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$\mathcal{I}_2$-deferred statistical convergence for sequences of sets

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Abstract. The main purpose of this study is presented the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence in the Wijsman sense for double sequences of sets. Also, to investigate the relations between these concepts and then to prove some theorems associated with the concept of $\mathcal{I}_2$-deferred statistical convergence in the Wijsman sense for double sequences of sets is purposed.

1. Introduction and Backgrounds

Long after the concept of deferred Cesàro mean for real (or complex) sequences was introduced by Agnew [1], Küçükaslan and Yılmaztürk [10] presented the concept of deferred statistical convergence for single sequences. After this, the concept of deferred $\mathcal{I}$-convergence was given by Şengül et al. [18]. For double sequences, the concepts of deferred Cesàro summability and deferred statistical convergence were studied by Dağadur and Sezgek [4, 5, 17].

Over the years, on the various convergence concepts for sequences of sets have been studied by many authors. One of them, discussed in this study, is the concept of convergence in the Wijsman sense [3, 12, 21]. The concepts of convergence and Cesàro summability in the Wijsman sense for double sequences of sets were introduced by Nuray et al. [14] and after that, for double sequences of sets, the concepts of statistical convergence and $\mathcal{I}$-convergence in the Wijsman sense were given by Nuray et al. [13, 15].

For sequences of sets, the concepts of strongly deferred Cesàro summability and deferred statistical convergence in the Wijsman sense were introduced by Altınok et al. [2]. Recently, for double sequences of sets, the concepts of deferred Cesàro summability and deferred statistical convergence were studied by Ulusu and Gülle [20]. See [7, 8] for more details.

Now, before introducing new concepts, we remind some fundamental definitions and notations (see, [4, 6, 9, 11, 13, 14, 16, 19, 20]).

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The natural density of K_2 is defined by

$$\delta(K_2) = \lim_{m,n\to\infty} \frac{1}{mn}\left|\left\{(i,j)\in K_2 : i\le m,\ j\le n\right\}\right|$$

where K_2 is a subset of $\mathbb{N}\times\mathbb{N}$ and $\left|\{(i,j)\in K_2 : i\le m,\ j\le n\}\right|$ denotes the number of elements of K_2 that does not exceed m and n.

A double sequence (x_{ij}) is statistically convergent to L if the set $\left\{(i,j) : |x_{ij}-L|\ge\delta\right\}$ has natural density zero, that is, for every $\delta>0$

$$\lim_{m,n\to\infty} \frac{1}{mn}\left|\left\{(i,j) : i\le m,\ j\le n,\ |x_{ij}-L|\ge\delta\right\}\right| = 0.$$

A family of sets $\mathcal{I}\subseteq 2^{\mathbb{N}}$ (the power set of $\mathbb{N}$) is called an ideal if and only if (i_1) $\emptyset\in\mathcal{I}$, (i_2) $E\cup F\in\mathcal{I}$ for each $E,F\in\mathcal{I}$, (i_3) $F\in\mathcal{I}$ for each $E\in\mathcal{I}$ and $F\subseteq E$.

An ideal $\mathcal{I}\subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N}\notin\mathcal{I}$ and a non-trivial ideal $\mathcal{I}\subseteq 2^{\mathbb{N}}$ is called admissible if $\{i\}\in\mathcal{I}$ for each $i\in\mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2\subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in\mathbb{N}$. Obviously, a strongly admissible ideal is admissible.

All through this study, except where otherwise stated, $\mathcal{I}_2\subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is conceived strongly admissible ideal.

In a metric space (Y,ρ), the distance function $d(y,U)=d_y(U)$ is defined by $d_y(U)=\inf_{u\in U}\rho(y,u)$ for any non-empty $U\subseteq Y$ and any $y\in Y$.

On a non-empty set Y, for a function $f:\mathbb{N}\to 2^Y$, the sequence $\{U_i\}=\{U_1,U_2,\ldots\}$ is called sequences of sets.

All through this study, (Y,ρ) is conceived a metric space and U, U_{ij} are conceived any non-empty closed subsets of Y.

