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$L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators involving the Weinstein transform

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Abstract. In this paper, an $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators associated with class of symbol S^0 are proven by utilizing the theory of the Weinstein transform. Using the aforesaid theory various properties and boundedness results on $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - type Sobolev spaces are given.

1. Introduction

The Weinstein transformation is the generalization of the Fourier and Hankel transform and its kernel is the product of complex exponential function and normalized Bessel function of the first kind. Using the theory of the Weinstein transformation many important problems of partial-differential equations, signal processing, wavelets, mechanics and mathematical sciences have been studied by many authors [10, 11, 18, 21]. Recently, an integral representation of pseudo-differential operators involving the Weinstein transform was investigated by [18] and discussed many properties. By the paper [18] utilizing the theory of the Weinstein transform authors gave the application of pseudo-differential operators in the heat equation. This theory is useful to study L^p_α - boundedness of pseudo-differential operators involving the Weinstein transform.

Kohn and Nirenberg [9], Hörmander [4] and others developed proper calculus of pseudo-differential operators by exploiting the theory of Fourier transformation. Calderón and Vaillancourt [1] proved the L^2- boundedness of M-order pseudo-differential operators associated with symbol $p(x_1, x_2, \xi) \in S^M_{\rho,\delta_1,\delta_1}$ for $0 \leq \rho \leq \delta_1, \delta_2 < 1$ and $\frac{M}{n} \geq \frac{(\delta_1+\delta_2)}{2} - \rho$, Fefferman [3] proved sharp L^p- boundedness results for pseudo-differential operators in the class $S^m_{\rho,\delta}$, Illner [7] discussed the L^p- boundedness of pseudo-differential operators with symbol $p(x, \xi, y) \in S^\mu_{\rho,\delta,\epsilon}$ for $\mu \leq (\rho - 1)(n + 1)$, Cato [8] considered the L^2- boundedness for pseudo-differential operators with symbol $a(x, \xi)$ lies in $S^0_{\rho,\rho}$ for $0 < \rho < 1$, Nagase [12] investigated the L^p-boundedness of pseudo-differential operators with non-regular symbols, Hwang and Lee [6] studied the L^p - boundedness of pseudo-differential operator with symbol class $S^m_{0,0}$ for $m = -n|1/p - 1/2|$, Wong [22, p. 77] proved that the $L^p(\mathbb{R}^n)$ - boundedness of pseudo-diffrential operators T_σ for $\sigma \in S^0$, Kumar and Ruzhansky

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[20] investigated the $L^p - L^q$ - boundedness of (k,a) - Fourier multipliers and using this theory they gave applications to nonlinear equations. These L^p - boundedness results made strong calculus of pseudo-differential operators involving Fourier transform with different types of symbol classes. Exploiting the Hankel transform theory Pathak and Upadhyay [15], established the L^p_μ - boundedness results of pseudo-differential operators with symbol class H^m. Using L^p_μ - boundedness properties authors proved $h_{\mu,\alpha}$ is bounded linear operator from $W^{m,p}_\mu \to W^{0,p}_\mu$ and $W^{s,p}_\mu \to W^{s-m,p}_\mu$. These spaces are defined in [14]. Upadhyay and Singh [19], discussed the mapping properties of wavelet transform and found important observations. Saoudi [16], found the boundedness and compactness of localization operators in the Weinstein setting. Motivated by the results of [22] and [15] our main objective of this paper is to investigate the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators associated with certain class of symbols involving the Weinstein transform.

Contents of this paper are organized in the following way:
Section 1 is introductory which describes the brief history regarding the L^p_α - boundedness of pseudo-differential operators. Section 2 provides basic notations, definitions, lemmas and theorems that are useful in other subsequent sections. In section 3, an integral representation of pseudo-differential operators and other properties are obtained. The L^p_α- boundedness of the pseudo-differential operators T_σ associated with symbol $\sigma \in S^0$ involving the Weinstein transform techniques is obtained. In the last section, L^p_α - type Sobolev space of order r is defined and boundedness of the pseudo-differential operators T_σ from $\mathcal{H}^{r,p}_\alpha \to \mathcal{H}^{r-m,p}_\alpha$ is given.

2. Preliminaries

Some standard notations are given below:
- $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty) = \{(x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$.
- $\mathbb{N}^{n+1}_0 = \{(a_1, a_2, \cdots, a_n, a_{n+1}) \in \mathbb{N}^{n+1} : a_j \in \mathbb{N} \cup \{0\}, \forall j = 1, 2, \cdots, n+1\}$.
- $x = (x', x_{n+1}) = (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}_+$.
- $\langle x', y' \rangle = \sum_{j=1}^n x_j y_j$.
- $||x||^2 = \sum_{j=1}^{n+1} x_j^2$.
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}}$, for $x \in \mathbb{R}^{n+1}_+$ and $\alpha \in \mathbb{N}^{n+1}_0$.
- $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} D_{n+1}^{\alpha_{n+1}}$, for $\alpha \in \mathbb{N}^{n+1}_0$.
- $\hat{J}_\alpha$, the normalized Bessel function of the first kind.
- $C^\infty_c(\mathbb{R}^{n+1}_+)$, the space of C^∞- functions on $\mathbb{R}^{n+1}_+$ with compact support.
- $C^k(\mathbb{R}^{n+1}_+)$, the space of C^k- functions on $\mathbb{R}^{n+1}_+$.
- $S_*(\mathbb{R}^{n+1})$, the space of C^∞ - functions on $\mathbb{R}^{n+1}$, even with respect to the last variable and rapidly decreasing together with their derivatives.
- $S'_*(\mathbb{R}^{n+1})$, the space of tempered distributions on $\mathbb{R}^{n+1}$.

For $f \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, the tempered distribution is denoted and defined as

$$\langle f, u \rangle = \int_{\mathbb{R}^{n+1}_+} f(x) u(x) d\mu_\alpha(x), \quad u \in S_*(\mathbb{R}^{n+1}) \tag{1}$$

where

$$d\mu_\alpha(x) = A_\alpha x_{n+1}^{2\alpha+1} dx, \tag{2}$$

and

$$A_\alpha = \frac{1}{(2\pi)^{\frac{n}{2}} 2^\alpha \Gamma(\alpha+1)}, \tag{3}$$

dx is the Lebesgue measure on $\mathbb{R}_+^{n+1}$.

- The Weinstein operator on $\mathbb{R}_+^{n+1}$ for $(n+1)$ variables is defined by

$$\Delta_{\alpha,n} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}} = \Delta_n + L_\alpha, \quad \alpha > -\frac{1}{2}, \tag{4}$$

where

$$\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \tag{5}$$

is the Laplacian operator defined on $\mathbb{R}^n$ for the first n variables and

$$L_\alpha = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}}, \tag{6}$$

is the Bessel operator defined on $(0,\infty)$ for the last variale .

- $L^p_\alpha(\mathbb{R}_+^{n+1})$, denotes the space of Lebesgue measurable functions on $\mathbb{R}_+^{n+1}$ such that

$$\|\phi\|_{L^p_\alpha} = \left(\int_{\mathbb{R}_+^{n+1}} |\phi(x)|^p d\mu_\alpha(x) \right)^{1/p} < \infty, \;\; 1 \leq p < \infty. \tag{7}$$

Definition 2.1. *The Weinstein transform of $\phi \in L^1_\alpha(\mathbb{R}_+^{n+1})$ is defined by*

$$(\mathcal{F}_\alpha \phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \phi(x) d\mu_\alpha(x), \quad \forall \xi \in \mathbb{R}_+^{n+1} \tag{8}$$

where $\mu_\alpha(x)$ is the measure on $\mathbb{R}_+^{n+1}$ given by (2).

