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Rough statistical convergence and rough ideal convergence in random 2-normed spaces

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Abstract. This study's main goal is to define approximate statistical convergence in spaces with probabilistic norms. The idea of convergence in random 2-normed space is more generalized as a result of our demonstrations of some fundamental features and examples of convergence in linear spaces with norms. More specifically, we demonstrate the findings for sets of statistical limit points and sets of cluster points of approximate statistically convergent sequences in these spaces. Additionally, we extend the idea of rough convergence by applying the idea of ideals, which automatically expands the original ideas of rough statistical convergence and rough convergence. We define the collection of rough ideal limit points and demonstrate a number of outcomes related to this collection.

1. Introduction

Fast [13] and Steinhaus [37] independently established the idea of statistical convergence for real number sequences in the same year 1951, and other expansions and implementations of this idea have subsequently been studied by numerous writers, including [7, 17, 22]. Kostyrko et al. description's of I-convergence, which is one of its intriguing generalizations, is cited in [19]. Balcerzak et al. [5], who recently explored I-convergence for sequences of functions, are cited in this sentence.

Menger [23] proposed the idea of statistical metric spaces, which was later explored by Schweizer and Sklar [34, 35], giving rise to the theory of probabilistic normed spaces [15]. It offers a vital technique for broadly utilizing the deterministic outcomes of normed linear spaces. It has many other extremely useful applications, such as the study of boundedness [11], convergence of random variables [12], continuity features [1], topological spaces [15], linear operators [16], etc.

Rough convergence is concerned with the approximate numerical resolution of any real-world problem. It aids in confirming the accuracy of computer program solutions and drawing conclusions from scientific research. As an intriguing generalization of the typical convergence for sequences on finite dimensional normed linear spaces, Phu [30] first developed the rough convergence. Later, Phu introduced it on infinite dimensional normed linear spaces [31]. In addition to developing the concept of rough convergence, he made contributions to the closeness and convexity of the rough limit set.

The convexity and proximity of the set of rough statistical limit points and rough cluster points of a sequence were two more factors that Aytar [4] looked into. Maity [24] showed rough statistical convergence

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of order α ($0 < \alpha \leq 1$) in normed linear spaces and discussed several significant findings for the set of rough statistical limit points of order α, both of which were inspired by the work of Aytar. In [25], Maity established the notion of pointwise rough statistical convergence and rough statistical Cauchy sequences for real valued functions. Malik and Maity first established rough convergence for double sequences in normed linear spaces in their article [26]. Later, the authors expanded on this concept and described rough statistical convergence for double sequences in their article [27].

Many researchers have been inspired to employ ideals concepts by this idea. Rough I-convergence was presented by Pal et al., [29], using the ideals of $\mathbb{N}$. Later, in [28], Malik et al. expanded this idea of rough I-convergence to rough I-statistical convergence and discussed certain topological features of the set of all rough I-statistical limits of sequences in normed linear spaces. In various contexts, statistical convergence as well as generalized statistical convergence can be used to further investigate, generalize, and apply the rough convergence; see [2–4, 30], and [31].

We introduce the idea of rough statistical convergence in the random 2-normed linear spaces in this study. Additionally, we use the idea of ideals to broaden the idea of rough convergence.

This essay is structured as follows: We provide some preliminary definitions and findings of random 2-normed spaces in the section that follows. The notions of rough convergence, rough Cauchy sequence, and the set of rough limit points of a sequence are introduced in Section 3, and we created rough convergence criteria for this set in RTN. We later prove that this set is convex and closed. Finally, we look into how rough convergence and rough Cauchy sequence relate to RTN. The idea of rough convergence is expanded upon in this section using the concept of ideals, which logically goes beyond previous understandings of rough convergence and rough statistical convergence. We define the group of rough ideal limit points and show some results that are connected to this group.

2. Definitions, notations and preliminary results

In this section, we will review some fundamental definitions and notations that serve as the foundation for the current work (see [5, 10, 16, 22, 33–37]).

A distribution function is an element of $\mathscr{D}^+$, where

$$\mathscr{D}^+ = \{f : \mathbb{R} \to (0,1); f \text{ is left-continuous, nondecreasing, } f(0) = 0 \text{ and } f(+\infty) = 1\}$$

and the subset $\mathscr{W}^+ \subseteq \mathscr{D}^+$ is the set $\mathscr{W}^+ = \{f \in \mathscr{D}^+ : l^- f(+\infty) = 1\}$ Here $l^- f(+\infty)$ denotes the left limit of the function f at the point x. The space $\mathscr{D}^+$ is partially ordered by the usual pointwise ordering of functions, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, ε_a is a distribution function defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a; \\ 1, & \text{if } x > a. \end{cases}$$

The set $\mathscr{D}$, as well as its subsets, can be partially ordered by the usual pointwise order: in this order, ε_a is the maximal element in $\mathscr{D}^+$.

Definition 2.1. A triangle function is a binary operation on $\mathscr{D}^+$, namely a function $\gamma : \mathscr{D}^+ \times \mathscr{D}^+ \to \mathscr{D}^+$, that is associative, commutative, nondecreasing, and has ε_a as unit; that is, for all $f, g, h \in \mathscr{D}^+$, we have:

(i) $\gamma(\gamma(f,g),h) = \gamma(f,\gamma(g,h))$,
(ii) $\gamma(f,g) = \gamma(g,f)$,
(iii) $\gamma(f,g) \leq \gamma(f,h)$ whenever $g \leq h$,
(iv) $\gamma(f,\varepsilon_a) = f$

Definition 2.2. A t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c, d \in [0,1]$ we have: (i) $*$ is associative and commutative, (ii) $*$ is continuous, (iii) $a * 1 = a$, (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

The concept of a 2-normed space was first introduced by Gähler [9].

Definition 2.3. Let X be a linear space of a dimension m, where $2 \leq m < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ satisfying the following conditions: for every $x, y \in X$, (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$. In this case, $(X, \|.,.\|$ is called a 2-normed space.

Example 2.4. Take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$ = the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \text{ where } x = (x_1, x_2), y = (y_1, y_2).$$

Recently, Golet [10] introduced the notion of a random 2-normed space as follows.

Definition 2.5. Let X be a linear space of a dimension greater than one, γ a triangle function, and $\psi : X \times X \to \mathscr{D}^+$. Then ψ is called a probabilistic 2-norm on X and (X, ψ, γ) a probabilistic 2-normed space if the following conditions are satisfied:

(i) $\psi_{x,y}(t) = \varepsilon_0(t)$ if x and y are linearly dependent, where $\psi_{x,y}(t)$ denotes the value of $\psi_{x,y}$ at $t \in \mathbb{R}$,
(ii) $\psi_{x,y}(t) \neq \varepsilon_0(t)$ if x and y are linearly independent,
(iii) $\psi_{x,y} = \psi_{y,x}$ for every x, y in X,
(iv) $\psi_{\alpha x,y}(t) = \psi_{x,y}\left(\frac{t}{|\alpha|}\right)$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,
(v) $\psi_{x+y,z} \geq \gamma(\psi_{x,z}, \psi_{y,z})$ whenever $x, y, z \in X$.
If (v)is replaced by
(v′) $\psi_{x+y,z}(t_1 + t_2) \geq \psi_{x,z}(t_1) * \psi_{y,z}(t_2)$, for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}^+$,

then triple $(X, \psi, *)$ is called a random 2-normed space (for short, RTN-space).

Remark 2.6. Note that every 2-normed space $(X, \|.,.\|)$ can be made a random 2-normed space in a natural way, by setting $\psi_{x,y}(t) = \varepsilon_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$ and $a * b = \min\{a, b\}$, $a, b \in [0, 1]$.

Definition 2.7. Let $(X, \psi, *)$ be a RTN. A triple sequence $\{x_{mnk}\}$ of element of X is said to be statistically convergent to $\zeta \in X$ with respect to RTN if for any $\epsilon, \lambda \in (0, 1)$ and each $z \in X$, we have $\delta(A(\lambda)) = 0$, where

$$A(\lambda) = \left\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk} - \zeta, z}(t) \leq 1 - \lambda\right\}.$$

In this case, ζ is called the statistical limit of the sequence x.

Definition 2.8. Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. A triple sequence $x = \{x_{mnk}\}$ of element of X is said to be r-convergent to $\zeta \in X$ with respect to RTN, denoted by $x \xrightarrow{r} x$, if for any $\epsilon, \lambda \in (0, 1)$ and each $z \in X$, there exists $N_0 \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that for all $m, n, k \geq N_0$ we have $\psi_{x_{mnk} - \zeta, z}(r + \epsilon) > 1 - \lambda$. In this case ζ is called an r-limit of x.

Remark 2.9. We consider r-limit set x which is denoted by LIM_x^r and is defined by

$$\mathrm{st}_3^\psi - LIM_x^r := \{\zeta \in X : x \xrightarrow{r} x\}.$$

Definition 2.10. Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. A triple sequence $x = \{x_{mnk}\}$ is said to be r-convergent if $\mathrm{st}_3^\psi - LIM_x^r \neq \emptyset$ and r is called a rough convergence degree of x. If $r = 0$, then it is ordinary ψ-convergence of triple sequence.

Definition 2.11. Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. A triple sequence $x = \{x_{mnk}\}$ of element of X is said to be r-statistically convergent to $\zeta \in X$ with respect to RTN, denoted by $st_3^\psi\text{-}\lim x = \zeta$, if for any $\epsilon, \lambda \in (0, 1)$ and each $z \in X$, we have $\delta(A(\lambda)) = 0$, where

$$A(\lambda) = \left\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk} - \zeta, z}(r + \epsilon) > 1 - \lambda\right\}.$$

In this case ζ is called r^ψ-statistical limit of x. If $r = 0$ then it is ordinary ψ-statistical convergent of triple sequence.