A double sequence $\{U_{ij}\}$ is (in the Wijsman sense);

i. convergent to set U provided that

$$\lim_{i,j\to\infty} d_y(U_{ij}) = d_y(U),$$

ii. $\mathcal{I}_2$-convergent to set U provided that for every $\delta>0$

$$\left\{(i,j)\in\mathbb{N}^2 : \left|d_y(U_{ij})-d_y(U)\right|\ge\delta\right\}\in\mathcal{I}_2,$$

iii. strongly $\mathcal{I}_2$-Cesàro summable to set U provided that for every $\delta>0$

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{mn}\sum_{i,j=1,1}^{m,n}\left|d_y(U_{ij})-d_y(U)\right|\ge\delta\right\}\in\mathcal{I}_2,$$

iv. $\mathcal{I}_2$-statistically convergent to set U provided that for every $\delta>0$, $\gamma>0$

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{mn}\left|\left\{(i,j) : i\le m,\ j\le n,\ \left|d_y(U_{ij})-d_y(U)\right|\ge\delta\right\}\right|\ge\gamma\right\}\in\mathcal{I}_2$$

for each $y\in Y$. For (ii) and (iv) concepts, the notations $U_{ij}\xrightarrow{W(\mathcal{I}_2)}U$ and $U_{ij}\xrightarrow{WS(\mathcal{I}_2)}U$ are used, respectively.

For a double sequence $x = (x_{ij})$, the deferred Cesàro mean $D_{\phi,\psi}$ is defined by

$$(D_{\phi,\psi}x)_{mn} = \frac{1}{\phi_m\psi_n}\sum_{i=p_m+1}^{q_m}\sum_{j=r_n+1}^{s_n} x_{ij} = \frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} x_{ij},$$

where (p_m), (q_m), (r_n), (s_n) are non-negative integer sequences satisfying following conditions:

$$p_m < q_m,\ \lim_{m\to\infty} q_m = \infty;\ \ r_n < s_n,\ \lim_{n\to\infty} s_n = \infty \tag{1}$$

$$q_m - p_m = \phi_m;\ \ s_n - r_n = \psi_n. \tag{2}$$

Note here that the method $D_{\phi,\psi}$ is openly regular for any selection of the sequences (p_m), (q_m), (r_n), (s_n).

All through this study, except where otherwise stated, (p_m), (q_m), (r_n), (s_n) are conceived non-negative integer sequences satisfying (1) and (2).

A double sequence $\{U_{ij}\}$ is (in the Wijsman sense);

i. deferred Cesàro summable to set U provided that

$$\lim_{m,n\to\infty}\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} d_y(U_{ij}) = d_y(U),$$

ii. strongly deferred Cesàro summable to set U provided that

$$\lim_{m,n\to\infty}\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} \left|d_y(U_{ij}) - d_y(U)\right| = 0,$$

iii. deferred statistical convergent to set U provided that for every $\delta > 0$

$$\lim_{m,n\to\infty}\frac{1}{\phi_m\psi_n}\left|\left\{(i,j) : p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| = 0$$

for each $y \in Y$.

By double lacunary sequence, we mean that a double sequence $\theta_2 = \{(k_m, l_n)\}$ of two increasing integer sequences (k_m) and (l_n) such that

$$k_0 = 0,\ h_m = k_m - k_{m-1} \to \infty \quad \text{and} \quad l_0 = 0,\ \bar{h}_n = l_n - l_{n-1} \to \infty \quad \text{as} \quad m,n \to \infty.$$

2. New Concepts

In this section, we present the concepts of $\mathcal{I}_2$-deferred Cesàro summability and $\mathcal{I}_2$-deferred statistical convergence in the Wijsman sense for double sequences of sets.

Unless otherwise specified, from this point on, summability and convergence concepts will be considered in the Wijsman sense for double sequences of sets.