From [2, 10], we take the following results which are useful in this paper:

1. **Parseval formula:** Let $\phi, \psi \in S_*(\mathbb{R}^{n+1})$, then

$$\int_{\mathbb{R}_+^{n+1}} \phi(x)\overline{\psi(x)} d\mu_\alpha(x) = \int_{\mathbb{R}_+^{n+1}} \mathcal{F}_\alpha(\phi)(\xi) \overline{\mathcal{F}_\alpha(\psi)(\xi)} d\mu_\alpha(\xi). \tag{9}$$

2. **Plancherel formula:** Let $\phi \in S_*(\mathbb{R}^{n+1})$, then

$$\|\mathcal{F}_\alpha \phi\|_{L^2_\alpha} = \|\phi\|_{L^2_\alpha}. \tag{10}$$

3. **Inversion formula:** Let $\phi \in L^1_\alpha(\mathbb{R}_+^{n+1})$ such that $\mathcal{F}_\alpha(\phi) \in L^1_\alpha(\mathbb{R}_+^{n+1})$, then

$$\phi(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) (\mathcal{F}_\alpha \phi)(\xi) d\mu_\alpha(\xi), \;\; a.e. \;\; \xi \in \mathbb{R}_+^{n+1}. \tag{11}$$

4. A map $\mathcal{F}_\alpha : \mathcal{S}_*(\mathbb{R}_+^{n+1}) \to \mathcal{S}_*(\mathbb{R}_+^{n+1})$ is a topological isomorphism and

$$\mathcal{F}_\alpha^{-1} \phi(\xi) = \mathcal{F}_\alpha \phi(-\xi), \quad \text{for all} \;\; \xi \in \mathbb{R}_+^{n+1}. \tag{12}$$

5. Let $u(x,\xi) = e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})$, then

$$\Delta_{\alpha,n} u(x,\xi) = (-\|\xi\|^2) u(x,\xi). \tag{13}$$

6. Let $\phi \in S_*(\mathbb{R}^{n+1})$ and $k \in \mathbb{N}_0$, then

$$[\mathcal{F}_\alpha(\Delta^k_{\alpha,n} \phi)](\xi) = (-1)^k \|\xi\|^{2k} (\mathcal{F}_\alpha \phi)(\xi). \tag{14}$$

Theorem 2.2. *Let $f \in S_*'(\mathbb{R}_+^{n+1})$, then from ([13], p. 109), we have*

$$\langle \mathcal{F}_\alpha(f), u\rangle = \langle f, \mathcal{F}_\alpha(u)\rangle, \quad u \in S_*(\mathbb{R}^{n+1}), \tag{15}$$

and

$$\langle \mathcal{F}_\alpha^{-1}(f), u\rangle = \langle f, \mathcal{F}_\alpha^{-1}(u)\rangle, \quad u \in S_*(\mathbb{R}^{n+1}). \tag{16}$$

Definition 2.3. *Let $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1})$. Then from ([18], p. 25) the translation of $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1})$ is defined by*

$$(\tau_x^\alpha \phi)(y) = \int_{\mathbb{R}_+^{n+1}} \phi(z)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z), \tag{17}$$

where

$$\mathcal{D}_\alpha(x,y,z) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',t'\rangle}\hat{j}_\alpha(x_{n+1}t_{n+1})e^{-i\langle y',t'\rangle}\hat{j}_\alpha(y_{n+1}t_{n+1}) \times e^{-i\langle z',t'\rangle}\hat{j}_\alpha(z_{n+1}t_{n+1})d\mu_\alpha(t), \tag{18}$$

is well defined and makes sense by the following relation from ([18], p. 24)

$$\mathcal{D}_\alpha(x,y,z) = \delta(z'+y'-x')D(x_{n+1},y_{n+1},z_{n+1}), \quad \forall x,y,z \in \mathbb{R}_+^{n+1}, \tag{19}$$

where $x' = (x_1,\cdots,x_n),\ y' = (y_1,\cdots,y_n),\ z' = (z_1,\cdots,z_n)$ and

$$\int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z) = 1, \tag{20}$$

for $x = (x', x_{n+1}),\ y = (y', y_{n+1})$ and $z = (z', z_{n+1})$.

Definition 2.4. *Let $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1})$ and $\psi \in L_\alpha^1(\mathbb{R}_+^{n+1})$. Then from ([18], p. 25) the Weinstein convolution of $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1}), \psi \in L_\alpha^1(\mathbb{R}_+^{n+1})$ is given by*

$$(\phi *_w \psi)(x) = \int_{\mathbb{R}_+^{n+1}} (\tau_x^\alpha \phi)(y)\psi(y)d\mu_\alpha(y). \tag{21}$$

By using (17) *and* (21) *the Weinstein convolution of $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1}), \psi \in L_\alpha^1(\mathbb{R}_+^{n+1})$ is defined by*

$$(\phi *_w \psi)(x) = \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\alpha(x,y,z)\phi(y)\psi(z)d\mu_\alpha(y)d\mu_\alpha(z). \tag{22}$$

Proposition 2.5. *From [13], we have*
1. For all $\phi, \psi \in L_\alpha^1(\mathbb{R}_+^{n+1})$, then $\phi *_w \psi \in L_\alpha^1(\mathbb{R}_+^{n+1})$, and we have

$$\mathcal{F}_\alpha(\phi *_w \psi) = \mathcal{F}_\alpha(\phi)\mathcal{F}_\alpha(\psi). \tag{23}$$

2. Let $p, q, r \in [1,\infty]$, such that $\frac{1}{p}+\frac{1}{q}-\frac{1}{r} = 1$. Then for all $\phi \in L_\alpha^p(\mathbb{R}_+^{n+1})$ and $\psi \in L_\alpha^q(\mathbb{R}_+^{n+1})$ the function $\phi *_w \psi$ belongs $L_\alpha^r(\mathbb{R}_+^{n+1})$, we estimate the following norm

$$\|\phi *_w \psi\|_{L_\alpha^r} \le \|\phi\|_{L_\alpha^p}\|\psi\|_{L_\alpha^q}. \tag{24}$$

Definition 2.6. *The symbol class S^m is the set of all functions $\sigma : C^\infty(\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{n+1}) \to \mathbb{C}$, $m \in \mathbb{R}$, such that $\forall q \in \mathbb{N}_0$ and $\beta, \gamma \in \mathbb{N}_0^{n+1}$, there exists a constant $C_{\beta,\gamma} > 0$ depending only on β and γ, satisfying*

$$\left|D_\xi^\beta D_x^\gamma \sigma(x,\xi)\right| \le C_{\beta,\gamma}(1+\|\xi\|^2)^{m-|\beta|}(1+\|x\|^2)^{-q}. \tag{25}$$

Definition 2.7. *Let $\sigma : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \to \mathbb{C}$ be a symbol. Then for $\phi \in S_*(\mathbb{R}^{n+1})$, the pseudo-differential operator T_σ associated with symbol $\sigma \in S^m$ is defined by*

$$\Big(T_\sigma\phi\Big)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)(\mathcal{F}_\alpha\phi)(\xi)d\mu_\alpha(\xi). \tag{26}$$

Lemma 2.8. *Let $s \in \mathbb{N}_0$. From [18], we have*

(i) If σ is a C^∞- function, then

$$\Delta_{\alpha,n}^s\sigma(x,\xi) = \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{|\delta'|\le s-j}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}x_{n+1}^{r-s}D_x^{2\delta'+r}\sigma(x,\xi), \tag{27}$$

where $E'_{\alpha,r}$ for $r \in \{0,1,\cdots,s\}$ is a constant depending only on α, $2\delta'+r = (2\delta_1,\cdots,2\delta_n,r) \in \mathbb{N}_0^{n+1}$ and $2|\delta'|+r = 2\delta_1+\cdots+2\delta_n+r$.

(ii) If f,σ are C^∞- function, then

$$\Delta_{\alpha,n}^s\Big[f(x)\sigma(x,\xi)\Big] = \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(s-j)}\sum_{|\delta'|\le s-j}\binom{s}{j}\binom{r}{q}\binom{s-j}{\delta_1,\cdots,\delta_n}\frac{1}{\rho'!}E'_{\alpha,r} \times \xi_{n+1}^{r-s}\Big[D_x^{\rho'+q}f(x)\Big]\Big[D_x^{\rho'+2\delta'+r-q}\sigma(x,\xi)\Big], \tag{28}$$

where $E'_{\alpha,r}$ for $r \in \{0,1,\cdots,s\}$ is a constant depending only on α, $\rho'+q = (\rho_1,\cdots,\rho_n,q)$, $\rho'+2\delta'+r-q = (\rho_1+2\delta_1,\cdots,\rho_n+2\delta_n,r-q) \in \mathbb{N}_0^{n+1}$ and $|\rho'|+q = \rho_1+\cdots+\rho_n+q$, $|\rho'|+2|\delta'|+r-q = \rho_1+2\delta_1+\cdots\rho_n+2\delta_n+r-q$.

