14 \geq L_2.$$

*The path of the functions* $\frac{g_2(x)}{x}$*,* $g_2'(x)$ *appear in Figure 5 through the interval* $x \in [-100, 100]$.

*Moreover, the function*

$$|P(t, x)| = \frac{e^{-t}}{1+x^2} \leq |e^{-t}| \leq 1 = M, \quad |G(t, y(t))| = \frac{e^{-t}}{1+y^2} \leq |e^{-t}| \leq 1 = N.$$

*Therefore, we get*

$$M \int_0^\infty G(s)ds \leq 1 = D_1.$$

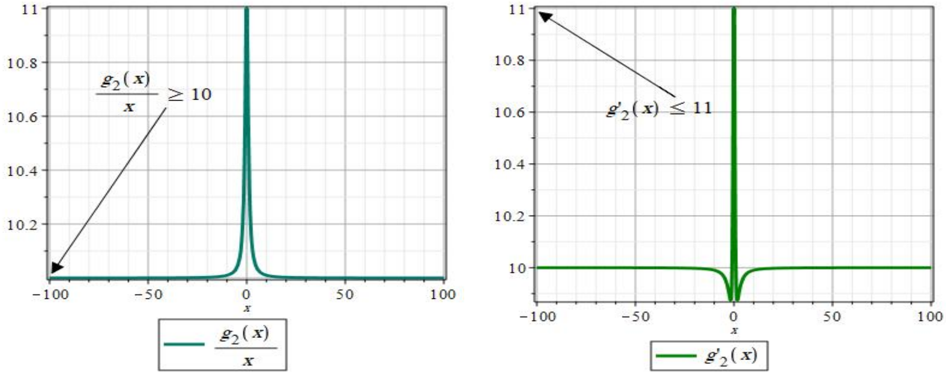



Figure 5: Trajectory of $\frac{g_2(x)}{x}, g_2'(x)$

.

*Now, we have*

$$\int_0^\infty G(s)ds = \int_0^\infty e^{-s}ds = 1, \quad \int_0^\infty P(s)ds = \int_0^\infty e^{-s}ds = 1,$$

$$M\int_0^\infty G(s)ds + N\int_0^\infty P(s)ds \leq 2 = D_2,$$

$$\sigma x(t - h(t)) = \tfrac{1}{2}\sin(x(t - \tfrac{1}{2}e^{-t})), \ \ \textit{then} \ \ \sigma = \tfrac{1}{2} \ \ \textit{and} \ \ h(t) = \tfrac{1}{2}e^{-t},$$

*which implies that*

$$|h'(t)| = \frac{1}{2}e^{-t} \leq \frac{1}{2}.$$

*Figure 6, shows the behaviour of the function* $\frac{1}{2}\sin(x(t - \frac{1}{2}e^{-t}))$ *through the intervals* $t \in [-5, 5]$, $x \in [-1, 1]$, *and it proves that* $|h'(t)| \leq \frac{1}{2}$ *on the interval* $[0, 30]$.