Definition 2.1. *The double sequence $\{U_{ij}\}$ is said to be $\mathcal{I}_2$-deferred Cesàro summable to the set U if for every $\delta > 0$*

$$\left\{(m,n) \in \mathbb{N}^2 : \left|\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\} \in \mathcal{I}_2$$

for each $y \in Y$ and this case is denoted in $U_{ij} \xrightarrow{WD(\mathcal{I}_2)} U$ format.

Definition 2.2. *The double sequence $\{U_{ij}\}$ is said to be strongly $\mathcal{I}_2$-deferred Cesàro summable to the set U if for every $\delta > 0$*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m \psi_n} \sum_{\substack{i=p_m+1 \\ j=r_n+1}}^{q_m, s_n} \left| d_y(U_{ij}) - d_y(U) \right| \geq \delta \right\} \in \mathcal{I}_2$$

for each $y \in Y$ and this case is denoted in $U_{ij} \stackrel{WD[\mathcal{I}_2]}{\longrightarrow} U$ format.

The set of all double sequences of sets that strongly $\mathcal{I}_2$-deferred Cesàro summable is denoted by $\{WD[\mathcal{I}_2]\}$.

Remark 2.3. *$WD(\mathcal{I}_2)$ and $WD[\mathcal{I}_2]$-summability concepts;*

(i) For $p_m = 0$, $q_m = m$ and $r_n = 0$, $s_n = n$, match with Wijsman $\mathcal{I}_2$-Cesàro and Wijsman strongly $\mathcal{I}_2$-Cesàro summability concepts in [19], respectively.

(ii) For $p_m = k_{m-1}$, $q_m = k_m$ and $r_n = l_{n-1}$, $s_n = l_n$ ($\{(k_m, l_n)\}$ states double lacunary sequence), match with Wijsman $\mathcal{I}_2$-lacunary and Wijsman strongly $\mathcal{I}_2$-lacunary convergence concepts in [6], respectively.

(iii) For the ideal $\mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$), match with Wijsman deferred Cesàro and Wijsman strongly deferred Cesàro summability concepts in [20], respectively.

Definition 2.4. *The double sequence $\{U_{ij}\}$ is said to be $\mathcal{I}_2$-deferred statistical convergent to the set U if for every $\delta, \gamma > 0$*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m \psi_n} \left| \left\{ (i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ \left| d_y(U_{ij}) - d_y(U) \right| \geq \delta \right\} \right| \geq \gamma \right\} \in \mathcal{I}_2$$

for each $y \in Y$ and this case is denoted in $U_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$ format.

The set of all double sequences of sets that $\mathcal{I}_2$-deferred statistical convergent is denoted by $\{WDS(\mathcal{I}_2)\}$.

Remark 2.5. *$WDS(\mathcal{I}_2)$-convergence concept;*

(i) For $p_m = 0$, $q_m = m$ and $r_n = 0$, $s_n = n$, matches with Wijsman $\mathcal{I}_2$-statistical convergence concept in [6].

(ii) For $p_m = k_{m-1}$, $q_m = k_m$ and $r_n = l_{n-1}$, $s_n = l_n$ ($\{(k_m, l_n)\}$ states double lacunary sequence), matches with Wijsman $\mathcal{I}_2$-lacunary statistical convergence concept in [6].

(iii) For the ideal $\mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$), matches with Wijsman deferred statistical convergence concept in [20].

3. Main Results

In this section, we first investigate the relations between $WD[\mathcal{I}_2]$-summability and $WDS(\mathcal{I}_2)$-convergence concepts.

Theorem 3.1. *If a double sequence $\{U_{ij}\}$ is strongly $\mathcal{I}_2$-deferred Cesàro summable to a set U, then this sequence is $\mathcal{I}_2$-deferred statistical convergent to same set.*