Lemma 2.9. *The Schwartz space $S_*(\mathbb{R}^{n+1})$ is a subspace of $L_\alpha^p(\mathbb{R}_+^{n+1})$, for all $p \in [1,\infty]$ and $\alpha > -\frac{1}{2}$.*

Proof. The proof of the result is obvious. □

Theorem 2.10. *Let $p \in [1,\infty)$ and $\alpha > -\frac{1}{2}$. Then the Schwartz space $S_*(\mathbb{R}^{n+1})$ is dense in $L_\alpha^p(\mathbb{R}_+^{n+1})$.*

Proof. Let $f \in S_*(\mathbb{R}^{n+1})$, then by Lemma 2.9, $f \in L_\alpha^p(\mathbb{R}_+^{n+1})$. Also, from the defintion of $S_*(\mathbb{R}^{n+1})$, we have

$$(1+\|x\|^2)^k f(x) \in S_*(\mathbb{R}^{n+1}), \quad \forall k \in \mathbb{N}.$$

Let $\{f_r\}_{r\ge1}$ be a sequence of functions in $S_*(\mathbb{R}^{n+1})$ such that $f_r \to 0$ in $S_*(\mathbb{R}^{n+1})$ as $r \to \infty$. Then, we have

$$\|f_r\|_{k,0} \to 0, \text{ as } r \to \infty,$$

where

$$\|f_r\|_{k,0} = \sup_{x\in R_+^{n+1}}(1+\|x\|^2)^k|f_r(x)|. \tag{29}$$

Now, we find

$$\begin{aligned}\|f_r\|^p_{L_\alpha^p(\mathbb{R}_+^{n+1})} &= \int_{\mathbb{R}_+^{n+1}}|f_r(x)|^p d\mu_\alpha(x)\\ &= \int_{\mathbb{R}_+^{n+1}}(1+\|x\|^2)^k|f_r(x)|^p\frac{d\mu_\alpha(x)}{(1+\|x\|^2)^k}.\end{aligned}$$

Using (29), the last expression becomes

$$\begin{aligned}\|f_r\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)} &= \|f^p_r\|_{k,0}\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-k}d\mu_\alpha(x)\\ &= \|f^p_r\|_{k,0}A_\alpha\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-k}x^{2\alpha+1}_{n+1}dx\\ &\leq \|f^p_r\|_{k,0}A_\alpha\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-k+2\alpha+1}dx.\end{aligned}$$

If we chosse $k > 2\alpha + n + 2$, then there exists a positive constant $C_{\alpha,k}$ such that

$$\|f_r\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,k}\|f^p_r\|_{k,0}.$$

Therefore, $f_r \to 0$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ space as $r \to \infty$. □

3. An Integral Representation of Pseudo-Differential Operators

An integral representation of the pseudo-differential operators T_σ associated with a symbol $\sigma \in S^m$ involving the Weinstein transform is obtained and its various properties studied.

Lemma 3.1. *Let $\alpha > -\frac{1}{2}$ and σ be a symbol in S^0. Define*

$$K(x,z) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1})\sigma(x,\xi)d\mu_\alpha(\xi), \tag{30}$$

as a distribution in $S'_*(\mathbb{R}^{n+1}_+)$. Then

(i) for each fixed $x \in \mathbb{R}^{n+1}_+$, $K(x,\cdot)$ is a function defined on $\mathbb{R}^{n+1}_+$,

(ii) for large values of $k \in \mathbb{N}_0$, there exists a constant $C_{\alpha,k}$ such that

$$|K(x,z)| \leq C_{\alpha,k}(1+\|x\|^2)^{-q}(1+\|z\|^2)^{-k}. \tag{31}$$

Proof. For $k \in \mathbb{N}$, (30) can be written as

$$K(x,z) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1})(1+\|z\|^2)^{-k}(1-\Delta_{\alpha,n})^k\sigma(x,\xi)d\mu_\alpha(\xi).$$

Invoking Binomial Theorem, we get

$$\begin{aligned}K(x,z) = \int_{\mathbb{R}^{n+1}_+} & e^{i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1})(1+\|z\|^2)^{-k}\\ &\times\left(\sum_{r=0}^{k}\binom{k}{r}(-1)^r\Delta^r_{\alpha,n}\sigma(x,\xi)\right)d\mu_\alpha(\xi).\end{aligned}$$

Therefore,

$$|K(x,z)| \leq (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\binom{k}{r}\int_{\mathbb{R}^{n+1}_+}|\Delta^r_{\alpha,n}\sigma(x,\xi)|d\mu_\alpha(\xi).$$

In view of Lemma 2.8, the last expression becomes

$$|K(x,z)| \leq (1+\|z\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l}$$
$$\times \int_{\mathbb{R}_+^{n+1}} \xi_{n+1}^{l-r} \left|D_\xi^{2\delta'+l}\sigma(x,\xi)\right| d\mu_\alpha(\xi)$$
$$\leq (1+\|z\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha$$
$$\times \int_{\mathbb{R}_+^{n+1}} |\xi_{n+1}|^{l-r+2\alpha+1} \left|D_\xi^{2\delta'+l}\sigma(x,\xi)\right| d\xi.$$

Using the fact that $\sigma \in S^0$, we obtain

$$|K(x,z)| \leq (1+\|z\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha C_{2\delta'+l,0}$$
$$\times \int_{\mathbb{R}_+^{n+1}} |\xi_{n+1}|^{l-r+2\alpha+1} (1+\|x\|^2)^{-q} (1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.$$

For large values of l, the above expression becomes

$$|K(x,z)| \leq (1+\|z\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha C_{2\delta'+l,0}$$
$$\times \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{l-r+2\alpha+1} (1+\|x\|^2)^{-q} (1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.$$

Therefore,

$$|K(x,z)| \leq (1+\|z\|^2)^{-k} (1+\|x\|^2)^{-q} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}$$
$$\times E'_{\alpha,l} A_\alpha C_{2\delta'+l,0} \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-r-2|\delta'|+2\alpha+1} d\xi.$$

It gives

$$|K(x,z)| \leq (1+\|z\|^2)^{-k} (1+\|x\|^2)^{-q} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \binom{k}{r}\binom{r}{j}$$
$$\times E'_{\alpha,l} A_\alpha C_{l,0} \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-r+2\alpha+1} d\xi.$$

Choosing $r > 2\alpha + \frac{n}{2} + \frac{3}{2}$, there exists a constant $C_{\alpha,k}$ such that

$$|K(x,z)| \leq C_{\alpha,k} (1+\|x\|^2)^{-q} (1+\|z\|^2)^{-k}.$$

□

Theorem 3.2. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in S^m$. Then for all $u \in S_*(\mathbb{R}^{n+1})$, the pseudo-differential operator T_σ can be written as*

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} K(x,z)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z)\Big) u(y)d\mu_\alpha(y), \tag{32}$$

in the distributional sense.