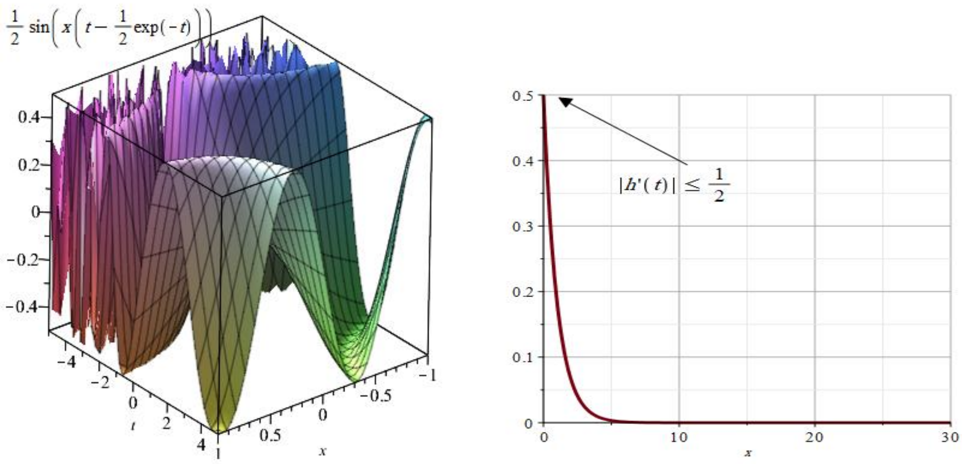



Figure 6: Trajectory of $\frac{1}{2}\sin(x(t - \frac{1}{2}e^{-t}))$ and $h'(t)$

*Also the function*

$$2 \leq f(t,y) = 2 + 100\sin^2 y + \frac{\sin^2 y}{1+t^2} \leq 102, \text{ then } f_1 = 2,\ f_2 = 102.$$

*Figure 7, shows the behavior of the function* $f(t,y)$ *through the interval* $t, y \in [0, 90]$.

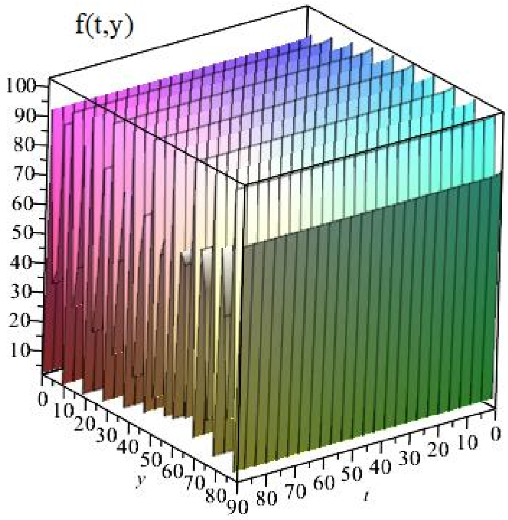



Figure 7: behaviour of $f(t,y)$

.

*Thus, we conclude the following estimates*

$$2\sigma^2 = \frac{1}{2} < 2c_1\delta_1 - a_1 f_1 - b_1 k - 2 - D_1 = 147.8,$$

$$D_2 = 2 < b_1 - a_0 f_2 - 2c_2 g_0 - 3 = 165,$$

*and*

$$D_1 = 1 < a_1 f_1 - 3 = 18.2.$$

*Hence, we obtain*

$$a_1 f_1 > 2.$$

*Thus, all assumptions of Theorem 2.1 are satisfied.*

## 3. ***Numerical Simulations***

*Here, we study the behaviour of the solution for equation* (2.14) *using a numerical method based on the Euler-Maruyama which enables us to get approximate numerical solutions for the considered system.*

*We illustrate the stability and the boundedness of the solutions for different values of the stepsize h of the numerical method and we chose the initial solution of* $(x(t), y(t), z(t))$ *be* $(x(0) = 1, y(0) = 1, z(0) = 1)$.

*The results can be distilled into the following :*

- *Figure 8, shows the behaviour of the solutions with* $h = 0.007, \sigma = 0.5$ *and we note that the error value* $\varepsilon = 7.4 \times 10^{-7}$, *makes it obvious that we have a stable system.*
- *If we change the value of h, as* $h = 0.009$, $h = 0.05$ *with* $\sigma = 0.5$, *we obtain Figures 9 and 10, respectively, with the error value being* $\varepsilon = 5.1 \times 10^{-4}$ *and* $\varepsilon = 0.02$. *We note that all solutions are stable, as can be seen.*