Proof. Assume that $U_{ij} \stackrel{WD[\mathcal{I}_2]}{\longrightarrow} U$. For every $\delta > 0$, we can write

$$\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} \left|d_y(U_{ij}) - d_y(U)\right| \geq \sum_{\substack{i=p_m+1\\ j=r_n+1\\ |d_y(U_{ij})-d_y(U)|\geq\delta}}^{q_m,s_n} \left|d_y(U_{ij}) - d_y(U)\right|$$

$$\geq \delta \left|\left\{(i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \geq \delta\right\}\right|$$

and so, we have

$$\frac{1}{\delta}\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} \left|d_y(U_{ij}) - d_y(U)\right| \geq \frac{1}{\phi_m\psi_n}\left|\left\{(i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \geq \delta\right\}\right|$$

for each $y \in Y$. Hence, for every $\gamma > 0$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \geq \delta\right\}\right| \geq \gamma\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n} \left|d_y(U_{ij}) - d_y(U)\right| \geq \gamma\,\delta\right\} \in \mathcal{I}_2.$$

Thus, by the assumption and the condition (i_3), we get $U_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$. □

Corollary 3.2. *If $U_{ij} \stackrel{W(\mathcal{I}_2)}{\longrightarrow} U$, then $U_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$.*

The converse of Theorem 3.1 is not true in general. We can consider the following example by taking $\mathcal{I}_2 = \mathcal{I}_2^f$ (the ideal of density zero sets of $\mathbb{N}$) to explain this situation.

Example 3.3. *Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:*

$$U_{ij} := \begin{cases} \{i^2 j^2\} & ; \quad \begin{matrix} q_m - [|\sqrt{\phi_m}|] < i \leq q_m \\ s_n - [|\sqrt{\psi_n}|] < j \leq s_n \end{matrix} \\ \{0\} & ; \quad \text{otherwise.} \end{cases} ,\quad (i,j) \in \mathbb{N}\times\mathbb{N},$$

This sequence is not bounded. Also, this sequence is $\mathcal{I}_2$-deferred statistical convergent in the Wijsman sense to the set $U = \{0\}$, but is not strongly $\mathcal{I}_2$-deferred Cesàro summable in the Wijsman sense.

A double sequence $\{U_{ij}\}$ is said to be bounded if $\sup_{i,j}\{d_y(U_{ij})\} < \infty$, that is for an $\mathcal{M} > 0$, $|d_y(U_{ij}) - d_y(U)| \leq \mathcal{M}$ for each $y \in Y$ and all $(i,j) \in \mathbb{N}^2$. Also, L_∞^2 denotes the set of all bounded double sequences of sets.

Theorem 3.4. *If a double sequence $\{U_{ij}\} \in L_\infty^2$ is $\mathcal{I}_2$-deferred statistical convergent to a set U, then this sequence is strongly $\mathcal{I}_2$-deferred Cesàro summable to same set.*

Proof. Assume that $\{U_{ij}\} \in L_\infty^2$ and $U_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$. For every $\delta > 0$, we can write

$$\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n}\left|d_y(U_{ij})-d_y(U)\right|$$

$$=\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1\\ |d_y(U_{ij})-d_y(U)|\geq\delta}}^{q_m,s_n}\left|d_y(U_{ij})-d_y(U)\right|+\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1\\ |d_y(U_{ij})-d_y(U)|<\delta}}^{q_m,s_n}\left|d_y(U_{ij})-d_y(U)\right|$$

$$\leq\frac{\mathcal{M}}{\phi_m\psi_n}\left|\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ \left|d_y(U_{ij})-d_y(U)\right|\geq\delta\right\}\right|+\delta$$

for each $y \in Y$. Hence, for every $\gamma > 0$ we have

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\phi_m\psi_n}\sum_{\substack{i=p_m+1\\ j=r_n+1}}^{q_m,s_n}\left|d_y(U_{ij})-d_y(U)\right|\geq\gamma\right\}$$

$$\subseteq\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\phi_m\psi_n}\left|\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ \left|d_y(U_{ij})-d_y(U)\right|\geq\delta\right\}\right|\geq\frac{\gamma}{\mathcal{M}}\right\}\in\mathcal{I}_2.$$

Thus, by the assumption and the condition (i_3), we get $U_{ij} \stackrel{WD[\mathcal{I}_2]}{\longrightarrow} U$. □

By Theorem (3.1) and Theorem (3.4), we get the following corollary.