3. Rough statistical convergence in RTN

In this section, we introduce the concepts of rough convergence, rough Cauchy sequence, and the set of rough limit points of a sequence, and we produce rough convergence criteria associated with this set in RTN. We later demonstrate that this set is both closed and convex. Finally, we investigate the relationships between rough convergence and rough Cauchy sequences in RTN.

Theorem 3.1. *Let $(X, \psi, *)$ be a RTN. A triple sequence $x = \{x_{mnk}\} \in X$, $r_1 \geq 0$ and $r_2 > 0$. $x = \{x_{mnk}\}$ is $(r_1 + r_2)$-convergent to ζ in X if and only if there exists a sequence $\{y_{mnk}\}$ such that for every $\lambda \in (0,1)$ and each $z \in X$,*

$$y_{mnk} \xrightarrow{r_1^{\psi}} \zeta \text{ and } \psi_{x_{mnk}-y_{mnk},z}(r_2) > 1 - \lambda, m, n, k \in \mathbb{N}. \tag{1}$$

Proof. Suppose that (1) is true. Given $\lambda \in (0,1), \epsilon > 0$, and for all $z \in X$. Choose $\eta \in (0,1)$ so that $(1-\eta) * (1-\eta) > 1 - \lambda$. Then $y_{mnk} \xrightarrow{r_1^{\psi}} \zeta$ means that there exists an n_λ such that

$$\psi_{y_{mnk}-\zeta,z}(r_1 + \epsilon) > 1 - \eta,$$

whenever $m, n, k \geq n_\lambda$. Since $\psi_{x_{mnk}-y_{mnk},z}(r_2) > 1 - \eta$ then, for $m, n, k \geq n_\lambda$, we have

$$\psi_{x_{mnk}-\zeta,z}(r_1 + r_2 + \epsilon) \geq \psi_{x_{mnk}-y_{mnk},z}(r_2) * \psi_{y_{mnk}-\zeta,z}(r_1 + \epsilon) > (1-\eta) * (1-\eta) > 1 - \lambda.$$

Hence, x_{mnk} is $r_1 + r_2$-convergent to ζ.

Now we let x_{mnk} is $r_1 + r_2$-convergent to ζ. We define a sequence $\{y_{mnk}\}$ for each $\lambda \in (0,1)$ and for every $z \in X$ as following:

$$y_{mnk} = \begin{cases} \zeta, & \text{if } \|x_{mnk} - \zeta, z\| \leq r_2; \\ x_{mnk} + r_2 \frac{(\zeta - x_{mnk})}{\|\zeta - x_{mnk}, z\|}, & \text{if } \|x_{mnk} - \zeta, z\| > r_2. \end{cases}$$

Then, we have

$$\psi_{y_{mnk}-\zeta,z}(r_2) = \frac{r_2}{r_2 + \left\|y_{mnk} - \zeta, z\right\|} = \begin{cases} 1, & \text{if } \psi_{x_{mnk}-\zeta,z}(r_2) \geq 1 - \lambda; \\ \frac{r_2}{\|x_{mnk}-\zeta,z\|}, & \text{if } \psi_{x_{mnk}-\zeta,z}(r_2) < 1 - \lambda. \end{cases}$$

and so

$$\psi_{y_{mnk}-\zeta,z}(r_2) \geq 1 - \lambda$$

for every $m, n, k \in \mathbb{N}$ and every $z \in X$. Since $\zeta \in LIM_{\psi}^{r_1+r_2} x$, we have

$$\liminf \psi_{x_{mnk}-\zeta,z}(r_1 + r_2) \geq 1 - \lambda,$$

and so

$$\liminf \psi_{y_{mnk}-\zeta,z}(r_1) \geq 1 - \lambda$$

for every $\lambda \in (0,1)$ and each $z \in X$. Hence we have $y_{mnk} \xrightarrow{r_1^{\psi}} \zeta$. Therefore, the proof is complete. □

Definition 3.2. Let $(X, \psi, *)$ be a RTN. A sequence $\{x_{mnk}\}$ in X is said to be rough statistically bounded with respect to the norm ψ for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists a real number $M > 0$ such that for each $z \in X$,

$$\delta\left(\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk},z}(M) \leq 1 - \lambda\}\right) = 0.$$

We discover the following intriguing findings on rough statistical convergence in RTN in light of the aforementioned criteria.

Theorem 3.3. *Let $(X, \psi, *)$ be a RTN. A triple sequence $x = \{x_{mnk}\}$ in X is statistically bounded in RTN if and only if $st_3^{\psi} - LIM_x^r \neq \emptyset$ for $r > 0$.*

Proof. Let $\{x_{mnk}\}$ be statistically bounded in X. Then, for every $\epsilon > 0, \lambda \in (0,1)$ and some $r > 0$ and each $z \in X$, there exists a real number $M > 0$ such that

$$\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk},z}(M) \leq 1-\lambda\}\right) = 0.$$

Let $K = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk},z}(M) \leq 1-\lambda\}$. For $(m,n,k) \in K^c$ we have $\psi_{x_{mnk},z}(M) > 1-\lambda$. Also

$$\begin{array}{rcl}\psi_{x_{mnk},z}(r+M) & \geq & \psi_{0,z}(r) * \psi_{x_{mnk},z}(M) \\ & > & 1 * (1-\lambda) > 1-\lambda.\end{array}$$

Hence, $0 \in \mathrm{st}_3^{\psi} - LIM_x^r$ and so $\mathrm{st}_3^{\psi} - LIM_x^r \neq \emptyset$.

For the converse, Let $\mathrm{st}_3^{\psi} - LIM_x^r \neq \emptyset$ for some $r > 0$. Then there exists $\zeta \in X$ such that $\zeta \in \mathrm{st}_3^{\psi} - LIM_x^r$ For every $\epsilon > 0, \lambda \in (0,1)$ and each $z \in X$ we have

$$\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \leq 1-\lambda\}\right) = 0.$$

Therefore, almost all elements x are contained in some ball with center ζ which implies that the sequence $x = \{x_{mnk}\}$ is statistically bounded in X. □

Corollary 3.4. *Let $(X,\psi,*)$ be a RTN and $x = \{x_{mnk}\} \in X$ be a triple sequence. If $x' = x'_{mnk}$ is a subsequence of x_{mnk} then,*

$$st_3^{\psi} - LIM_x^r \subseteq st_3^{\psi} - LIM_{x'}^r$$

in $(X,\psi,*)$.

Example 3.5. Let $X = \mathbb{R}^3$. We define the probabilistic norm ψ for $x \in X$, $t \in \mathbb{R}$ as $\psi_{x,z}(t) = \frac{t}{t+\|x,z\|}$ for each $z \in X$. Then $(X,\psi,*)$ be a RTN under the t-norm $*$ which is defined as $x * y = \min\{x,y\}$. Then, define a sequence $x = \{x_{mnk}\}$ as

$$x_{mnk} = \begin{cases} (-1)^{mnk}, & \text{if } (m,n,k) \neq (i^2, j^2, l^2) \ \ (i,j,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}; \\ mnk, & \text{otherwise.}\end{cases}$$

Then

$$\mathrm{st}_3^{\psi} - LIM_x^r = \begin{cases} \emptyset, & \text{if } r < 1; \\ [1-r, r-1], & \text{otherwise.}\end{cases}$$

and $\mathrm{st}_3^{\psi} - LIM_x^r = \emptyset$ for all $r \geq 0$. Thus, this sequence is divergent in ordinary sense as it is unbounded. Also, the sequence is not rough convergent in RTN for any r.

Theorem 3.6. *Let $(X,\psi,*)$ be a RTN and $x = \{x_{mnk}\} \in X$ be a triple sequence. For all $r \geq 0$, the r-limit set $st_3^{\psi} - LIM_x^r$ of an arbitrary sequence $\{x_{mnk}\}$ is closed.*

Proof. If $\mathrm{st}_3^{\psi} - LIM_x^r = \emptyset$, then nothing to prove. Assume that $\mathrm{st}_3^{\psi} - LIM_x^r \neq \emptyset$ for some $r > 0$ and consider $y = \{y_{mnk}\}$ be a convergent sequence in $\mathrm{st}_3^{\psi} - LIM_x^r$ with respect to the norm ψ to $y_0 \in X$. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta)*(1-\eta) > 1-\lambda$. Then for every $\epsilon > 0$ and $\eta \in (0,1)$ and each $z \in X$ there exist $m_1, n_1, k_1 \in \mathbb{N}$ such that

$$\psi_{y_{mnk}-y_0,z}(\epsilon/2) 1-\eta \text{ for all } m \geq m_1, n \geq n_1, k \geq k_1$$

Let us choose $y_{stp} \in \mathrm{st}_3^{\psi} - LIM_x^r$ with $s > m_1, t > n_1, p > k_1$ such that

$$\delta\left(\left\{(m,nk) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{stp},z}(r+\epsilon/2) \leq 1-\eta\right\}\right) = 0. \tag{2}$$

For $(i,j,l) \in \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{stp},z}(r+\epsilon/2) > 1-\eta\}$, we have $\psi_{x_{ijl}-y_{stp},z}(r+\epsilon/2) > 1-\eta$. Hence, we have

$$\begin{array}{rcl}\psi_{x_{ijl}-y_0,z}(r+\epsilon) & \geq & \psi_{x_{ijl}-y_{stp},z}(r+\epsilon/2) * \psi_{y_{stp}-y_0,z}(\epsilon/2) \\ & > & (1-\eta)*(1-\eta) > 1-\lambda.\end{array}$$