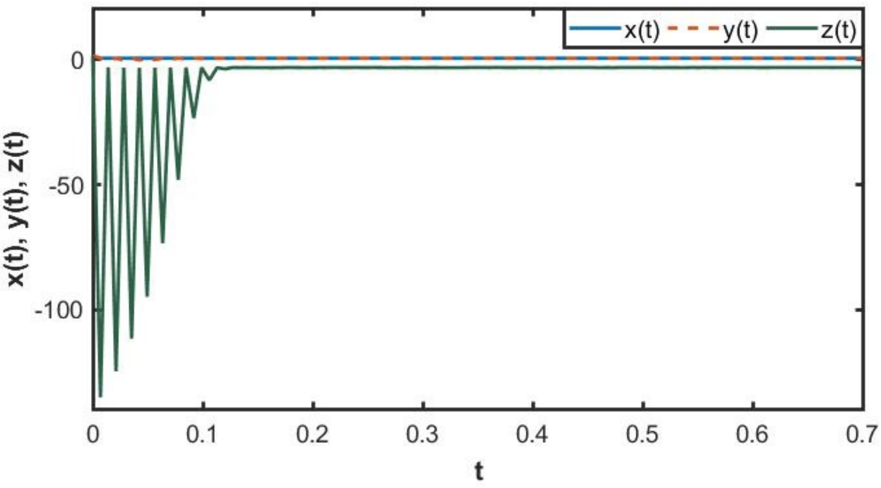



Figure 8: The behaviour of the solution of (2.14) with $h = 0.007$ *and* $\sigma = 0.5$.

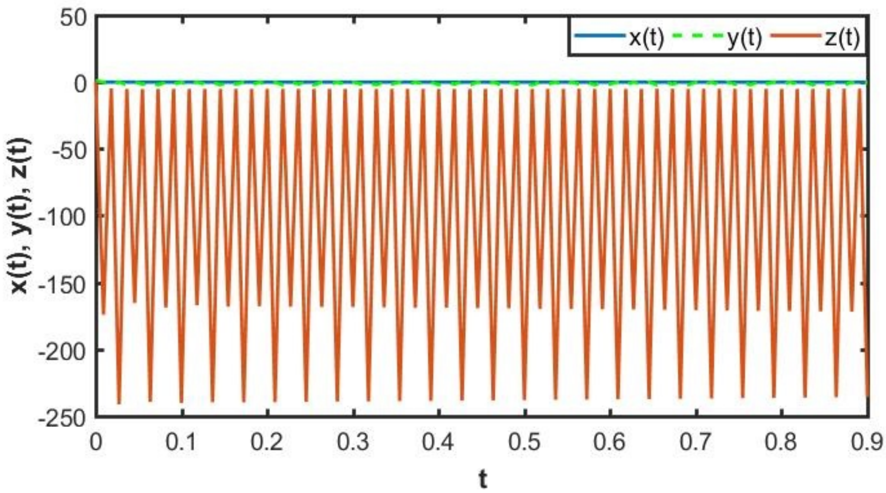



Figure 9: The path of the stochastic solutions of (2.14) with $h = 0.009$ *and* $\sigma = 0.5$.

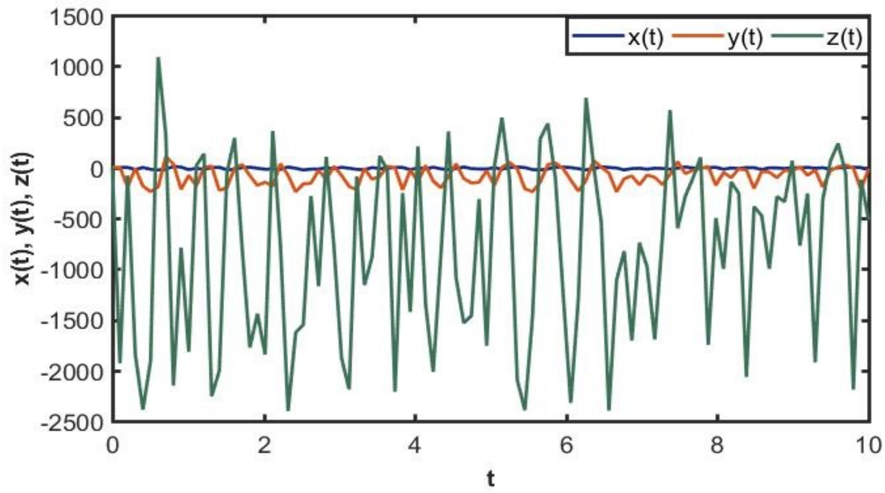



Figure 10: The path of the stochastic solutions of (2.14) with $h = 0.05$ *and* $\sigma = 0.5$.