Corollary 3.5. $L_\infty^2 \bigcap \left\{WD[\mathcal{I}_2]\right\} = L_\infty^2 \bigcap \left\{WDS(\mathcal{I}_2)\right\}$.

Now, we secondly prove some theorems associated with the concept of $\mathcal{I}_2$-deferred statistical convergence in the Wijsman sense for double sequences of sets.

Theorem 3.6. *Let $\{T_{ij}\}$, $\{U_{ij}\}$ and $\{V_{ij}\}$ be double sequences such that $T_{ij} \subset U_{ij} \subset V_{ij}$ for all $(i,j) \in \mathbb{N}^2$. Then,*

$$T_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U \text{ and } V_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U \Rightarrow U_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U.$$

Proof. Suppose that $T_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$, $V_{ij} \stackrel{WDS(\mathcal{I}_2)}{\longrightarrow} U$ and $T_{ij} \subset U_{ij} \subset V_{ij}$ for all $(i,j) \in \mathbb{N}^2$. For each $y \in Y$,

$$T_{ij} \subset U_{ij} \subset V_{ij} \Rightarrow d_y(V_{ij}) \leq d_y(U_{ij}) \leq d_y(T_{ij})$$

is hold. Hence, for every $\delta > 0$ we can write

$$\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ \left|d_y(U_{ij})-d_y(U)\right|\geq\delta\right\}$$

$$=\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ d_y(U_{ij})\geq d_y(U)+\delta\right\}$$

$$\cup\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ d_y(U_{ij})\leq d_y(U)-\delta\right\}$$

$$\subset\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ d_y(T_{ij})\geq d_y(U)+\delta\right\}$$

$$\cup\left\{(i,j):p_m<i\leq q_m,\ r_n<j\leq s_n,\ d_y(V_{ij})\leq d_y(U)-\delta\right\}$$

and so,

$$\frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right|$$

$$< \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(T_{ij}) - d_y(U)\right| \ge \delta\right\}\right|$$

$$+\frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(V_{ij}) - d_y(U)\right| \ge \delta\right\}\right|.$$

Hence, for every $\gamma > 0$ we have

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\subset \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(T_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(V_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\}.$$

Thus, by the assumption and the conditions (i_2) and (i_3), we get $U_{ij} \overset{WDS(\mathcal{I}_2)}{\longrightarrow} U$. □

Theorem 3.7. *Let* $\left(\frac{p_m}{\phi_m}\right)$ *and* $\left(\frac{r_n}{\psi_n}\right)$ *be bounded, then*

$$U_{ij} \overset{WS(\mathcal{I}_2)}{\longrightarrow} U \Rightarrow U_{ij} \overset{WDS(\mathcal{I}_2)}{\longrightarrow} U.$$

Proof. First of all, since $\left(\frac{p_m}{\phi_m}\right)$ is bounded, there exists an $\alpha > 0$ such that $\frac{p_m}{\phi_m} < \alpha$ for all $m \in \mathbb{N}$. So, we write

$$\frac{p_m}{\phi_m} < \alpha \ \Rightarrow\ \frac{\phi_m}{q_m} > \frac{1}{1+\alpha}.$$

Similarly, for all $n \in \mathbb{N}$, following inequalities can be obtained

$$\frac{r_n}{\psi_n} < \beta \ \Rightarrow\ \frac{\psi_n}{s_n} > \frac{1}{1+\beta}.$$

Suppose that $U_{ij} \overset{WS(\mathcal{I}_2)}{\longrightarrow} U$. For every $\delta > 0$, we can write

$$\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\} \subseteq \left\{(i,j): i \le q_m,\ j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}$$

for each $y \in Y$, and so

$$\frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right|$$

$$\le \frac{q_m s_n}{\phi_m\psi_n}\frac{1}{q_m s_n}\left|\left\{(i,j): i \le q_m,\ j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right|.$$

Hence, for every $\gamma > 0$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m \psi_n}\left|\left\{(i,j) : p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{q_m s_n}\left|\left\{(i,j) : i \le q_m,\ j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \frac{\gamma}{(1+\alpha)(1+\beta)}\right\}.$$

Thus, by the assumption and the condition (i_3), we get $U_{ij} \overset{WDS(\mathcal{I}_2)}{\longrightarrow} U$. □

The following theorems will be considered under the restrictions:

$$p_m \le p'_m < q'_m \le q_m \quad \text{and} \quad r_n \le r'_n < s'_n \le s_n$$

for all $m,n \in \mathbb{N}$, where all of these are sequences of non-negative integers.