Proof. Exploiting the concept of ([2], p. 266), we can see that Schwartz space $S_*(\mathbb{R}^{n+1})$ is invariant under the translation operator τ_x^α, $x \in \mathbb{R}_+^{n+1}$. Then for all $u \in S_*(\mathbb{R}^{n+1})$, we have $\mathcal{F}_\alpha(\tau_x^\alpha u) \in S_*(\mathbb{R}^{n+1})$.
From (17), we have

$$(\tau_x^\alpha u)(z) = \int_{\mathbb{R}_+^{n+1}} u(y)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y).$$

In view of (18), above expression becomes

$$\begin{aligned}(\tau_x^\alpha u)(z) = \int_{\mathbb{R}_+^{n+1}} u(y)\Big(\int_{\mathbb{R}_+^{n+1}} & e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle y',\xi'\rangle}\hat{\jmath}_\alpha(y_{n+1}\xi_{n+1}) \\ & \times e^{-i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1})d\mu_\alpha(\xi)\Big)d\mu_\alpha(y).\end{aligned}$$

Therefore,

$$\begin{aligned}(\tau_x^\alpha u)(z) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1}) \\ &\quad \times \Big(\int_{\mathbb{R}_+^{n+1}} u(y)e^{-i\langle y',\xi'\rangle}\hat{\jmath}_\alpha(y_{n+1}\xi_{n+1})d\mu_\alpha(y)\Big)d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}\hat{\jmath}_\alpha(z_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi).\end{aligned}$$

Invoking (8), we get

$$(\tau_x^\alpha u)(z) = \mathcal{F}_\alpha\Big[e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi)\Big](z).$$

Therefore, we get

$$(\mathcal{F}_\alpha u)(\xi) = \Big[e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})\Big]^{-1}\mathcal{F}_\alpha^{-1}(\tau_x^\alpha u)(\xi) \in S_*(\mathbb{R}^{n+1}). \tag{33}$$

In the distributional sense, the pseudo-differential operator T_σ can be defined as

$$(T_\sigma u)(x) = \Big\langle e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi), (\mathcal{F}_\alpha u)(\xi)\Big\rangle.$$

Invoking (33), above expression becomes

$$\begin{aligned}(T_\sigma u)(x) &= \Big\langle e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi), [e^{i\langle x',\xi'\rangle}\hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})]^{-1}\mathcal{F}_\alpha^{-1}(\tau_x^\alpha u)(\xi)\Big\rangle \\ &= \Big\langle \sigma(x,\xi), \mathcal{F}_\alpha^{-1}(\tau_x^\alpha u)(\xi)\Big\rangle \\ &= \Big\langle \mathcal{F}_\alpha^{-1}\Big(\sigma(x,\xi)\Big)(z), (\tau_x^\alpha u)(z)\Big\rangle.\end{aligned}$$

From (30), we get

$$\begin{aligned}(T_\sigma u)(x) &= \Big\langle K(x,z), (\tau_x^\alpha u)(z)\Big\rangle \\ &= \int_{\mathbb{R}_+^{n+1}} K(x,z)(\tau_x^\alpha u)(z)d\mu_\alpha(z).\end{aligned}$$

Invoking (17), the last expression yields

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} K(x,z)\Big(\int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\alpha(x,y,z)u(y)d\mu_\alpha(y)\Big)d\mu_\alpha(z).$$

□

Theorem 3.3. *Let $\alpha > -\frac{1}{2}$ and $\theta \in C^k(\mathbb{R}_+^{n+1}), k \in \mathbb{N}$. Assume that for $\beta \in \mathbb{N}_0^{n+1}$, there exists a positive constant B such that*

$$\left|D_\xi^\beta \theta(\xi)\right| \leq B(1 + \|\xi\|^2)^{-|\beta|}, \quad |\beta| \leq k. \tag{34}$$

If

$$\psi(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})\theta(\xi)d\mu_\alpha(\xi), \tag{35}$$

then $\psi \in L_\alpha^p(\mathbb{R}_+^{n+1})$, for $1 \leq p < \infty$.

Proof. For $k \in \mathbb{N}$, (35) can be written as

$$\psi(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})(1 + \|x\|^2)^{-k}(1 - \Delta_{\alpha,n})^k\theta(\xi)d\mu_\alpha(\xi).$$

Invoking Binomial Theorem, we get

$$\psi(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{\jmath}_\alpha(x_{n+1}\xi_{n+1})(1 + \|x\|^2)^{-k} \times \Big(\sum_{r=0}^{k} \binom{k}{r}(-1)^r \Delta_{\alpha,n}^r \theta(\xi)\Big)d\mu_\alpha(\xi).$$

Therefore,

$$|\psi(x)| \leq (1 + \|x\|^2)^{-k} \sum_{r=0}^{k} \binom{k}{r} \int_{\mathbb{R}_+^{n+1}} |\Delta_{\alpha,n}^r \theta(\xi)| d\mu_\alpha(\xi).$$

In view of (27), the last expression becomes

$$|\psi(x)| \leq (1 + \|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'| \leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} \times \int_{\mathbb{R}_+^{n+1}} \xi_{n+1}^{l-r} \left|D_\xi^{2\delta'+l}\theta(\xi)\right| d\mu_\alpha(\xi).$$

Thus, we obtain

$$|\psi(x)| \leq (1 + \|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'| \leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha \times \int_{\mathbb{R}_+^{n+1}} |\xi_{n+1}|^{l-r+2\alpha+1} \left|D_\xi^{2\delta'+l}\theta(\xi)\right| d\xi.$$

Using the assumption (34), we obtain

$$|\psi(x)| \leq (1+\|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha B \times \int_{\mathbb{R}_+^{n+1}} |\xi_{n+1}|^{l-r+2\alpha+1}(1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.$$

For large values of l, above yields

$$|\psi(x)| \leq (1+\|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha B \times \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{l-r+2\alpha+1}(1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.$$

Therefore,

$$|\psi(x)| \leq (1+\|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} \times E'_{\alpha,l} A_\alpha B \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-r-2|\delta'|+2\alpha+1} d\xi.$$

It gives

$$|\psi(x)| \leq (1+\|x\|^2)^{-k} \sum_{r=0}^{k} \sum_{j=0}^{r} \sum_{l=1}^{2j} \binom{k}{r}\binom{r}{j} \times E'_{\alpha,l} A_\alpha B \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-r+2\alpha+1} d\xi.$$

Choosing $r > 2\alpha + \frac{n}{2} + \frac{3}{2}$, there exists a constant $B_{\alpha,k}$ such that

$$|\psi(x)| \leq B_{\alpha,k}(1+\|x\|^2)^{-k}.$$

Therefore,

$$\|\psi\|_{L^p_\alpha(\mathbb{R}_+^{n+1})} \leq B_{\alpha,k}\|(1+\|x\|^2)^{-k}\|_{L^p_\alpha(\mathbb{R}_+^{n+1})}.$$

For large values of $k \in \mathbb{N}$, we have $\psi \in L^p_\alpha(\mathbb{R}_+^{n+1})$. □

Lemma 3.4. *Let $\alpha > -\frac{1}{2}$ and $\theta \in C^k(\mathbb{R}_+^{n+1})$ which satisfies (34), then we prove*

$$(\psi *_w u)(x) = \mathcal{F}_\alpha^{-1}[\theta(\xi)(\mathcal{F}_\alpha u)(\xi)](x), \quad u \in S_*(\mathbb{R}^{n+1}) \tag{36}$$

where $\psi(x)$ is given in (35).