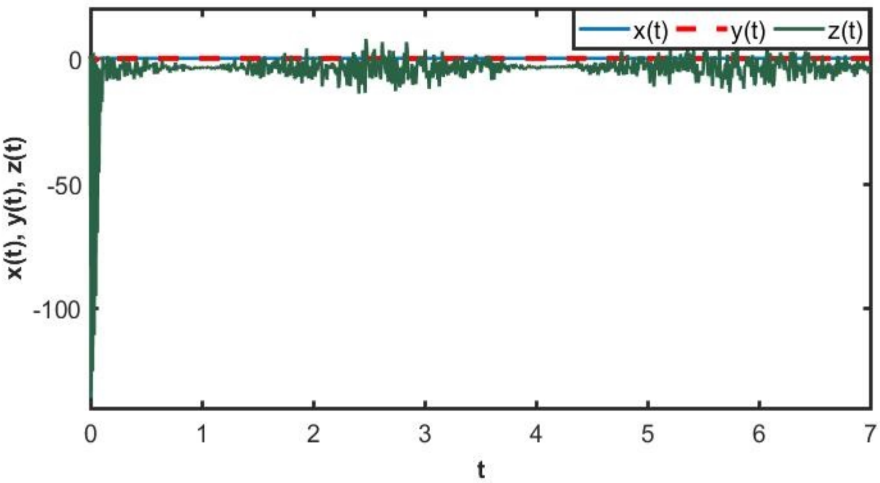



Figure 11: Stochastic evolution of (2.14) with $h = 0.005$ *and* $\sigma = 50$ .

- *More clearly Figures 11 and 12, illustrate the behaviour of solutions when* $\sigma = 50$, *with* $h = 0.005$ *and* $h = 0.01$, *respectively. It may be observed that the stochasticity increases as the noise level rises and it reaches the maximum level of stability as in Figure 12.*
- *On the other hand, Figure 13, shows the behaviour of the solutions when* $h = 0.18$, $\sigma = 0.5$, *we note that, when the value of* $h$ *increases, the value of* $\varepsilon$ *increases and we suddenly don't have a stable system, but we can see that when* $h$ *is very small as in Figure 14, the behavior of solution seems stable and bounded.*

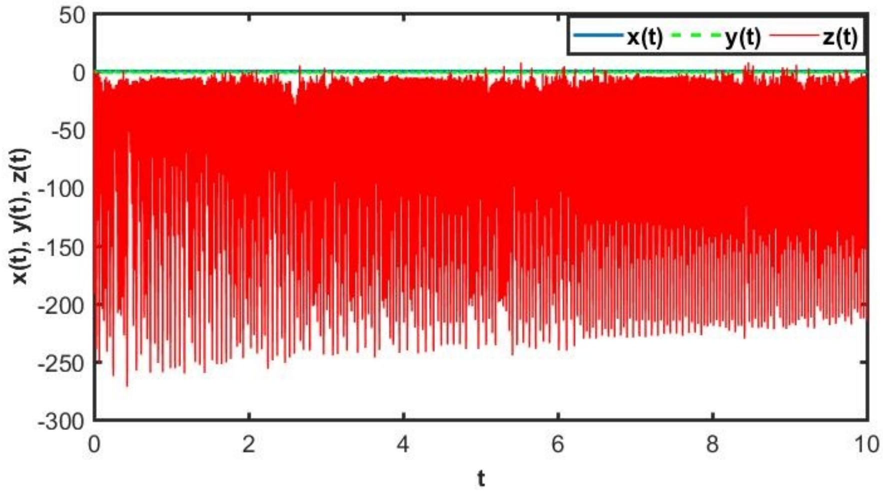



Figure 12: The stochastic behaviour of (2.14) with $\sigma = 50$ *and* $h = 0.01$.