Theorem 3.8. *If* $\left(\frac{\phi_m \psi_n}{\phi'_m \psi'_n}\right)$ *is bounded, then*

$$\left\{WDS(\mathcal{I}_2)\right\}_{[\phi,\psi]} \subseteq \left\{WDS(\mathcal{I}_2)\right\}_{[\phi',\psi']}.$$

Proof. First of all, since $\left(\frac{\phi_m \psi_n}{\phi'_m \psi'_n}\right)$ is bounded, there exists an $\eta > 0$ such that $\frac{\phi_m \psi_n}{\phi'_m \psi'_n} < \eta$ for all $m,n \in \mathbb{N}$. Assume that $\{U_{ij}\} \in \{WDS(\mathcal{I}_2)\}_{[\phi,\psi]}$ and $U_{ij} \longrightarrow U\ (WDS(\mathcal{I}_2)_{[\phi,\psi]})$. For every $\delta > 0$ since

$$\left\{(i,j) : p'_m < i \le q'_m,\ r'_n < j \le s'_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\} \subset \left\{(i,j) : p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}$$

for each $y \in Y$, we can write

$$\frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \le q'_m,\ r'_n < j \le s'_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right|$$

$$\le \frac{\phi_m \psi_n}{\phi'_m \psi'_n}\left(\frac{1}{\phi_m \psi_n}\left|\left\{(i,j) : p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right|\right).$$

Hence, for every $\gamma > 0$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \le q'_m,\ r'_n < j \le s'_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m \psi_n}\left|\left\{(i,j) : p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \frac{\gamma}{\eta}\right\}.$$

Thus, by the assumption and the condition (i_3), we get $U_{ij} \longrightarrow U(WDS(\mathcal{I}_2)_{[\phi',\psi']})$ and $\{U_{ij}\} \in \{WDS(\mathcal{I}_2)\}_{[\phi',\psi']}$. Consequently, $\{WDS(\mathcal{I}_2)\}_{[\phi,\psi]} \subseteq \{WDS(\mathcal{I}_2)\}_{[\phi',\psi']}$. □

Theorem 3.9. *If the sets* $\{i : p_m < i \le p'_m\}$, $\{i : q'_m < i \le q_m\}$, $\{j : r_n < j \le r'_n\}$, $\{j : s'_n < j \le s_n\}$ *are finite for all* $m,n \in \mathbb{N}$, *then*

$$\{WDS(\mathcal{I}_2)\}_{[\phi',\psi']} \subseteq \{WDS(\mathcal{I}_2)\}_{[\phi,\psi]}.$$

Proof. Let $\{U_{ij}\} \in \{WDS(\mathcal{I}_2)\}_{[\phi',\psi']}$ and $U_{ij} \longrightarrow U$ $(\{WDS(\mathcal{I}_2)\}_{[\phi',\psi']})$. Then, for every $\delta, \gamma > 0$ and each $y \in Y$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \leq q'_m,\ r'_n < j \leq s'_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \geq \delta\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2.$$