Proof. From Theorem 3.3, we have $\psi \in L^p_\alpha(\mathbb{R}_+^{n+1})$, $1 \leq p < \infty$. Then by density of $S_*(\mathbb{R}^{n+1})$ in $L^p_\alpha(\mathbb{R}_+^{n+1})$ and Propostion 2.5, we get $\psi *_w u \in L^p_\alpha(\mathbb{R}_+^{n+1})$, for all $u \in S_*(\mathbb{R}^{n+1})$. Therefore, for a.e. $x \in \mathbb{R}_+^{n+1}$, we can find

$$(\psi *_w u)(x) = \mathcal{F}_\alpha^{-1}[\mathcal{F}_\alpha(\psi *_w u)(\xi)](x).$$

Using (23), above expression becomes

$$(\psi *_w u)(x) = \mathcal{F}_\alpha{}^{-1}[\mathcal{F}_\alpha(\psi)(\xi)\mathcal{F}_\alpha(u)(\xi)](x). \tag{37}$$

We can write (35),

$$\psi(x) = \mathcal{F}_\alpha{}^{-1}\Big(\theta(\xi)\Big)(x).$$

Therefore, from (8) we have

$$\Big(\mathcal{F}_\alpha\psi\Big)(\xi) = \theta(\xi). \tag{38}$$

Hence, from (37) and (38), we obtain

$$(\psi *_w u)(x) = \mathcal{F}_\alpha{}^{-1}[\theta(\xi)(\mathcal{F}_\alpha u)(\xi)](x).$$

□

Theorem 3.5. *Let $\alpha > -\frac{1}{2}$ and $\theta \in C^k(\mathbb{R}^{n+1}_+), k \in \mathbb{N}$ which satisfies (34). Then for $1 \le p < \infty$, there exists a constant $C_{\alpha,k}$ such that*

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le C_{\alpha,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad u \in S_*(\mathbb{R}^{n+1}) \tag{39}$$

where

$$\Big(T_\theta u\Big)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{j}_\alpha(x_{n+1}\xi_{n+1})\theta(\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi). \tag{40}$$

Proof. Using (11), (40) can be expressed as

$$\Big(T_\theta u\Big)(x) = \mathcal{F}_\alpha^{-1}\Big[\theta(\xi)(\mathcal{F}_\alpha u)(\xi)\Big], \quad u \in S_*(\mathbb{R}^{n+1}).$$

From Lemma 3.4, the last expression becomes

$$\Big(T_\theta u\Big)(x) = (\psi *_w u)(x).$$

Invoking the inequality (24), we get

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le \|\psi\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Using the fact that $\psi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, we can find a positive constant $C_{\alpha,k}$ such that

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le C_{\alpha,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

□

Theorem 3.6. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in S^0$. Then for $1 < p < \infty$, the pseudo-differential operator $T_\sigma : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is a bounded linear operator.*

Proof. Let us denote

$$\mathbb{Z}^n \times \mathbb{N}_0 = \{(x_1, x_2, \cdots, x_n, x_{n+1}) : x_j \in \mathbb{Z}, 1 \le j \le n, x_{n+1} \in \mathbb{N}_0\},$$

and $M = (m, m_1)$, for $m \in \mathbb{Z}^n, m_1 \in \mathbb{N}_0$. Then we write $\mathbb{R}^{n+1}_+$ as a union of Q_M with disjoint interiors, i.e.,

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty) = \bigcup_{M\in\mathbb{Z}^n\times\mathbb{N}_0} Q_M,$$

where Q_M is the product of n-dimensional cube with center at m, edges of lenth one, parallel to the coordinate axes and the interval $[m_1, m_1 + 1]$.
Let η be the smooth function defined on $\mathbb{R}_+^{n+1}$ such that

$$\eta(x) = 1, \quad \forall x \in Q_0$$

and

$$\left|D_x^\gamma \eta(x)\right| \leq C_\gamma, \quad \forall \gamma \in \mathbb{N}_0^{n+1} \tag{41}$$

where C_γ is a finite constant depends on γ.
Now, define

$$\sigma_m(x, \xi) = \eta(x - m)\sigma(x, \xi), \quad \forall x, \xi \in \mathbb{R}_+^{n+1}. \tag{42}$$

Then, from (26) we have

$$\left(T_{\sigma_m} u\right)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma_m(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi). \tag{43}$$

Using (42), we get

$$\begin{aligned}\left(T_{\sigma_m} u\right)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \eta(x - m)\sigma(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\ &= \eta(x - m) \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi).\end{aligned}$$

Taking (26) we can get

$$\left(T_{\sigma_m} u\right)(x) = \eta(x - m)\left(T_\sigma u\right)(x). \tag{44}$$

Now, we have

$$\int_{Q_M} \left|\left(T_\sigma u\right)(x)\right|^p d\mu_\alpha(x) \leq \int_{\mathbb{R}_+^{n+1}} \left|\eta(x - m)\left(T_\sigma u\right)(x)\right|^p d\mu_\alpha(x).$$

Therefore, from (44) we find

$$\int_{Q_M} \left|\left(T_\sigma u\right)(x)\right|^p d\mu_\alpha(x) \leq \int_{\mathbb{R}_+^{n+1}} \left|\left(T_{\sigma_m} u\right)(x)\right|^p d\mu_\alpha(x). \tag{45}$$

Since $\sigma_m(x, \xi)$ has compact support in variable x and applying inversion formula of the Weinstein transform (11) we obtain

$$\sigma_m(x, \xi) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \lambda' \rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \left(\mathcal{F}_\alpha \sigma_m\right)(\lambda, \xi) d\mu_\alpha(\lambda), \tag{46}$$

where

$$\left(\mathcal{F}_\alpha \sigma_m\right)(\lambda, \xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \lambda' \rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \sigma_m(x, \xi) d\mu_\alpha(x). \tag{47}$$

Invoking (46) in (43), we get

$$\begin{aligned}\left(T_{\sigma_m} u\right)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \\ &\quad \times \left(\int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \lambda' \rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \left(\mathcal{F}_\alpha \sigma_m\right)(\lambda, \xi) d\mu_\alpha(\lambda) \right) (\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \lambda' \rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \\ &\quad \times \left(\int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \left(\mathcal{F}_\alpha \sigma_m\right)(\lambda, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \right) d\mu_\alpha(\lambda).\end{aligned}$$

Hence,

$$\Big(T_{\sigma_m}u\Big)(x) = \int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(T_\lambda u)(x)d\mu_\alpha(\lambda), \tag{48}$$

where

$$\Big(T_\lambda u\Big)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi). \tag{49}$$

Remaining proof of the theorem requires the following lemma.

Lemma 3.7. *Let $\alpha > -\frac{1}{2}$ and σ_m be defined in (42). Then for $\beta \in \mathbb{N}_0^{n+1}, N \in \mathbb{N}_0$, there exists a positive constant $C_{\alpha,N}$ such that*

$$\Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \le C_{\alpha,N}(1+\|\lambda\|^2)^{-N}(1+\|\xi\|^2)^{-|\beta|}, \quad |\beta| \le k. \tag{50}$$

Proof. Using (47), we have

$$D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})D_\xi^\beta\sigma_m(x,\xi)d\mu_\alpha(x).$$

Invoking (42), we get

$$D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})\eta(x-m)D_\xi^\beta\sigma(x,\xi)d\mu_\alpha(x).$$

Using (13) for $N \in \mathbb{N}_0$, we have

$$\begin{aligned} D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) = \int_{\mathbb{R}_+^{n+1}} & e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(1+\|\lambda\|^2)^{-N} \\ & \times (1-\Delta_{\alpha,n})^N\Big(\eta(x-m)D_\xi^\beta\sigma(x,\xi)\Big)d\mu_\alpha(x). \end{aligned}$$

Invoking Binomial Theorem, we get

$$\begin{aligned} D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) = \int_{\mathbb{R}_+^{n+1}} & e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(1+\|\lambda\|^2)^{-N} \\ & \times \sum_{r=0}^{N}\binom{N}{r}(-1)^r\Delta_{\alpha,n}^r\Big(\eta(x-m)D_\xi^\beta\sigma(x,\xi)\Big)d\mu_\alpha(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \le (1+\|\lambda\|^2)^{-N} & \sum_{r=0}^{N}\binom{N}{r} \\ & \times \int_{\mathbb{R}_+^{n+1}}\Big|\Delta_{\alpha,n}^r\Big(\eta(x-m)D_\xi^\beta\sigma(x,\xi)\Big)\Big|d\mu_\alpha(x). \end{aligned} \tag{51}$$

From ([13], Lemma 2.1), we have

$$\Big|\Delta_{\alpha,n}^r f(x)\Big| \le \sum_{|\gamma|\le r} C_{\alpha,r}\Big|D_x^{2\gamma}f(x)\Big|, \quad \forall f \in C_c^\infty(\mathbb{R}_+^{n+1}). \tag{52}$$

Using (52) we can write (51) in the following way:

$$\Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \leq (1+\|\lambda\|^2)^{-N}\sum_{r=0}^{N}\sum_{|\gamma|\leq r}\binom{N}{r}C_{\alpha,r} \times\int_{\mathbb{R}_+^{n+1}}\Big|D_x^{2\gamma}\Big(\eta(x-m)D_\xi^\beta\sigma(x,\xi)\Big)\Big|d\mu_\alpha(x).$$