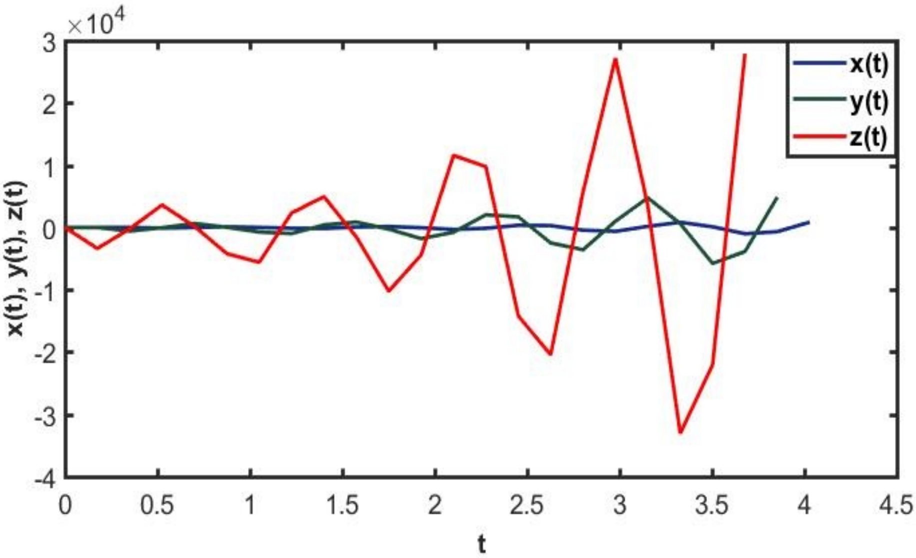



Figure 13: the instability behaviour for (2.14) with $h = 0.18$ *and* $\sigma = 0.5$

.

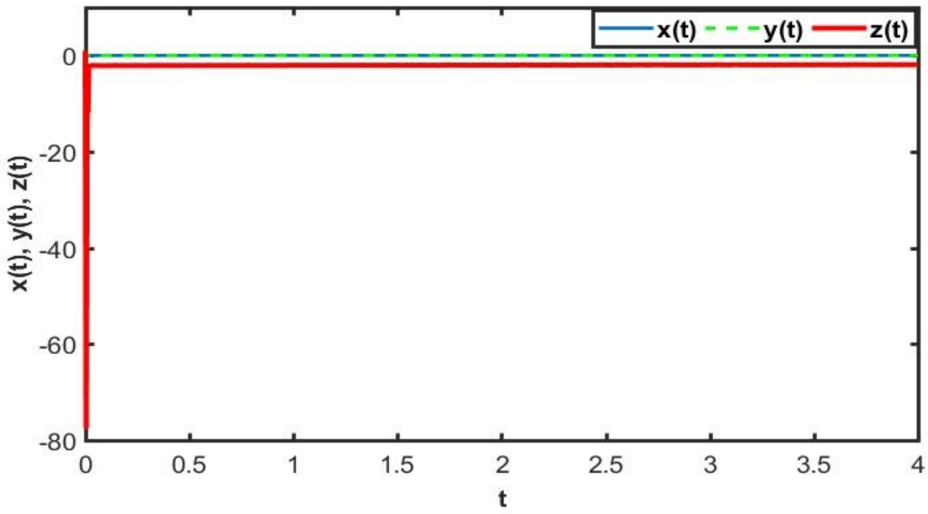



Figure 14: Trajectory of the solution of (2.14) with $\sigma = 0.5$ *and* $h = 0.002$

.

## 4. Conclusion

The significance of stochastic IDE of the form (1.1) lies in the fact that they arise in many situations. For example, equations of this kind occur in the stochastic formulation of problems in reactor dynamics, due to the complex random nature of the situation, the phenomenon being studied should be more realistically considered in a stochastic framework, resulting in a stochastic integro-differential equation, also in the study of the growth of biological populations, and in many other problems occurring in the general areas of biology, physics, and engineering.

Now all of the above will be summarized in the following points :

- The third-order non-autonomous stochastic SIDE with time delay that appeared in (1.1) has been considered.
- To reach the goal of this paper, by constructing a new suitable LF, we established the sufficient conditions of Theorem 2.1 to study the continuability and boundedness of solutions for (1.1).
- As an application, a numerical example was proposed to perform the given results all functions were drawn, next, we use the Euler-Maruyama method to prove an approximate numerical solution for the considered system and also orbits of the numerical solutions were drawn with assigned initial conditions to demonstrate the effectiveness of the obtained results.

Besides, to the best of our information, we did not see any previous paper in the literature showing the behaviors of the paths of the considered (VIDEs), clearly, as in our work. This paper may be the first attempt in the literature on the topic of that kind of retarded (VIDEs). From all of the above, we can conclude the new and novel properties of the present paper and the Motivation to study this paper.

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