Hence $(i,j,l) \in \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-y_0,z}(r+\epsilon) > 1-\lambda\}$. Now we have the following inclusion

$$\begin{aligned}
&\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{stp},z}(r+\epsilon/2) > 1-\eta\}\\
&\subseteq \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-y_0,z}(r+\epsilon) > 1-\lambda\}, i.e,\\
&\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-y_0,z}(r+\epsilon) \le 1-\lambda\}\\
&\subseteq \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{stp},z}(r+\epsilon/2) \le 1-\eta\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-y_0,z}(r+\epsilon) \le 1-\lambda\}\right)\\
&\le \delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{stp},z}(r+\epsilon/2) \le 1-\eta\}\right)
\end{aligned}$$

Using (2), we obtain

$$\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-y_0,z}(r+\epsilon) \le 1-\lambda\}\right) = 0$$

and so $y_0 \in \mathrm{st}_3^{\psi} - LIM_x^r$. □

Theorem 3.7. *Let $(X,\psi,*)$ be a RTN and $x = \{x_{mnk}\} \in X$ be a triple sequence. If $y_0 \in LIM_{\psi}^{r_0} x_{mnk}$ and $y_1 \in LIM_{\psi}^{r_1} x_{mnk}$, then $y_\alpha := \alpha y_0 + (1-\alpha)y_1 \in LIM_{\psi}^{\alpha r_0+(1-\alpha)r_1} x_{mnk}$, for $\alpha \in [0,1]$.*

Proof. Given $\epsilon > 0, \lambda \in (0,1), r_0, r_1 > 0$ and $z \in X$. Choose $\eta \in (0,1)$ so that $(1-\eta)*(1-\eta) > 1-\lambda$. By definition there exist $m(\lambda), n(\lambda)$ and $k(\lambda)$ such that $m > m(\lambda), n > n(\lambda)$ and $k > k(\lambda)$ implies that

$$\psi_{x_{mnk}-y_0,z}\,(r_0+\epsilon) > 1-\eta \text{ and } \psi_{x_{mnk}-y_1,z}\,(r_1+\epsilon) > 1-\eta$$

which implies that

$$\begin{aligned}
\psi_{x_{mnk}-y_\alpha,z}(\alpha(r_0+\epsilon)+(1-\alpha)(r_1+\epsilon)) &\ge \psi_{\alpha(x_{mnk}-y_0),z}(\alpha(r_0+\epsilon)) * \psi_{(1-\alpha)\left(x_{mnk}-y_1\right),z}((1-\alpha)(r_1+\epsilon))\\
&= \psi_{x_{mnk}-y_0,z}\left(\frac{\alpha(r_0+\epsilon)}{\alpha}\right) * \psi_{x_{mnk}-y_1,z}\left(\frac{(1-\alpha)(r_1+\epsilon)}{1-\alpha}\right)\\
&= \psi_{x_{mnk}-y_0,z}(r_0+\epsilon) * \psi_{x_{mnk}-y_1,z}(r_1+\epsilon)\\
&> (1-\eta)*(1-\eta) > 1-\lambda.
\end{aligned}$$

Hence, we have

$$y_\alpha \in LIM_{\psi}^{\alpha r_0+(1-\alpha)r_1} x_{mnk}.$$

□

By setting $r = r_0 = r_1$ in Theorem 3.7, we have

Corollary 3.8. *Let $(X,\psi,*)$ be a RTN and $x = \{x_{mnk}\} \in X$ be a triple sequence. Then $st_3^{\psi} - LIM_x^r$ is convex.*

Theorem 3.9. *Let $(X,\psi,*)$ be a RTN and $x = \{x_{mnk}\}, y = \{y_{mnk}\} \in X$ be triple sequences. If $x \xrightarrow{r^{\psi}} \zeta_1$ and $y \xrightarrow{r^{\psi}} \zeta_2$, then*

(i) $x + y \xrightarrow{r^{\psi}} \zeta_1 + \zeta_2$ and

(ii) $cx \xrightarrow{r^{\psi}} c\zeta_1$ for every $c \in \mathbb{R}$.

Proof. (i) Given $\epsilon > 0, \lambda \in (0,1), r_0, r_1 > 0$ and $z \in X$. Choose $\eta \in (0,1)$ so that $(1-\eta) * (1-\eta) > 1-\lambda$. By definition there exist $m(\lambda), n(\lambda)$ and $k(\lambda)$ such that $m > m(\lambda), n > n(\lambda)$ and $k > k(\lambda)$ implies that $\psi_{x_{mnk}-\zeta_1,z}(r_1+\epsilon) > 1-\eta$. Also there exist $r(\lambda), s(\lambda)$ and $t(\lambda)$ such that $m > r(\lambda), n > s(\lambda)$ and $k > t(\lambda)$ implies that $\psi_{y_{mnk}-\zeta_2,z}(r_2+\epsilon) > 1-\eta$. Let $i = \max\{m(\lambda), r(\lambda)\}$, $j = \max\{n(\lambda), s(\lambda)\}$, $p = \max\{k(\lambda), t(\lambda)\}$ and $r = r_1 + r_2$. For every $m > i, n > j, k > p$ and each $z \in X$, we have

$$\begin{aligned} \psi_{x_{mnk}+y_{mnk}-(\zeta_1+\zeta_2),z}(r+2\epsilon) &\geq \psi_{x_{mnk}-\zeta_1,z}(r_1+\epsilon) * \psi_{y_{mnk}-\zeta_2,z}(r_2+\epsilon) \\ &> (1-\eta) * (1-\eta) > 1-\lambda \end{aligned}$$

and so $\{x_{mnk} + y_{mnk}\} \xrightarrow{r^{\psi}} \zeta_1 + \zeta_2$.

(ii) It is clear for $c = 0$. Let $c \neq 0$ and assume that $x \xrightarrow{r^{\psi}} \zeta_1$. Given $\epsilon > 0, \lambda \in (0,1), r > 0$ and every $z \in X$. By the definition of r^{ψ}-convergence there exist $m(\lambda), n(\lambda)$ and $k(\lambda)$ such that $m > m(\lambda), n > n(\lambda)$ and $k > k(\lambda)$ yields that

$$\psi_{x_{mnk}-\zeta_1,z}\left(\frac{r+\epsilon}{|c|}\right) > 1-\lambda.$$

According to this, for all $m > m(\lambda), n > n(\lambda)$ and $k > k(\lambda)$ and every $z \in X$, we can write

$$\begin{aligned} \psi_{cx_{mnk}-c\zeta,z}(r+\epsilon) &= \psi_{x_{mnk}-\zeta,z}\left(\frac{r+\epsilon}{|c|}\right) \\ &> 1-\lambda. \end{aligned}$$

Therefore, $cx \xrightarrow{r^{\psi}} c\zeta_1$. $\square$

Theorem 3.10. *Let $(X, \psi, *)$ be a RTN. A triple sequence $x = \{x_{mnk}\}$ in X is rough statistically convergent to $\zeta \in X$ with respect to the norm ψ for some non-negative number r if there exists a sequence $y = \{y_{mnk}\}$ in X, which is statistically convergent to $\zeta \in X$ with respect to the norm ψ and for every $\lambda \in (0,1)$ and each $z \in X$ we have $\psi_{x_{mnk}-y_{mnk},z}(r) > 1-\lambda$ for all $m, n, k \in \mathbb{N}$.*

Proof. Let $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$. Suppose that $y_{mnk} \xrightarrow{r^{\psi}} \zeta$ and $\psi_{x_{mnk}-y_{mnk},z}(r) > 1-\lambda$ for all $m, n, k \in \mathbb{N}$. For given $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Define

$$\begin{aligned} A &= \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{y_{mnk}-\zeta,z}(\epsilon) \leq 1-\eta\right\} \text{ and} \\ B &= \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{mnk},z}(r) \leq 1-\eta\right\}. \end{aligned}$$

Obviously, $\delta(A) = 0$ and $\delta(B) = 0$. For $(m,n,k) \in A^c \cap B^c$, we have

$$\begin{aligned} \psi_{x_{mnk}-\zeta,z}(r+\epsilon) &\geq \psi_{y_{mnk}-\zeta,z}(\epsilon) * \psi_{x_{mnk}-y_{mnk},z}(r) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Then $\psi_{x_{mnk}-\zeta,z}(r+\epsilon) > 1-\lambda$ for all $(m,n,k) \in A^c \cap B^c$. This implies that

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \leq 1-\lambda\} \subseteq A \cup B$$

and so

$$\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \leq 1-\lambda\}\right) \leq \delta(A) + \delta(B).$$

Consequently,

$$\delta\left(\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \leq 1-\lambda\}\right) = 0$$

and this implies that $x_{mnk} \xrightarrow{r^{\psi}} \zeta$. $\square$

Theorem 3.11. *Let $(X, \psi, *)$ be a RTN and $x = \{x_{mnk}\}$ be a triple sequence in X. Then there does not exist elements $y, w \in st_3^{\psi} - LIM_x^r$ for some $r > 0$ and every $\lambda \in (0,1)$ and $z \in X$ such that $\psi_{y-w,z}(mr) \leq 1-\lambda$ for $m > 2$.*

Proof. We will use contradiction to prove our result. Assume there exists elements $y, w \in \mathrm{st}_3^{\psi} - LIM_x^r$ such that

$$\psi_{y-w,z}(mr) \leq 1 - \lambda \text{ for } m > 2. \tag{3}$$

As $y, w \in \mathrm{st}_3^{\psi} - LIM_x^r$. For given $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1 - \lambda$. Then for every $\epsilon > 0, \eta \in (0,1)$ and $z \in X$ we have $\delta(E_1) = 0$ and $\delta(E_2) = 0$, where

$$\begin{aligned} E_1 &= \left\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y,z}(r+\epsilon) > 1 - \eta\right\} \text{ and} \\ E_2 &= \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-w,z}(r+\epsilon) > 1 - \eta\}. \end{aligned}$$

For $(m,n,k) \in E_1^c \cap E_2^c$ we have

$$\begin{aligned} \psi_{y-w,z}(2r+2\epsilon) &\geq \psi_{x_{mnk}-y,z}(r+\epsilon) * \psi_{x_{mnk}-w,z}(r+\epsilon) \\ &> (1-\eta) * (1-\eta) > 1 - \lambda. \end{aligned}$$

Hence, $\psi_{y-w,z}(2r+2\epsilon) > 1 - \lambda$. Then (3) ensures that

$$\psi_{y-w,z}(mr) > 1 - \lambda \text{ for } m > 2$$

which is a contradiction. Then there does not exist elements $y, w \in \mathrm{st}_3^{\psi} - LIM_x^r$ for some $r > 0$ and every $\lambda \in (0,1)$ and $z \in X$ such that $\psi_{y-w,z}(mr) \leq 1 - \lambda$ for $m > 2$. □

The statistical cluster point of a sequence in RTN is then defined, and certain results are established in relation to it.