By Leibnitz formula, the last expression becomes

$$\Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \leq (1+\|\lambda\|^2)^{-N}\sum_{r=0}^{N}\sum_{|\gamma|\leq r}\sum_{\delta\leq 2\gamma}\binom{N}{r}\binom{2\gamma}{\delta}C_{\alpha,r} \times\int_{\mathbb{R}_+^{n+1}}\Big|D_x^{\delta}\eta(x-m)\Big|\Big|D_x^{2\gamma-\delta}D_\xi^\beta\sigma(x,\xi)\Big|d\mu_\alpha(x).$$

Now, $\sigma\in S^0$ and from (41), we get

$$\Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \leq (1+\|\lambda\|^2)^{-N}\sum_{r=0}^{N}\sum_{|\gamma|\leq r}\sum_{\delta\leq 2\gamma}\binom{N}{r}\binom{2\gamma}{\delta}C_{\alpha,r}C_\delta C_{2\gamma-\delta,\beta} \times\int_{\mathbb{R}_+^{n+1}}(1+\|x\|^2)^{-q}(1+\|\xi\|^2)^{-|\beta|}d\mu_\alpha(x).$$

For each $q\in\mathbb{N}$, there exists a constant $C_{\alpha,N}$ depends only on α and N such that

$$\Big|D_\xi^\beta\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\Big| \leq C_{\alpha,N}(1+\|\lambda\|^2)^{-N}(1+\|\xi\|^2)^{-|\beta|}.$$

□

Hence, from Theorem 3.5 and Lemma 3.7, the operator $u\to T_\lambda u$ defined on $S_*(\mathbb{R}^{n+1})$ by (49) can be extened to a bounded linear operator on $L_\alpha^p(\mathbb{R}_+^{n+1})$.

Now, we proof the following Lemma which is useful in our investigation.

Lemma 3.8. *The operator* (49) *can be expressed in the following form*

$$\Big(T_\lambda u\Big)(x)=\int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}}\mathcal{F}_\alpha{}^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big](z)\mathcal{D}_\alpha(x,y,z)u(y)d\mu_\alpha(y)d\mu_\alpha(z), \tag{53}$$

where

$$\mathcal{F}_\alpha{}^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big](z)=\int_{\mathbb{R}_+^{n+1}}e^{i\langle z',\xi'\rangle}\hat{J}_\alpha(z_{n+1}\xi_{n+1})\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)d\mu_\alpha(\xi), \tag{54}$$

inversion formula is taken with respect to the second variable ξ.

Proof. From (8) and (49), we have

$$\Big(T_\lambda u\Big)(x)=\int_{\mathbb{R}_+^{n+1}}e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) \times\Big(\int_{\mathbb{R}_+^{n+1}}e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})u(y)d\mu_\alpha(y)\Big)d\mu_\alpha(\xi).$$

Therefore,

$$\Big(T_\lambda u\Big)(x) = \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) e^{-i\langle y',\xi'\rangle} \hat{J}_\alpha(y_{n+1}\xi_{n+1}) \times \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi)\Big) u(y) d\mu_\alpha(y).$$

After using (18), we find the following expression

$$\Big(T_\lambda u\Big)(x) = \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1}) \mathcal{D}_\alpha(x,y,z) \times \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi) d\mu_\alpha(z)\Big) u(y) d\mu_\alpha(y).$$

Exploiting Fubini's Theorem, we get

$$\Big(T_\lambda u\Big)(x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1}) \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi)\Big) \times \mathcal{D}_\alpha(x,y,z) u(y) d\mu_\alpha(y) d\mu_\alpha(z).$$

Applying inverse the Weinstein transform (11) with respect to second varibale ξ on aforesaid expression, we yields

$$\Big(T_\lambda u\Big)(x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} \mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big](z) \mathcal{D}_\alpha(x,y,z) u(y) d\mu_\alpha(y) d\mu_\alpha(z),$$

where

$$\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big](z) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1}) \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi).$$

□

Lemma 3.9. *Let $\alpha > -\frac{1}{2}$ and $N \in \mathbb{N}_0$, then $\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big]$ defined in* (54) *satisfies the following inequality*

$$\Big|\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big]\Big| \le B_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}(1+\|x\|^2)^{-k}, \quad \forall k \in \mathbb{N}_0. \tag{55}$$

Proof. From (54), we have

$$\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big] = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi).$$

Taking (14), we find

$$\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big] = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) (1+\|x\|^2)^{-k} \times (1-\Delta_{\alpha,n})^k \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi).$$

From Binomial Theorem, we get

$$\mathcal{F}_\alpha^{-1}\Big[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\Big] = (1+\|x\|^2)^{-k} \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \times \sum_{r=0}^{k} \binom{k}{r} (-1)^r \Delta_{\alpha,n}^r \Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi) d\mu_\alpha(\xi).$$

Therefore,

$$\left|\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right| \leq (1+\|x\|^2)^{-k}\sum_{r=0}^{k}\binom{k}{r}\int_{\mathbb{R}_+^{n+1}}\left|\Delta_{\alpha,n}^{r}\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\right|d\mu_\alpha(\xi).$$

Taking (52), the above expression becomes

$$\left|\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right| \leq (1+\|x\|^2)^{-k}\sum_{r=0}^{k}\sum_{|\gamma|\leq r}\binom{k}{r}C_{\alpha,r}\int_{\mathbb{R}_+^{n+1}}\left|D_\xi^{2\gamma}\Big(\mathcal{F}_\alpha\sigma_m\Big)(\lambda,\xi)\right|d\mu_\alpha(\xi).$$

With help of Lemma 3.7, we find

$$\begin{aligned}\left|\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right| &\leq (1+\|x\|^2)^{-k}\sum_{r=0}^{k}\sum_{|\gamma|\leq r}\binom{k}{r}C_{\alpha,r}C_{\alpha,N}\\ &\quad\times\int_{\mathbb{R}_+^{n+1}}(1+\|\lambda\|^2)^{-N}(1+\|\xi\|^2)^{-2|\gamma|}d\mu_\alpha(\xi)\\ &\leq (1+\|x\|^2)^{-k}(1+\|\lambda\|^2)^{-N}\sum_{r=0}^{k}\sum_{|\gamma|\leq r}\binom{k}{r}C_{\alpha,r}C_{\alpha,N}A_\alpha\\ &\quad\times\int_{\mathbb{R}_+^{n+1}}(1+\|\xi\|^2)^{-2|\gamma|+2\alpha+1}d\xi.\end{aligned}$$

Choosing $|\gamma| > \alpha + \frac{n}{4} + \frac{3}{4}$, there exisits a constant $C_{\alpha,N,k}$ such that

$$\left|\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right| \leq C_{\alpha,N,k}(1+\|x\|^2)^{-k}(1+\|\lambda\|^2)^{-N}.$$

□

Now, we have to prove that

$$\|T_\lambda u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} = \left(\int_{\mathbb{R}_+^{n+1}}|T_\lambda u(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} < \infty.$$

In view of (53), we have

$$\begin{aligned}\|T_\lambda u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} = \Bigg(\int_{\mathbb{R}_+^{n+1}}\Bigg|\int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}}&\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right](z)\mathcal{D}_\alpha(x,y,z)u(y)\\ &\times d\mu_\alpha(y)d\mu_\alpha(z)\Bigg|^p d\mu_\alpha(x)\Bigg)^{\frac{1}{p}}.\end{aligned}$$

Exploiting the Weinstein convolution (22), we get

$$\begin{aligned}\|T_\lambda u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} &= \left(\int_{\mathbb{R}_+^{n+1}}\left|\Big(\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]*_w u\Big)(x)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}\\ &= \left\|\Big(\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]*_w u\Big)(x)\right\|_{L_\alpha^p(\mathbb{R}_+^{n+1})}.\end{aligned}$$

Applying (24), we obtain

$$\|T_\lambda u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} \leq \left\|\mathcal{F}_\alpha{}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right\|_{L_\alpha^1(\mathbb{R}_+^{n+1})}\|u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})}.$$