Also, for every $\delta > 0$ and each $y \in Y$, since

$$\begin{aligned}
&\left\{(i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad = \left\{(i,j) : p_m < i \leq p'_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : p_m < i \leq p'_m,\ r'_n < j \leq s'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : p_m < i \leq p'_m,\ s'_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : p'_m < i \leq q'_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : p'_m < i \leq q'_m,\ r'_n < j \leq s'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : p'_m < i \leq q'_m,\ s'_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : q'_m < i \leq q_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : q'_m < i \leq q_m,\ r'_n < j \leq s'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\} \\
&\quad \cup \left\{(i,j) : q'_m < i \leq q_m,\ s'_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{\phi_m \psi_n}\left|\left\{(i,j) : p_m < i \leq q_m,\ r_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad \leq \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p_m < i \leq p'_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p_m < i \leq p'_m,\ r'_n < j \leq s'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p_m < i \leq p'_m,\ s'_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \leq q'_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \leq q'_m,\ r'_n < j \leq s'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : p'_m < i \leq q'_m,\ s'_n < j \leq s_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right| \\
&\quad + \frac{1}{\phi'_m \psi'_n}\left|\left\{(i,j) : q'_m < i \leq q_m,\ r_n < j \leq r'_n,\ |d_y(U_{ij}) - d_y(U)| \geq \delta\right\}\right|
\end{aligned}$$

$$+\frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): q'_m < i \le q_m,\ r'_n < j \le s'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right|$$

$$+\frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): q'_m < i \le q_m,\ s'_n < j \le s_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right|$$

and so, for every $\gamma > 0$ and each $y \in Y$

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p_m < i \le p'_m,\ r_n < j \le r'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p_m < i \le p'_m,\ r'_n < j \le s'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p_m < i \le p'_m,\ s'_n < j \le s_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p'_m < i \le q'_m,\ r_n < j \le r'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p'_m < i \le q'_m,\ r'_n < j \le s'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): p'_m < i \le q'_m,\ s'_n < j \le s_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): q'_m < i \le q_m,\ r_n < j \le r'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): q'_m < i \le q_m,\ r'_n < j \le s'_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}$$

$$\cup\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi'_m\psi'_n}\left|\left\{(i,j): q'_m < i \le q_m,\ s'_n < j \le s_n,\ |d_y(U_{ij}) - d_y(U)| \ge \delta\right\}\right| \ge \gamma\right\}.$$

If the sets $\{i : p_m < i \le p'_m\}$, $\{i : q'_m < i \le q_m\}$, $\{j : r_n < j \le r'_n\}$, $\{j : s'_n < j \le s_n\}$ are finite for all $n, m \in \mathbb{N}$ in the above expression, by the assumption and the conditions (i_2) and (i_3), we get

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\phi_m\psi_n}\left|\left\{(i,j): p_m < i \le q_m,\ r_n < j \le s_n,\ \left|d_y(U_{ij}) - d_y(U)\right| \ge \delta\right\}\right| \ge \gamma\right\} \in \mathcal{I}_2,$$

i.e., $U_{ij} \longrightarrow U\,(\{WDS(\mathcal{I}_2)\}_{[\phi,\psi]})$ and $\{U_{ij}\} \in \{WDS(\mathcal{I}_2)\}_{[\phi,\psi]}$. Consequently, $\{WDS(\mathcal{I}_2)\}_{[\phi',\psi']} \subseteq \{WDS(\mathcal{I}_2)\}_{[\phi,\psi]}$. □

By Theorem (3.1), Theorem (3.4), Theorem (3.8) and Theorem (3.9), we get the following corollary.

Corollary 3.10.

i. Let $\left(\frac{\phi_m\,\psi_n}{\phi'_m\,\psi'_n}\right)$ *be bounded. If a double sequence* $\{U_{ij}\}$ *is* $WD[\mathcal{I}_2]_{[\phi,\psi]}$*-summable to a set* U*, then this sequence is* $WDS(\mathcal{I}_2)_{[\phi',\psi']}$*-convergent to same set.*

ii. Let the sets $\{i : p_m < i \leq p'_m\}$*,* $\{i : q'_m < i \leq q_m\}$*,* $\{j : r_n < j \leq r'_n\}$*,* $\{j : s'_n < j \leq s_n\}$ *be finite for all* $m, n \in \mathbb{N}$*. If a double sequence* $\{U_{ij}\} \in L^2_\infty$ *is* $WDS(\mathcal{I}_2)_{[\phi',\psi']}$*-convergent to a set* U*, then this sequence is* $WD[\mathcal{I}_2]_{[\phi,\psi]}$*-summable to same set.*

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