Definition 3.12. Let $(X, \psi, *)$ be a RTN. Then $\mu \in X$ in X is called rough statistical cluster point of the sequence $x = \{x_{mnk}\}$ with respect the norm ψ if for every $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$ and for some $r \geq 0$,

$$\delta\left(\left\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-\mu,z}(r+\epsilon) > 1 - \lambda\right\}\right) > 0,$$

In this case, μ is known as r^{ψ}-cluster point of a sequence $x = \{x_{mnk}\}$.

Let $\mathrm{st}_3^{\psi} - C_x^r$ denotes the set of all r^{ψ}-cluster points of a sequence $x = \{x_{mnk}\}$. If $r = 0$ then we get ordinary statistical cluster point defined by Karakus [17] i.e. $\mathrm{st}_3^{\psi} - C_x^r = \mathrm{st}_3^{\psi} - C_x$.

Theorem 3.13. *Let $(X, \psi, *)$ be a RTN. Then, the set $st_3^{\psi} - C_x^r$ of any sequence $x = \{x_{mnk}\}$ is closed for some non-negative real number r.*

Proof. If $\mathrm{st}_3^{\psi} - C_x^r = \emptyset$, then there is nothing to prove. Assume that $\mathrm{st}_3^{\psi} - C_x^r \neq \emptyset$. Then, take a sequence $\{y_{mnk}\} \subseteq \mathrm{st}_3^{\psi} - C_x^r$ such that $y_{mnk} \xrightarrow{r^{\psi}} \nu$. It is sufficient to show that $\nu \in \mathrm{st}_3^{\psi} - C_x^r$. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1 - \lambda$. As $y_{mnk} \xrightarrow{r^{\psi}} \nu$, then for every $\epsilon > 0, \eta \in (0,1)$ and $z \in X$ there exist $m_\eta, n_\eta, k_\eta \in \mathbb{N}$ such that $\psi_{y_{mnk}-\nu,z}(\epsilon) > 1 - \eta$ for all $m \geq m_\eta, n \geq n_\eta$ and $k \geq k_\eta$.

Now choose $m_0, n_0, k_0 \in \mathbb{N}$ such that $m_0 \geq m_\eta, n_0 \geq n_\eta$ and $k_0 \geq k_\eta$. Then, we have $\psi_{y_{m_0n_0k_0}-\nu,z}(\epsilon) > 1 - \eta$. Again as $y = \{y_{mnk}\} \subseteq \mathrm{st}_3^{\psi} - C_x^r$, we have $y_{m_0n_0k_0} \in \mathrm{st}_3^{\psi} - C_x^r$. Hence,

$$\delta\left(\left\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_{m_0n_0k_0},z}(r+\epsilon) > 1 - \eta\right\}\right) > 0. \tag{4}$$

Choose $(i,j,l) \in \left\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_{m_0n_0k_0},z}(r+\epsilon) > 1 - \eta\right\}$, then we have

$$\psi_{x_{ijl}-y_{m_0n_0k_0},z}(r+\epsilon) > 1 - \eta.$$

Now

$$\begin{aligned}\psi_{x_{ijl}-\nu,z}(r+2\epsilon) &\geq \psi_{x_{ijl}-y_{m_0n_0k_0},z}(r+\epsilon)*\psi_{y_{m_0n_0k_0}-\nu,z}(\epsilon)\\ &> (1-\eta)*(1-\eta)>1-\lambda.\end{aligned}$$

Thus $(i,j,l)\in\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(r+2\epsilon)>1-\lambda\}$. Therefore

$$\begin{aligned}&\left\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-y_{m_0n_0k_0},z}(r+\epsilon)>1-\eta\right\}\\ &\subseteq\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(r+2\epsilon)>1-\lambda\}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\delta\left(\left\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-y_{m_0n_0k_0},z}(r+\epsilon)>1-\eta\right\}\right)\\ &\leq\delta\left(\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(r+2\epsilon)>1-\lambda\}\right).\end{aligned}\tag{5}$$

Using equation (4), we obtain that the set on left side of (5) has natural density more than 0. Therefore,

$$\delta\left(\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(r+2\epsilon)>1-\lambda\}\right)>0$$

and so $\nu\in \mathrm{st}_3^{\psi}-C_x^r$. □

Theorem 3.14. *Let $(X,\psi,*)$ be a RTN. Let $st_3^{\psi}-C_x^r$ be the set of all statistical cluster points of a triple sequence $x=\{x_{mnk}\}\in X$ and r be some non-negative real number. Then, for an arbitrary $\nu\in st_3^{\psi}-C_x^r$ and $\lambda\in(0,1)$ and $z\in X$ we have*

$$\psi_{\zeta-\nu,z}(r)>1-\lambda \text{ for all } \zeta\in st_3^{\psi}-C_x^r.$$

Proof. For $\lambda\in(0,1)$ choose $\eta\in(0,1)$ such that $(1-\eta)*(1-\eta)>1-\lambda$. Let $\nu\in \mathrm{st}_3^{\psi}-C_x$. Then, for every $\epsilon>0,\eta\in(0,1)$ and $z\in X$ we have

$$\delta\left(\{\psi_{x_{mnk}-\nu,z}(\epsilon)>1-\eta\}\right)>0.\tag{6}$$

Now we will show that if for $\zeta\in X$ we have $\psi_{\zeta-\nu,z}(r)>1-\eta$ then $\zeta\in \mathrm{st}_3^{\psi}-C_x^r$. Let $(i,j,l)\in\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(\epsilon)>1-\eta\}$ then $\psi_{x_{ijl}-\nu,z}(\epsilon)>1-\eta$. Now

$$\begin{aligned}\psi_{x_{ijl}-\zeta,z}(r+\epsilon) &\geq \psi_{x_{ijl}-\nu,z}(\epsilon)*\psi_{\nu-\zeta,z}(r)\\ &> (1-\eta)*(1-\eta)>1-\lambda.\end{aligned}$$

We have $\psi_{x_{ijl}-\zeta,z}(r+\epsilon)>1-\lambda$. Thus

$$(i,j,l)\in\left\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{ijl}-\zeta,z}(r+\epsilon)>1-\lambda\right\}.$$

Now the next inclusion holds.

$$\begin{aligned}&\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(\epsilon)>1-\eta\}\\ &\subseteq\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\zeta,z}(r+\epsilon)>1-\lambda\}.\end{aligned}$$

So,

$$\begin{aligned}&\delta\left(\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\nu,z}(\epsilon)>1-\eta\}\right)\\ &\leq\delta\left(\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\zeta,z}(r+\epsilon)>1-\lambda\}\right).\end{aligned}$$

Using equation (6) we get

$$\delta\left(\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\zeta,z}(r+\epsilon)>1-\lambda\}\right)>0$$

and hence $\zeta\in \mathrm{st}_3^{\psi}-C_x^r$. □

Theorem 3.15. *Let $(X, \psi, *)$ be a RTN. If $\overline{B}^r_\lambda(c) = \{x \in X : \psi_{x-c,z}(r) \geq 1-\lambda\}$ represents the closure of open ball $B^r_\lambda(c) = \{x \in X : \psi_{x-c,z}(r) > 1-\lambda\}$ for some $r > 0$, $\lambda \in (0,1)$ and $z \in X$ and fixed $c \in X$ then $st^\psi_3 - C^r_x = \bigcup_{c \in st^\psi_3 - C_x} \overline{B}^r_\lambda(c)$.*

Proof. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Let $\nu \in \bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$ then there exists $c \in \mathrm{st}^\psi_3 - C_x$ for some $r > 0$ and every $\eta \in (0,1)$ and $z \in X$ such that

$$\psi_{c-\nu,z}(r) > 1-\eta.$$

Fix $\epsilon > 0$. Since $c \in \mathrm{st}^\psi_3 - C_x$ then there exists a set $E = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-c,z}(\epsilon) > 1-\eta\}$ with $\delta(E) > 0$. Now, for $(m,n,k) \in E$

$$\begin{aligned} \psi_{x_{mnk}-\nu,z}(r+\epsilon) &\geq \psi_{x_{mnk}-c,z}(\epsilon) * \psi_{c-\nu,z}(r) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

This implies that

$$\delta\left(\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-\nu,z}(r+\epsilon)\}\right) > 0.$$

Hence, $\nu \in \mathrm{st}^\psi_3 - C^r_x$. Therefore, $\bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c) \subseteq \mathrm{st}^\psi_3 - C^r_x$.