From (55), the last inequality becomes

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq B_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}\|(1+\|x\|^2)^{-k}\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Choosing large enough $k \in \mathbb{N}$, we get a positive constant $C_{\alpha,N,k}$ such that

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad N \in \mathbb{N}_0. \tag{56}$$

From (48), we have

$$\begin{aligned}\|T_{\sigma_m} u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} &= \Big(\int_{\mathbb{R}^{n+1}_+} |T_{\sigma_m} u(x)|^p d\mu_\alpha(x)\Big)^{\frac{1}{p}} \\ &= \Big(\int_{\mathbb{R}^{n+1}_+}\Big|\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle}\hat{j}_\alpha(x_{n+1}\lambda_{n+1})(T_\lambda u)(x)d\mu_\alpha(\lambda)\Big|^p d\mu_\alpha(x)\Big)^{\frac{1}{p}}.\end{aligned}$$

The last expression becomes after applying Minkowski's inequality

$$\begin{aligned}\|T_{\sigma_m} u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} &\leq \int_{\mathbb{R}^{n+1}_+}\Big(\int_{\mathbb{R}^{n+1}_+}\Big|(T_\lambda u)(x)\Big|^p d\mu_\alpha(x)\Big)^{\frac{1}{p}} d\mu_\alpha(\lambda) \\ &\leq \int_{\mathbb{R}^{n+1}_+} \|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} d\mu_\alpha(\lambda).\end{aligned}$$

In view of (56), we get

$$\|T_{\sigma_m} u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,N,k}\Big(\int_{\mathbb{R}^{n+1}_+}(1+\|\lambda\|^2)^{-N} d\mu_\alpha(\lambda)\Big)\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Choosing N sufficiently large, we get a positive constant $E_{\alpha,N,k}$ such that

$$\|T_{\sigma_m} u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq E_{\alpha,N,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{57}$$

We have, after using (45),

$$\int_{Q_M}\Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) \leq \|T_{\sigma_m} u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Using (57), the last inequality becomes

$$\int_{Q_M}\Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) \leq E^p_{\alpha,N,k}\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{58}$$

Now, let us define Q^{**}_M be the cube same as Q_M and twice edge lenght of Q_M. Also let Q^*_M be the another concentric cube with Q_M and Q^{**}_M satisfying $Q_M \subset Q^*_M \subset Q^{**}_M$.
Let ψ be the smooth function defined on $\mathbb{R}^{n+1}_+$, with compact support and satisfies the following properties:

(i) $0 \leq \psi(x) \leq 1, \quad \forall x \in \mathbb{R}^{n+1}_+$

(ii) $supp(\psi) \subseteq Q^{**}_M$,

(iii) and $\psi(x) = 1$ for all x in a neighbourhood of Q^*_M.

If we write $u = u_1 + u_2$, where $u_1 = \psi u$ and $u_2 = (1-\psi)u$. Then we have

$$\Big(T_\sigma u\Big)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{j}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\Big(\mathcal{F}_\alpha(u_1+u_2)\Big)(\xi)d\mu_\alpha(\xi).$$

Thus, we get

$$\begin{aligned}\Big(T_\sigma u\Big)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\Big(\mathcal{F}_\alpha u_1\Big)(\xi)d\mu_\alpha(\xi)\\ &+ \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\Big(\mathcal{F}_\alpha u_2\Big)(\xi)d\mu_\alpha(\xi)\\ &= \Big(T_\sigma u_1\Big)(x) + \Big(T_\sigma u_2\Big)(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{Q_M} \Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) &= \int_{Q_M} \Big|\Big(T_\sigma u_1\Big)(x) + \Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x)\\ &\le 2^p \int_{Q_M} \Big|\Big(T_\sigma u_1\Big)(x)\Big|^p d\mu_\alpha(x)\\ &\quad + 2^p \int_{Q_M} \Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x).\end{aligned}$$

Taking (58), we get

$$\begin{aligned}\int_{Q_M} \Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) &\le 2^p E^p_{\alpha,N,k} \|u_1\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}\\ &\quad + 2^p \int_{Q_M} \Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x).\end{aligned}$$

Since $u_1 = \psi u$ and $supp(\psi) \subseteq Q_M^{**}$, hence, we obtain

$$\begin{aligned}\int_{Q_M} \Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) &\le 2^p E^p_{\alpha,N,k} \int_{Q_M^{**}} |u(x)|^p d\mu_\alpha(x)\\ &\quad + 2^p \int_{Q_M} \Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x). \qquad (59)\end{aligned}$$

Taking the concept of Theorem 3.2, we have

$$(T_\sigma u_2)(x) = \int_{\mathbb{R}_+^{n+1}} \Big(\int_{\mathbb{R}_+^{n+1}} K(x,y)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y)\Big)u_2(z)d\mu_\alpha(z).$$

Since $u_2(z) = 0$ in a neighbourhood of $z \in Q_M^*$, therefore

$$\Big|(T_\sigma u_2)(x)\Big| \le \int_{\mathbb{R}_+^{n+1}-Q_M^*} \Big(\int_{\mathbb{R}_+^{n+1}} \Big|K(x,y)\Big|\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y)\Big)|u_2(z)|d\mu_\alpha(z).$$

In view of Lemma 3.1, the last inequality gives

$$\begin{aligned}\Big|(T_\sigma u_2)(x)\Big| &\le C_{\alpha,k} \int_{\mathbb{R}_+^{n+1}-Q_M^*} \Big(\int_{\mathbb{R}_+^{n+1}} (1+\|x\|^2)^{-q}(1+\|y\|^2)^{-k}\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y)\Big)\\ &\quad \times |u_2(z)|d\mu_\alpha(z).\end{aligned}$$

Using the fact

$$(1+\|x\|^2)^{-q} \le (1+\|m\|^2)^{-q}, \quad \forall x \in Q_M$$

the last inequality becomes

$$\begin{aligned}\Big|(T_\sigma u_2)(x)\Big| &\leq C_{\alpha,k}(1+\|m\|^2)^{-q}\int_{\mathbb{R}_+^{n+1}}\Big(\int_{\mathbb{R}_+^{n+1}}(1+\|y\|^2)^{-k}\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y)\Big)\\ &\quad\times|u(z)|d\mu_\alpha(z)\\ &\leq C_{\alpha,k}(1+\|m\|^2)^{-q}\Big(h*_w|u|\Big)(x),\end{aligned}$$

where $h(y)=(1+\|y\|^2)^{-k}$.
Therefore,

$$\int_{Q_M}\Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x)\leq C_{\alpha,k}^p(1+\|m\|^2)^{-qp}\int_{Q_M}\Big|\Big(h*_w|u|\Big)(x)\Big|^p d\mu_\alpha(x).$$

We get from (24),

$$\int_{Q_M}\Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x)\leq C_{\alpha,k}^p(1+\|m\|^2)^{-qp}\|h\|^p_{L^1_\alpha(\mathbb{R}_+^{n+1})}\|u\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}.$$

Choosing large values of $k\in\mathbb{N}$, we get a finite constant $C_{\alpha,k,p}$ such that

$$\int_{Q_M}\Big|\Big(T_\sigma u_2\Big)(x)\Big|^p d\mu_\alpha(x)\leq C_{\alpha,k,p}(1+\|m\|^2)^{-qp}\|u\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}. \tag{60}$$

Thus, from (59) and (60), we have

$$\begin{aligned}\int_{Q_M}\Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) &\leq 2^p E^p_{\alpha,N,k}\int_{Q_M^{**}}|u(x)|^p d\mu_\alpha(x)\\ &\quad+2^p C_{\alpha,k,p}(1+\|m\|^2)^{-qp}\|u\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}.\end{aligned}$$