For the converse, let $\nu \in \mathrm{st}^\psi_3 - C^r_x$. Then we try to show that $\nu \in \bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$. Suppose that $\nu \notin \bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$, i.e., $\nu \notin \overline{B}^r_\lambda(c)$ for all $c \in \mathrm{st}^\psi_3 - C_x$. Then $\psi_{\nu-c,z}(r) \leq 1-\lambda$ for every $c \in \mathrm{st}^\psi_3 - C_x$ and $z \in X$. By Theorem 3.14 for arbitrary $c \in \mathrm{st}^\psi_3 - C_x$ we have $\psi_{\nu-c,z}(r) > 1-\lambda$ for every $c \in \mathrm{st}^\psi_3 - C_x$ and $z \in X$ which is a contradiction to the assumption. Therefore, $\nu \in \bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$. Consequently, $\mathrm{st}^\psi_3 - C^r_x \subseteq \bigcup_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$. □

Theorem 3.16. *Let $(X, \psi, *)$ be a RTN. Let $x = \{x_{mnk}\}$ be a triple sequence in X. Then for every $\lambda \in (0,1)$, we have*

(i) If $c \in st^\psi_3 - C_x$, then $st^\psi_3 - LIM^r_x \subseteq \overline{B}^r_\lambda(c)$.
(ii) $st^\psi_3 - LIM^r_x = \bigcap_{c \in st^\psi_3 - C_x} \overline{B}^r_\lambda(c) = \{\zeta \in X : st^\psi_3 - C_x \subseteq \overline{B}^r_\lambda(c)\}$.

Proof. (i) Let $\epsilon > 0$. For given $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Consider $\zeta \in \mathrm{st}^\psi_3 - \mathrm{st}^\psi_3 - C^r_x$ and $c \in \mathrm{st}^\psi_3 - C^r_x$. For every $\epsilon > 0, \eta \in (0,1)$ and $z \in X$, define the sets

$$A = \{(m,n,k) \in \mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) > 1-\eta\} \text{ with } \delta(A) = 0,$$

and

$$B = \{(m,n,k) \in \mathbb{N} : \psi_{x_{mnk}-c,z}(\epsilon) > 1-\eta\} \text{ with } \delta(B) \neq 0.$$

Now if $(m,n,k) \in A \cap B$ we have

$$\begin{aligned} \psi_{\zeta-c,z}(r+2\epsilon) &\geq \psi_{x_{mnk}-c,z}(r+\epsilon) * \psi_{x_{mnk}-\zeta,z}(\epsilon) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Consequently, $\zeta \in \overline{B}^r_\lambda(c)$ and so $\mathrm{st}^\psi_3 - LIM^r_x \subseteq \overline{B}^r_\lambda(c)$.

(ii) By previous part we have $\mathrm{st}^\psi_3 - LIM^r_x \subseteq \bigcap_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$.

Assume that $m \in \bigcap_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c)$. Then $\psi_{m-c,z}(r) > 1-\lambda$ for $z \in X$ and for all $c \in \mathrm{st}^\psi_3 - C_x$. This implies that $\mathrm{st}^\psi_3 - C_x \subseteq \overline{B}^r_\lambda(c)$, i.e.,

$$\bigcap_{c \in \mathrm{st}^\psi_3 - C_x} \overline{B}^r_\lambda(c) \subseteq \left\{\zeta \in X : \mathrm{st}^\psi_3 - C_x \subseteq \overline{B}^r_\lambda(c)\right\}.$$

Further, let $m \notin \mathrm{st}^\psi_3 - LIM^r_x$ then for $\epsilon > 0$ and $z \in X$ we have

$$\delta\left(\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-m,z}(r+\epsilon) \leq 1-\lambda\}\right) \neq 0,$$

which implies that the existence of a statistical cluster point c of the sequence $x = \{x_{mnk}$ with $\psi_{x_{mnk}-y,z}(r+\epsilon) \leq 1-\lambda$. Hence $\mathrm{st}_3^{\psi} - C_x \nsubseteq \overline{B}^r_{\lambda}(c)$ and $m \notin \{\zeta \in X : \mathrm{st}_3^{\psi} - C^r_x \subseteq \overline{B}^r_{\lambda}(c)\}$. This implies that $\{\zeta \in X : \mathrm{st}_3^{\psi} - C_x \subseteq \overline{B}^r_{\lambda}(c)\} \subseteq \mathrm{st}_3^{\psi} - LIM^r_x$ and we obtain $\bigcap_{c\in \mathrm{st}_3^{\psi}-C_x} \overline{B}^r_{\lambda}(c) \subseteq \mathrm{st}_3^{\psi} - LIM^r_x$. Consequently,

$$\mathrm{st}_3^{\psi} - LIM^r_x = \bigcap_{c\in \mathrm{st}_3^{\psi}-C_x} \overline{B}^r_{\lambda}(c) = \left\{\zeta \in X : \mathrm{st}_3^{\psi} - C_x \subseteq \overline{B}^r_{\lambda}(c)\right\}.$$

□

Theorem 3.17. *Let $(X, \psi, *)$ be a RTN. Let $x = \{x_{mnk}\}$ be a triple sequence in X which is statistically convergent to $\zeta \in X$ with respect to the norm ψ then there exists $\lambda \in (0, 1)$ such that*

$$st_3^{\psi} - LIM^r_x = \overline{B}^r_{\lambda}(c)$$

for some $r > 0$.

Proof. Let $\epsilon > 0$. For given $\lambda \in (0, 1)$ choose $\eta \in (0, 1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Since $x_{mnk} \xrightarrow{r^{\psi}} \zeta$ then there is a set

$$K = \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-\zeta,z}(\epsilon) \leq 1-\eta\} \text{ with } \delta(K) = 0 \text{ and } z \in X.$$

Consider for $z \in X$, $w \in \overline{B}^r_{\lambda}(c) = \{q \in X : \psi_{q-\zeta,z}(r) \geq 1-\eta\}$. For $(m, n, k) \in K^c$ we have

$$\begin{aligned} \psi_{x_{mnk}-w,z} &\geq \psi_{x_{mnk}-\zeta,z}(\epsilon) * \psi_{w-\zeta,z}(r) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

This implies that $w \in \mathrm{st}_3^{\psi} - LIM^r_x$, that is, $\overline{B}^r_{\lambda}(c) \subseteq \mathrm{st}_3^{\psi} - LIM^r_x$. Also, $\mathrm{st}_3^{\psi} - LIM^r_x \subseteq \overline{B}^r_{\lambda}(c)$. Therefore, $\mathrm{st}_3^{\psi} - LIM^r_x = \overline{B}^r_{\lambda}(c)$. □

Theorem 3.18. *Let $(X, \psi, *)$ be a RTN. Let $x = \{x_{mnk}\}$ be a triple sequence in X which is statistically convergent with respect to the norm ψ then*

$$st_3^{\psi} - C^r_x = st_3^{\psi} - LIM^r_x$$

for some $r > 0$.

Proof. Suppose $x_{mnk} \xrightarrow{r^{\psi}} \zeta$. Then $\mathrm{st}_3^{\psi} - C_x = \{\zeta\}$. By Theorem 3.15 for some $r > 0$ and $\lambda \in (0, 1)$ we have $\mathrm{st}_3^{\psi} - C^r_x = \overline{B}^r_{\lambda}(c)$. Also by Theorem 3.17 we get $\mathrm{st}_3^{\psi} - LIM^r_x = \overline{B}^r_{\lambda}(c)$. Hence, $\mathrm{st}_3^{\psi} - C^r_x = \mathrm{st}_3^{\psi} - LIM^r_x$.

Conversely, let $\mathrm{st}_3^{\psi} - C^r_x = \mathrm{st}_3^{\psi} - LIM^r_x$. By Theorem 3.15 and Theorem 3.16(ii) we have

$$\bigcup_{c\in \mathrm{st}_3^{\psi}-C_x} \overline{B}^r_{\lambda}(c) = \bigcap_{c\in \mathrm{st}_3^{\psi}-C_x} \overline{B}^r_{\lambda}(c).$$

This implies that either $\mathrm{st}_3^{\psi} - C_x = \emptyset$ or $\mathrm{st}_3^{\psi} - C_x$ is a singleton set. Then

$$\mathrm{st}_3^{\psi} - LIM^r_x = \bigcap_{c\in \mathrm{st}_3^{\psi}-C_x} \overline{B}^r_{\lambda}(c) = \overline{B}^r_{\lambda}(\zeta)$$

for some $\zeta \in \mathrm{st}_3^{\psi} - C_x$, and it follows then by Theorem 3.17 that $\mathrm{st}_3^{\psi} - LIM^r_x = \{\zeta\}$. □

Definition 3.19. Let $(X, \psi, *)$ be a RTN and $x = \{x_{mnk}\} \in X$ be triple sequences. $\{x_{mnk}\}$ is said to be a rough Cauchy sequence with roughness degree τ, if

$$\forall \lambda \in (0, 1), \epsilon > 0, \exists i_{\lambda}, j_{\lambda}, k_{\lambda} : m, r \geq i_{\lambda}, n, s \geq j_{\lambda}, k, t \geq p_{\lambda} : \psi_{x_{mnk}-x_{rst},z}(\tau+\epsilon) \geq 1-\lambda$$

is hold for $\tau > 0$, $\zeta \in X$ and every $z \in X$. τ is also called a Cauchy degree of $\{x_{mnk}\}$.

Proposition 3.20. *Let $(X,\psi,*)$ be a RTN and $x=\{x_{mnk}\}\in X$ be triple sequences.*

(i) Assume $\tau'>\tau$. If τ is a Cauchy degree of a given sequence $x=\{x_{mnk}\}\in X$, so τ' is a Cauchy degree of $\{x_{mnk}\}$.

(ii) A sequence $x=\{x_{mnk}\}\in X$ is bounded if and only if there exists a $\tau\geq 0$ such that $\{x_{mnk}\}$ is a τ-Cauchy sequence in X.

Theorem 3.21. *Let $(X,\psi,*)$ be a RTN. If $\{x_{mnk}\}$ is rough convergent in X, i.e., $st_3^{\psi}-LIM_x^r\neq\emptyset$, if and only if $\{x_{mnk}\}$ is a τ-Cauchy sequence for every $\tau\geq 2r$.*

Proof. Let ζ be any point in $\mathrm{st}_3^{\psi}-LIM_x^r$. Then, for all $\epsilon>0,\lambda\in(0,1)$, choose $\eta\in(0,1)$ such that $(1-\eta)*(1-\eta)>1-\lambda$ and so by definition of r^{ψ}-convergent there exist $i_\lambda, j_\lambda, k_\lambda$ such that $m,r\geq i_\lambda, n,s\geq j_\lambda, k,t\geq p_\lambda$ implies that

$$\psi_{x_{mnk}-\zeta,z}(r+\epsilon)>1-\eta \text{ and } \psi_{x_{rst}-\zeta,z}(r+\epsilon)>1-\eta$$

for every $z\in X$. Consequently, for every $m,r\geq i_\lambda, n,s\geq j_\lambda, k,t\geq p_\lambda$ we have

$$\begin{aligned}\psi_{x_{mnk}-x_{rst},z}(2r+2\epsilon) &\geq \psi_{x_{mnk}-\zeta,z}(r+\epsilon)*\psi_{x_{rst}-\zeta,z}(r+\epsilon)\\ &> (1-\eta)*(1-\eta)>1-\lambda\end{aligned}$$

for every $z\in X$. Hence, $\{x_{mnk}\}$ is a τ-Cauchy sequence for every $\tau\geq 2r$. By Proposition 3.20, every $\tau\geq 2r$ is also a Cauchy degree of $\{x_{mnk}\}$.

Let $\{x_{mnk}\}$ be a rough Cauchy sequence in X. Since $\{x_{mnk}\}$ be a rough Cauchy sequence, then it is bounded and consequently it is rough convergent for $\tau>0$ and for every $z\in X$. □

4. Rough ideal convergence

In this section, we extend the notion of rough convergence by the notion of ideals, which naturally extends the notions of rough convergence and rough statistical convergence. We define the group of rough ideal limit points and show some results that are connected to this group. See the references provided by [2, 16, 18, 32] for information on the ideal limit points, ideal cluster points, statistical rough limit points, statistical rough cluster points, and statistical rough points of sequences in a fuzzy 2-normed space and probabilistic normed space, respectively.

Let G be a non empty set. Then a family of sets $I\subset 2^G$ (2^G is the power set of G) is said to be an *ideal on* G if for each $A_1,A_2\in I$ we have $A_1\cup A_2\in I$, and for each $A_1\in I$ and each $A_2\subset A_1$, we have $A_2\in I$. A non empty family of sets $F\subset 2^G$ is said to be *filter on G* if $\emptyset\notin F$, for each $A_1,A_2\in F$ we have $A_1\cap A_2\in F$ and for each $A_1\in F$ and each $A_2\supset A_1$, we have $A_2\in F$. An ideal I on G is called *non-trivial* if $I\neq\emptyset$ and $G\notin I$. It is clear that $I\subset 2^G$ is an non-trivial ideal on G if and only if $F=F(I)=:\{G\setminus K: K\in I\}$ is a filter on G. A non-trivial ideal $I\subset 2^G$ is called an *admissible ideal* if $I\supset\{\{x\}: x\in S\}$. In this paper we consider the case $G=\mathbb{N}$.

Definition 4.1. Let $(X,\psi,*)$ be a RTN and r be a non-negative real number. A triple sequence $x=\{x_{mnk}\}$ of elements of X is said to be rough ideal convergent or rI_3^{ψ}-convergent to ζ with respect to RTN, denoted by $x\xrightarrow{r-I_3^{\psi}}\zeta$, if for any $\epsilon,\lambda\in(0,1)$ and $z\in X$, we have

$$\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\zeta,z}(r+\epsilon)\leq 1-\lambda\}\in I_3.$$

In this case ζ is called $r-I_3^{\psi}$-limit of x and a triple sequence $x=\{x_{mnk}\}$ is called rough I_3^{ψ}-convergent to ζ with r as roughness of degree. If $r=0$ then it is ordinary I_3^{ψ}-convergent.

The same reasons as in [30] and [3] for presenting this idea. For instance assume that the sequence $y=\{y_{mnk}\}$ is I_3-convergent and can not be measured or calculated exactly and one has to do with an approximated (or I-approximated) sequence $x=\{x_{mnk}\}$ satisfying $\psi_{x_{mnk}-y_{mnk},z}(r)\geq 1-\lambda$ for all $m,n,k\in\mathbb{N}$ and every $\lambda\in(0,1)$ and $z\in X$ (or for almost all m,n,k that is $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-y_{mnk},z}(r)<1-\lambda\}\in I_3$). Then I_3-convergence of the sequence x is not assured, but as the inclusion $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{y_{mnk}-\xi,z}(\epsilon)\leq 1-\lambda\}\supseteq\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\psi_{x_{mnk}-\xi,z}(r+\epsilon)\leq 1-\lambda\}$ holds so the sequence $x=\{x_{mnk}\}$ is rI_3-convergent.

Remark 4.2. Let $(X, \psi, *)$ be a RTN and r be a non-negative real number.

(i) Generally, a triple sequence $y = y_{mnk}$ is not I_3^{ψ}-convergent in usual sense and

$$\psi_{x_{mnk}-y_{mnk},z}(r) \geq 1-\lambda$$

for all $(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ or

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{mnk},z}(r) \leq 1-\lambda\} \in I_3$$

for some $r > 0$ and each $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$. Then the triple sequence $x = \{x_{mnk}\}$ is $r - I_3^{\psi}$-convergent.

(ii) It is clear that $r - I_3^{\psi}$-limit of x is not necessarily unique.

(iii) Consider $r - I_3^{\psi}$-limit set of x which is denoted by I_3^{ψ}-$LIM_x^r = \{\zeta \in X : x \xrightarrow{r-I_3^{\psi}} \zeta\}$, then the triple sequence $x = \{x_{mnk}\}$ is said to be $r - I_3^{\psi}$-convergent if I_3^{ψ}-$LIM_x^r \neq \emptyset$ and r is called a rough I_3^{ψ}-convergence degree of x.

Theorem 4.3. *Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. For a triple sequence $x = \{x_{mnk}\}$ we have $diam(I_3^{\psi} - LIM_x^r) \leq 2r$. In general $diam(I_3^{\psi} - LIM_x^r)$ has no smaller bound.*

Proof. Let $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$. Choose $\eta \in (0,1)$ so that $(1-\eta) * (1-\eta) > 1-\lambda$. Assume that $diam(I_3^{\psi} - LIM_x^r) > 2r$. Then there exists $y, w \in I_3^{\psi} - LIM_x^r$ such that $\psi_{w-y,z}(2r) < 1-\lambda$. Put

$$\begin{aligned} A_1 &= \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y,z}(r+\epsilon) \leq 1-\eta\right\} \text{ and} \\ A_2 &= \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-w,z}(r+\epsilon) \leq 1-\eta\} \end{aligned}$$

Then $A_1, A_2 \in I_3$ and hence $M = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus (A_1 \cup A_2) \in F(I_3)$ and so $M \neq \emptyset$. Let $(m,n,k) \in M$. Now

$$\begin{aligned} \psi_{y-w,z}(2r+2\epsilon) &\geq \psi_{x_{mnk}-y,z}(r+\epsilon) * \psi_{x_{mnk}-w,z}(r+\epsilon) \\ &> (1-\eta) * (1-\eta) > 1-\lambda \end{aligned}$$

which is a contradiction. Thus $diam(I_3^{\psi} - LIM_x^r) \leq 2r$.

To prove the converse part, consider a sequence $x = \{x_{mnk}\}$ such that r-I_3^{ψ}-$\lim x = \zeta$. Let $\epsilon > 0, \lambda \in (0,1)$ be given. Choose $\eta \in (0,1)$ so that $(1-\eta) * (1-\eta) > 1-\lambda$. Then for $z \in X$, $A = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(\epsilon) \leq 1-\eta\} \in I_3$. Now for each $\xi \in \overline{B}^r(\zeta) = \{\xi \in X : \psi_{\xi-\zeta,z}(r) \leq 1-\eta\}$ we have

$$\begin{aligned} \psi_{x_{mnk}-\zeta,z}(r+\epsilon) &\geq \psi_{x_{mnk}-\zeta,z}(\epsilon) * \psi_{\zeta-\xi,z}(r) \\ &> (1-\eta) * (1-\eta) > 1-\lambda, \end{aligned}$$

whenever $(m,n,k) \notin A$. Which shows that $\xi \in I_3^{\psi} - LIM_x^r$ and consequently we can write $I_3^{\psi} - LIM_x^r = \overline{B}_{\lambda}^r(\zeta)$. This shows that in general upper bound $2r$ of the diameter of the set $I_3^{\psi} - LIM_x^r = \overline{B}_{\lambda}^r(\zeta)$ can not be decreased anymore. □

In [30] it was established that there exists a non-negative real number r such that LIM_x^r for a bounded sequence x. As LIM_x^r implies $I - LIM_x^r$, this result is also true for $I_3^{\psi} - LIM_x^r$ for any admissible ideal I_3. The converse implication is not generally true. For this we have the following result.