Summing over all $m\in\mathbb{Z}^n\times\mathbb{N}_0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}}\Big|\Big(T_\sigma u\Big)(x)\Big|^p d\mu_\alpha(x) &\leq 2^p E^p_{\alpha,N,k}\int_{\mathbb{R}_+^{n+1}}|u(x)|^p d\mu_\alpha(x)\\ &\quad+2^p C_{\alpha,k,p}\Big(\sum_{m\in\mathbb{Z}^n\times\mathbb{N}_0}(1+\|m\|^2)^{-qp}\Big)\|u\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}\\ &\leq 2^p\Big(E^p_{\alpha,N,k}+C_{\alpha,k,p}\sum_{m\in\mathbb{Z}^n\times\mathbb{N}_0}(1+\|m\|^2)^{-qp}\Big)\|u\|^p_{L^p_\alpha(\mathbb{R}_+^{n+1})}.\end{aligned}$$

For $q>1$ and $1<p<\infty$, we can find a finite constant $C=C(\alpha,k,N,p,q)$ such that

$$\|T_\sigma u\|_{L^p_\alpha(\mathbb{R}_+^{n+1})}\leq C\|u\|_{L^p_\alpha(\mathbb{R}_+^{n+1})},\quad u\in S_*(\mathbb{R}^{n+1}). \tag{61}$$

Since $S_*(\mathbb{R}^{n+1})$ is dense in $L^p_\alpha(\mathbb{R}_+^{n+1})$, therefore from (61), T_σ can be extended to a bounded linear operator on $L^p_\alpha(\mathbb{R}_+^{n+1})$. □

4. Sobolev Spaces

In this section, we introduce Bessel potential and $L^p_\alpha(\mathbb{R}_+^{n+1})$ - type Sobolev space of order r. Using $L^p_\alpha(\mathbb{R}_+^{n+1})$ - boundedness properties we find the various properties and boundedness results in Sobolev type space.

Definition 4.1. $\Big($***Bessel Potential***$\Big)$
Let $r \in \mathbb{R}$ and $\sigma(\xi) = (1+\|\xi\|^2)^{-r/2}$ be a symbol in S^{-r}. Then for $u \in S'_*(\mathbb{R}^{n+1})$, the Bessel potential of order r is defined by

$$(J_{r,\alpha}u)(x) = \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-r/2}(\mathcal{F}_\alpha u)(\xi)\Big](x). \tag{62}$$

Lemma 4.2. *Let $u \in S'_*(\mathbb{R}^{n+1})$, then we have*

(i) $J_{0,\alpha}u = u$,

(ii) $J_{r,\alpha}J_{t,\alpha}u = J_{r+t,\alpha}u$.

Proof.

(i) Taking $r = 0$ in (62), we get

$$(J_{0,\alpha}u)(x) = \mathcal{F}_\alpha{}^{-1}\Big[(\mathcal{F}_\alpha u)(\xi)\Big](x).$$

Since, $u \in S'_*(\mathbb{R}^{n+1})$, therefore we obtain

$$(J_{0,\alpha}u)(x) = u(x).$$

(ii) From the Definition 62, we have

$$\begin{aligned}(J_{r,\alpha}J_{t,\alpha}u)(x) &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-r/2}\Big(\mathcal{F}_\alpha(J_{t,\alpha}u)\Big)(\xi)\Big](x)\\ &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-r/2}(1+\|\xi\|^2)^{-t/2}(\mathcal{F}_\alpha u)(\xi)\Big](x)\\ &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-(r+t)/2}(\mathcal{F}_\alpha u)(\xi)\Big](x)\\ &= (J_{r+t,\alpha}u)(x).\end{aligned}$$

□

Definition 4.3. *For $r \in \mathbb{R}$ and $1 < p < \infty$, then the following space is defined*

$$\mathcal{H}^{r,p}_\alpha = \Big\{u \in S'_*(\mathbb{R}^{n+1}) : J_{-r,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)\Big\}. \tag{63}$$

The space $\mathcal{H}^{r,p}_\alpha$ forms a normed linear space with the following norm

$$\begin{aligned}\|u\|_{\mathcal{H}^{r,p}_\alpha} &= \|J_{-r,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}\\ &= \Big(\int_{\mathbb{R}^{n+1}_+}\Big|(J_{-r,\alpha}u)(\xi)\Big|^p d\mu_\alpha(\xi)\Big)^{1/p}.\end{aligned} \tag{64}$$

We call $\mathcal{H}^{r,p}_\alpha$ the $L^p_\alpha(\mathbb{R}^{n+1}_+)$- Sobolev space of order r.

If $r = 0$ then (63) becomes

$$\mathcal{H}^{0,p}_\alpha = L^p_\alpha(\mathbb{R}^{n+1}_+).$$

Theorem 4.4. *Let $r, t \in \mathbb{R}$ and $1 < p < \infty$, then the Weinstein potential $J_{t,\alpha}$ is an isometry of $\mathcal{H}^{r,p}_\alpha$ onto $\mathcal{H}^{r+t,p}_\alpha$. Moreover,*

$$\|J_{t,\alpha}u\|_{\mathcal{H}^{r+t,p}_\alpha} = \|u\|_{\mathcal{H}^{r,p}_\alpha}, \quad \forall\, u \in \mathcal{H}^{r,p}_\alpha. \tag{65}$$

Proof. Let $u \in \mathcal{H}^{r,p}_\alpha$. Then from (64), we have

$$\|J_{t,\alpha}u\|_{\mathcal{H}^{r+t,p}_\alpha} = \|J_{-r-t,\alpha}J_{t,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Using Lemma 4.2, we find

$$\|J_{t,\alpha}u\|_{\mathcal{H}_\alpha^{r+t,p}} = \|J_{-r,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

By invoking (64), above yields

$$\|J_{t,\alpha}u\|_{\mathcal{H}_\alpha^{r+t,p}} = \|u\|_{\mathcal{H}_\alpha^{r,p}}, \quad \forall\ u \in \mathcal{H}_\alpha^{r,p}.$$

The last expression follows that, $J_{t,\alpha} : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{r+t,p}$ is an isometry.
For each $v \in \mathcal{H}_\alpha^{r+t,p}$, take $u = J_{-t,\alpha}v$. Then, by Lemma 4.2, we get $v = J_{t,\alpha}u$.
Therefore, $u \in \mathcal{H}_\alpha^{r,p}$. Hence, $J_{t,\alpha} : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{r+t,p}$ is an onto isometry. □

Theorem 4.5. *Let $\sigma(x,\xi)$ be a symbol in $S^m, m \in \mathbb{R}$. Then for $r \in \mathbb{R}$ and $1 < p < \infty$, the pseudo-differential operator $T_\sigma : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{r-m,p}$ is a bounded linear operator.*

Proof. First, we consider the following operators:

$$J_{-r,\alpha} : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{0,p},$$

$$T_\sigma J_{m,\alpha} : \mathcal{H}_\alpha^{0,p} \to \mathcal{H}_\alpha^{0,p},$$

and

$$J_{r-m,\alpha} : \mathcal{H}_\alpha^{0,p} \to \mathcal{H}_\alpha^{r-m,p},$$

which are linear. Then, from Theorem 4.4, the operators $J_{-r,\alpha} : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{0,p}$ and $J_{r-m,\alpha} : \mathcal{H}_\alpha^{0,p} \to \mathcal{H}_\alpha^{r-m,p}$ are bounded. Also, by Theorem 3.6, $T_\sigma J_{m,\alpha} : L^p_\alpha \to L^p_\alpha$ is the bounded linear operator. Hence, $T_\sigma : \mathcal{H}_\alpha^{r,p} \to \mathcal{H}_\alpha^{r-m,p}$ is a bounded linear operator. □

Conclusion: Taking concepts of the papers of Fefferman [3], Illner [7], Cato [8], Nagase [12], Hörmander [5], Hwang-Lee [6], Wong [22] and Pathak and Upadhyay [15], authors are able to study the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness results of pseudo-differential operators associated with the Weinstein transform on a certain class of symbol S^0 and used the aforesaid theory on $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev spaces. The importance of the Weinstein transform is in the sense that by utilizing this theory authors got more general results and will be useful for applications of partial-differential equations and other problems related to pseudo-differential operators. The $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators provides the bridge between pseudo-differential operators and maximal - minimal pseudo-differential operators, which are useful in functional analysis, partial-differential equations and other areas of mathematics.

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