Definition 4.4. Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. A triple sequence $x = \{x_{mnk}\}$ is said to be I_3^{ψ}-bounded if there exists a real number $M > 0$ such that

$$A = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk},z}(M) \leq 1-\lambda\} \in I_3$$

for every $\lambda \in (0,1)$ and $z \in X$.

Theorem 4.5. *Let $(X, \psi, *)$ be a RTN. A triple sequence $x = \{x_{mnk}\}$ is I_3^{ψ}-bounded if and only if there is a non-negative real number r such that I_3^{ψ}-$LIM_x^r \neq \emptyset$.*

Proof. Let $x = \{x_{mnk}\}$ be an $\mathscr{I}_3^{\psi}$-bounded sequence. Then, for every $\epsilon > 0, \lambda \in (0,1)$ and some $r > 0$ and $z \in X$, there exists a real number $M > 0$ such that

$$A = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk},z}(M) \leq 1 - \lambda\} \in I_3.$$

For $(m,n,k) \in A^c$ we have $\psi_{x_{mnk},z}(M) > 1 - \lambda$. Also

$$\begin{aligned} \psi_{x_{mnk},z}(r + M) &\geq \psi_{0,z}(r) * \psi_{x_{mnk},z}(M) \\ &> 1 * (1 - \lambda) = 1 - \lambda. \end{aligned}$$

Hence, $0 \in I_3^{\psi} - LIM_x^r$ and so I_3^{ψ}-$LIM_x^r \neq \emptyset$.

For the converse, Let I_3^{ψ}-$LIM_x^r \neq \emptyset$ for some $r > 0$. Then there exists $\zeta \in X$ such that $\zeta \in I_3^{\psi} - LIM_x^r$. For every $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$ we have

$$\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r + \epsilon) \leq 1 - \lambda\} \in I_3.$$

Therefore, almost all elements x are contained in some ball with center ζ which implies that sequence $x = \{x_{mnk}\}$ is $\mathscr{I}_3^{\psi}$-bounded sequence in X. □

The r-I_3^{ψ}-limit set of a sequence is then given various topological and geometrical features.

Theorem 4.6. *Let $(X, \psi, *)$ be a RTN. The r-I_3^{ψ}-limit set I_3^{ψ}-LIM_x^r of a triple sequence $x = \{x_{mnk}\}$ is a closed set.*

Proof. If I_3^{ψ}-$LIM_x^r = \emptyset$, then nothing to prove. Assume that I_3^{ψ}-$LIM_x^r \neq \emptyset$ for some $r > 0$ and consider $y = \{y_{mnk}\}$ be a convergent sequence in I_3^{ψ}-LIM_x^r with respect to the norm ψ to $y_0 \in X$. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1 - \eta) * (1 - \eta) > 1 - \lambda$. Then for every $\epsilon > 0$ and $\eta \in (0,1)$ and $z \in X$ there exist $m_1, n_1, k_1 \in \mathbb{N}$ such that $\psi_{y_{mnk}-y_0,z}(\epsilon) > 1 - \eta$ for all $m \geq m_1, n \geq n_1$ and $k \geq k_1$. Let $(m_0, n_0, k_0) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $y_{m_0n_0k_0} \in \{y_{mnk}\} \subseteq I_3^{\psi}$-$LIM_x^r$. Therefore, the set $K = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_{m_0n_0k_0},z}(r + \epsilon) \leq 1 - \eta\} \in I_3$. Clearly, $E = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus K \in F(I_3)$ and hence $E \neq \emptyset$. Let $(p,q,l) \in E$ and choose $m_0 > m_1, n_0 > n_1, k_0 > k_1$. Consequently,

$$\begin{aligned} \psi_{x_{pql}-y_0,z}(r + 2\epsilon) &\geq \psi_{x_{pql}-y_{m_0n_0k_0},z}(r + \epsilon) * \psi_{y_{m_0n_0k_0}-y_0,z}(\epsilon) \\ &> (1 - \eta) * (1 - \eta) > 1 - \lambda. \end{aligned}$$

Hence

$$\begin{aligned} E &= \Big\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_{m_0n_0k_0},z}(r + \epsilon) > 1 - \eta\Big\} \\ &\subseteq \Big\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_0,z}(r + 2\epsilon) > 1 - \eta\Big\}. \end{aligned}$$

where $\Big\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_{m_0n_0k_0},z}(r + \epsilon) > 1 - \eta\Big\} \in F(I_3)$. Hence

$$\Big\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_0,z}(r + 2\epsilon) > 1 - \eta\Big\} \in F(I_3)$$

and so $\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk}-y_0,z}(r + 2\epsilon) \leq 1 - \eta\} \in I_3$. Therefore, the proof is complete. □

Theorem 4.7. *Let $(X, \psi, *)$ be a RTN. The r-I_3^{ψ}-limit set I_3^{ψ}-LIM_x^r of a triple sequence $x = \{x_{mnk}\}$ is convex.*

Proof. Let $\epsilon > 0$ and $\lambda \in (0,1)$ be given. Choose choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Let $y_0, y_1 \in I_3^{\psi}\text{-}LIM_x^r$. Then for $z \in X$, put

$$\begin{aligned} Q_1 &= \Big\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_0,z}(r+\epsilon) \le 1-\eta\Big\} \text{ and} \\ Q_2 &= \Big\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_1,z}(r+\epsilon) \le 1-\eta\Big\}. \end{aligned}$$

Then $Q_1 \cup Q_2 \in I_3$ which implies that $E = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus (Q_1 \cup Q_2) \in F(I_3)$ and so $E \neq \emptyset$. Now for all $(m,n,k) \in E$ and each $\alpha \in [0,1]$ we have

$$\begin{aligned} \psi_{x_{mnk}-(\alpha y_0+(1-\alpha)y_1),z}(r+\epsilon) &= \psi_{x_{mnk}-(\alpha y_0+(1-\alpha)y_1),z}(\alpha(r+\epsilon)+(1-\alpha)(r+\epsilon)) \\ &\ge \psi_{\alpha x_{mnk}-\alpha y_0,z}(\alpha(r+\epsilon)) * \psi_{(1-\alpha)x_{mnk}-(1-\alpha)y_0,z}((1-\alpha)(r+\epsilon)) \\ &= \psi_{x_{mnk}-y_0,z}\left(\alpha\frac{r+\epsilon}{\alpha}\right) * \psi_{x_{mnk}-y_0,z}\left((1-\alpha)\frac{r+\epsilon}{1-\alpha}\right) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Hence,

$$\Big\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-(\alpha y_0+(1-\alpha)y_1),z}(r+\epsilon) \le 1-\lambda\Big\} \in I_3$$

which implies that $\alpha y_0 + (1-\alpha)y_1 \in I_3^{\psi}\text{-}LIM_x^r$. Consequently, $I_3^{\psi}\text{-}LIM_x^r$ of the triple sequence $x = \{x_{mnk}\}$ is convex. □

Theorem 4.8. *Let $(X, \psi, *)$ be a RTN and r be a non-negative real number. Then a triple sequence $x = \{x_{mnk}\}$ is r-I_3^{ψ}-convergent to ζ if and only if there exists a sequence $y = \{y = y_{mnk}\}$ such that $I_3^{\psi}\text{-}\lim y = \xi$ and $\psi_{x_{mnk}-y_{mnk},z}(r) > 1-\lambda$ for all $(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and every $\lambda \in (0,1)$ and $z \in X$.*

Proof. Let $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$. Suppose that $y_{mnk} \to r^{\psi}\zeta$ and $\psi_{x_{mnk}-y_{mnk},z}(r) > 1-\lambda$ for all $m,n,k \in \mathbb{N}$. For given $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Define

$$\begin{aligned} A &= \Big\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{y_{mnk}-\zeta,z}(\epsilon) \le 1-\eta\Big\} \text{ and} \\ B &= \Big\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-y_{mnk},z}(r) \le 1-\eta\Big\}. \end{aligned}$$

Then $A \cup B \in I_3$ and hence $M = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus (A \cup B) \in F(I_3)$. For $(m,n,k) \in M$, we have

$$\begin{aligned} \psi_{x_{mnk}-\zeta,z}(r+\epsilon) &\ge \psi_{y_{mnk}-\zeta,z}(\epsilon) * \psi_{x_{mnk}-y_{mnk},z}(r) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Then $\psi_{x_{mnk}-\zeta,z}(r+\epsilon) > 1-\lambda$ for all $(m,n,k) \in M$. This implies that

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \le 1-\lambda\} \subseteq A \cup B$$

and so

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \le 1-\lambda\} \in I_3$$

and this implies that $x_{mnk} \xrightarrow{r-I_3^{\psi}} \zeta$.

Conversely, suppose that the given condition holds. For any $\epsilon > 0, \lambda \in (0,1)$ and $z \in X$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Then the set

$$A = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{y_{mnk}-\zeta,z}(\epsilon) \le 1-\eta\} \in I_3.$$

Observe that

$$\begin{aligned} \psi_{x_{mnk}-\zeta,z}(r+\epsilon) &\ge \psi_{x_{mnk}-y_{mnk},z}(r) * \psi_{y_{mnk}-\zeta,z}(\epsilon) \\ &> (1-\eta) * (1-\eta) > 1-\lambda \end{aligned}$$

for all $(m,n,k) \in M = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus A$. This shows that

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\zeta,z}(r+\epsilon) \leq 1-\lambda\} \subseteq A \in I_3.$$

Consequently, $x_{mnk} \xrightarrow{r-I_3^{\psi}} \zeta$. □

Definition 4.9. Let $(X,\psi,*)$ be a RTN and r be a non-negative real number. A point $\xi \in X$ is called an I_3^{ψ}-limit point of a triple sequence $x = \{x_{mnk}\}$ if there exists a set $K = \{(m_j,n_j,k_j) : j \in \mathbb{N}\} \in F(I_3)$ such that $\lim_{j\to\infty} x_{m_jn_jk_j} = \xi$. The set of all I_3^{ψ}-limit points of the sequence x will be denoted by I_3^{ψ}-LIM_x^r.

Definition 4.10. Let $(X,\psi,*)$ be a RTN and r be a non-negative real number. $\xi \in X$ is called an I_3^{ψ}-cluster point of a triple sequence $x = \{x_{mnk}\}$ if for any $\epsilon > 0$, $\lambda \in (0,1)$ and $z \in X$, the set $\{(m,n,k) : \psi_{x_{mnk}-\xi,z}(r+\epsilon) > 1-\lambda\} \notin I_3$. The set of all I_3^{ψ}-cluster points of x will be denoted by $I_3^{\psi} - C_x^r$.

Theorem 4.11. *Let $(X,\psi,*)$ be a RTN and r be a non-negative real number. For an arbitrary $\nu \in I_3^{\psi} - C_x^r$, where $x = \{x_{mnk}\}$ is a triple sequence in X, we have $\psi_{\nu-\xi,z}(r) \geq 1-\lambda$ for all $\xi \in I_3^{\psi}$-LIM_x^r and every $\lambda \in (0,1)$ and $z \in X$.*

Proof. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta)*(1-\eta) > 1-\lambda$. Let $\nu \in I_3^{\psi} - C_x^r$. Then, for every $\epsilon > 0, \eta \in (0,1)$ and $z \in X$ we have

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\nu,z}(\epsilon) > 1-\eta\} \notin I_3. \tag{7}$$

Now we will show that if for $\xi \in X$ we have $\psi_{\xi-\nu,z}(r) > 1-\eta$ then $\xi \in I_3^{\psi} - C_x^r$. Let $(i,j,l) \in \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\nu,z}(\epsilon) > 1-\eta\}$ then $\psi_{x_{ijl}-\nu,z}(\epsilon) > 1-\eta$. Now

$$\begin{aligned} \psi_{x_{ijl}-\xi,z}(r+\epsilon) &\geq \psi_{x_{ijl}-\nu,z}(\epsilon) * \psi_{\nu-\xi,z}(r) \\ &> (1-\eta)*(1-\eta) > 1-\lambda. \end{aligned}$$

We have $\psi_{x_{ijl}-\xi,z}(r+\epsilon) > 1-\lambda$. Thus

$$(i,j,l) \in \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{ijl}-\xi,z}(r+\epsilon) > 1-\lambda\right\}.$$

Now the next inclusion holds.

$$\begin{aligned} &\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\nu,z}(\epsilon) > 1-\eta\} \\ &\subseteq \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\xi,z}(r+\epsilon) > 1-\lambda\}. \end{aligned}$$

Using equation (7) we get

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{x_{mnk}-\xi,z}(r+\epsilon) > 1-\lambda\} \notin I_3$$

and hence $\xi \in I_3^{\psi} - C_x^r$. □

Theorem 4.12. *Let $(X,\psi,*)$ be a RTN, r be a non-negative real number and $\lambda \in (0,1)$. A triple sequence $x = \{x_{mnk}\}$ is I_3^{ψ}-convergent to ξ if and only if I_3^{ψ}-$LIM_x^r = \overline{B}_{\lambda}^r(\xi)$.*

Proof. In Theorem 4.3 we have already proved the necessity part. For the sufficiency, let I_3^{ψ}-$LIM_x^r = \overline{B}_{\lambda}^r(\xi) \neq \emptyset$. Thus the sequence $x = \{x_{mnk}\}$ is I_3^{ψ}-bounded. Suppose that x has another I_3^{ψ}-cluster point ζ different from ξ. The point $\bar{\xi} = \xi + \frac{r}{\|\xi-\zeta,z\|}(\xi-\zeta)$ satisfies $\psi_{\bar{\xi}-\xi,z}(r) < 1-\lambda$ for every $\lambda \in (0,1)$ and $z \in X$. Since $\zeta \in I_3^{\psi} - C_x^r$, it follows by Theorem 4.11 that $\zeta \notin I_3^{\psi}$-LIM_x^r. But this is impossible as $\psi_{\bar{\xi}-\xi,z}(r) > 1-\lambda$ and I_3^{ψ}-$LIM_x^r = \overline{B}_{\lambda}^r(\xi)$. Since ξ is the unique I_3^{ψ}-cluster point of x, it follows that x is I_3^{ψ}-convergent. □

Theorem 4.13. *Let $(X, \psi, *)$ be a RTN. Let $x = \{x_{mnk}\}$ be a triple sequence in X. Then for every $\lambda \in (0, 1)$, we have*

(i) If $c \in I_3^{\psi} - C_x$, then $I_3^{\psi} - LIM_x^r \subseteq \overline{B}_{\lambda}^r(c)$.

(ii) $I_3^{\psi} - LIM_x^r = \bigcap_{c \in I_3^{\psi} - C_x} \overline{B}_{\lambda}^r(c) = \{\zeta \in X : I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)\}$.

Proof. (i) Let $\epsilon > 0$. For given $\lambda \in (0, 1)$ choose $\eta \in (0, 1)$ such that $(1 - \eta) * (1 - \eta) > 1 - \lambda$. Consider $\zeta \in I_3^{\psi} - C_x^r$ and $c \in I_3^{\psi} - C_x$. For every $\epsilon > 0, \eta \in (0, 1)$ and $z \in X$, define the sets

$$A = \{(m, n, k) \in \mathbb{N} : \psi_{x_{mnk} - \zeta, z}(r + \epsilon) > 1 - \eta\} \notin I_3,$$

and

$$B = \{(m, n, k) \in \mathbb{N} : \psi_{x_{mnk} - c, z}(\epsilon) > 1 - \eta\} \notin I_3.$$

Now if $(m, n, k) \in A \cap B$ we have

$$\begin{aligned} \psi_{\zeta - c, z}(r + 2\epsilon) &\geq \psi_{x_{mnk} - c, z}(r + \epsilon) * \psi_{x_{mnk} - \zeta, z}(\epsilon) \\ &> (1 - \eta) * (1 - \eta) > 1 - \lambda. \end{aligned}$$

Consequently, $\zeta \in \overline{B}_{\lambda}^r(c)$ and so $I_3^{\psi} - LIM_x^r \subseteq \overline{B}_{\lambda}^r(c)$.

(ii) By previous part we have $I_3^{\psi} - LIM_x^r \subseteq \bigcap_{c \in I_3^{\psi} - C_x} \overline{B}_{\lambda}^r(c)$.

Assume that $m \in \bigcap_{c \in C_{\psi} x} \overline{B}_{\lambda}^r(c)$. Then $\psi_{m - c, z}(r) > 1 - \lambda$ for $z \in X$ and for all $c \in I_3^{\psi} - C_x$. This implies that $I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)$, i.e.,

$$\bigcap_{c \in I_3^{\psi} - C_x} \overline{B}_{\lambda}^r(c) \subseteq \left\{\zeta \in X : I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)\right\}.$$

Further, let $m \notin I_3^{\psi} - LIM_x^r$ then for $\epsilon > 0$ and $z \in X$ we have

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk} - m, z}(r + \epsilon) \leq 1 - \lambda\} \notin I_3,$$

which implies that the existence of a I_3^{ψ}-cluster point c of the sequence $x = \{x_{mnk}\}$ with $\psi_{x_{mnk} - y, z}(r + \epsilon) \leq 1 - \lambda$. Hence $I_3^{\psi} - C_x \not\subseteq \overline{B}_{\lambda}^r(c)$ and $m \notin \{\zeta \in X : I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)\}$. This implies that $\{\zeta \in X : I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)\} \subseteq I_3^{\psi} - LIM_x^r$ and we obtain $\bigcap_{c \in I_3^{\psi} - C_x} \overline{B}_{\lambda}^r(c) \subseteq I_3^{\psi} - LIM_x^r$. Consequently,

$$I_3^{\psi} - LIM_x^r = \bigcap_{c \in I_3^{\psi} - C_x} \overline{B}_{\lambda}^r(c) = \left\{\zeta \in X : I_3^{\psi} - C_x \subseteq \overline{B}_{\lambda}^r(c)\right\}.$$

□

Theorem 4.14. *Let $(X, \psi, *)$ be a RTN, r be a non-negative real number. Let $x = \{x_{mnk}\}$ be an I_3^{ψ}-bounded sequence. If $r = diam(I_3^{\psi} - C_x^r)$, then we have $I_3^{\psi} - C_x^r \subseteq I_3^{\psi} - LIM_x^r$.*

Proof. Let $c \notin I_3^{\psi} - LIM_x^r$. Then there exists an $\epsilon' > 0$ such that the set

$$K = \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{x_{mnk} - c, z}(r + \epsilon') \leq 1 - \lambda\} \notin I_3.$$

Since the sequence is I_3^{ψ}-bounded, there exists an I_3^{ψ}-cluster point c' such that $\psi_{c - c', z}(r + \epsilon'/2) < 1 - \lambda$. Consequently $c \notin I_3^{\psi} - C_x^r$ and the result follows. □

5. Conclusion

This work develops and investigates the notion of rough statistical convergence as well as ideals to widen the notion of rough convergence for triple sequences in the setting of random 2-normed space. Also, some fascinating results and connections between these ideas were shown. Definitions of I_3^*-rough convergence and I_3^*-rough-Cauchy of triple sequences in probabilistic 2-metric spaces will be given in further work, which will also look at some connections with the concepts discussed in this study. On the other hand, the lacunary sequence can benefit from these concepts.

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