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Dynamical behavior of solution of fifteenth-order rational difference equation

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Abstract. Discrete-time systems are sometimes used to explain natural phenomena that happen in non-linear sciences. We study the periodicity, boundedness, oscillation, stability, and certain exact solutions of nonlinear difference equations of generalized order in this paper. Using the standard iteration method, exact solutions are obtained. Some well-known theorems are used to test the stability of the equilibrium points. Some numerical examples are also provided to confirm the theoretical work's validity. The numerical component is implemented with Wolfram Mathematica. The method presented may be simply applied to other rational recursive issues.

In this research, we examine the qualitative behavior of rational recursive sequences provided that the initial conditions are arbitrary real numbers. We examine the behavior of solutions on graphs according to the state of their initial value

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{\pm x_{n-5}x_{n-11} \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}.$$

1. Introduction

Differential equations are often used to describe some natural phenomena when the time is continuous. However, some real life problems can be simply investigated using discrete-time equations [13, 26, 27, 29, 36]. Differential equations occur naturally in many nonlinear sciences, including ecology and economics. In such cases, the state of a phenomenon at a specific point in time completely predicts its state after a year. Dynamical systems theory is useful in discussing the behavior of some models without solving them. Most natural phenomena are studied using difference equations. Some researchers studied biological, economic, statistical, engineering, electrical, mechanical, thermal, physical, and nonlinear science problems using recursive equations [7, 8]. For example, difference equations have been used to investigate the size of a population, the drug in the blood system, and information transmission [14]. Furthermore, the development of digital devices has strongly influenced the use of recursive equations as approximations for ordinary and partial differential equations [28]. We get a difference equation when we discretize a specified

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differential equation. Euler's scheme, for example, can be obtained by discretizing a first order differential equation. Various scholars have extensively studied the qualitative behaviour of some recursive equations. Scientists have especially explored the stability, periodicity, bifurcation, boundedness, and solutions of several recursive equations. Some published papers are presented in this literature review. Almatrafi [3] found precise solutions to sixth order recursive equations. Under particular conditions, the author analyzed the stability of the critical points and the periodicity of the solutions. In [15], author examined the stability, some analytic solutions for a sixth order difference problem. Furthermore, Elsayed et al. [16], analyzed a twelfth order nonlinear difference equation of the periodicity, stability and solutions. More discussions about difference equations can be seen in Refs. [1–37].

This article discusses the solutions, stability and periodicity of the difference equations,

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{\pm x_{n-5}x_{n-11} \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \tag{1}$$

where the initial values are arbitrary positive real numbers. We generate precise solutions through classical iteration. Figures are produced to demonstrate numerical solutions.

Here, we display some basic definitions and some theorems which will be beneficial in our research. Let I be some interval of real numbers and let $f : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [25].

Definition 1.1. (Equilibrium Point)
A point $\overline{x} \in I$ is called an equilibrium point of (2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of (2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2. (Stability)

(a) The equilibrium point $\overline{x}$ of (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq k.$$

(b) The equilibrium point $\overline{x}$ of (2) is called locally asymptotically stable if $\overline{x}$ is a locally stable solution of (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) The equilibrium point $\overline{x}$ of (2) is called a global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) The equilibrium point $\overline{x}$ of (2) is called a global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of (2).

(e) The equilibrium point $\overline{x}$ of (2) is called unstable if $\overline{x}$ is not locally stable.

The linearized equation of (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 1.3. ([23]) *Assume that $p, q \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n \in \mathbb{N}_0.$$

Remark 1.4. Theorem 1.3 can be easily extended to general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n \in \mathbb{N}_0, \tag{3}$$

where, $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \mathbb{N}$. Then (3) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

2. First case

In this section, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers.

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11} + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \tag{4}$$

where,

$$x_{-14} = T, \quad x_{-13} = S, \quad x_{-12} = R, \quad x_{-11} = P, \quad x_{-10} = M, \quad x_{-9} = L, \quad x_{-8} = J, \quad x_{-7} = H, \quad x_{-6} = G, \quad x_{-5} = F, \quad x_{-4} = E, \quad x_{-3} = D, \quad x_{-2} = C, \quad x_{-1} = B, \quad x_0 = A. \tag{5}$$

Theorem 2.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of (4). Then,*

$$x_{18n+1} = \frac{CJT \prod_{i=0}^{n-1}(1 + (6i + 5)CJT)}{FP \prod_{i=0}^{n}(1 + (6i + 1)CJT)}, \qquad x_{18n+2} = \frac{BHS \prod_{i=0}^{n-1}(1 + (6i + 5)BHS)}{EM \prod_{i=0}^{n}(1 + (6i + 1)BHS)},$$

$$x_{18n+3} = \frac{AGR \prod_{i=0}^{n-1}(1 + (6i + 5)AGR)}{DL \prod_{i=0}^{n}(1 + (6i + 1)AGR)}, \qquad x_{18n+4} = \frac{T \prod_{i=0}^{n-1}(1 + (6i + 6)CJT)}{\prod_{i=0}^{n}(1 + (6i + 2)CJT)},$$

$$x_{18n+5} = \frac{S \prod_{i=0}^{n-1}(1 + (6i + 6)BHS)}{\prod_{i=0}^{n}(1 + (6i + 2)BHS)}, \qquad x_{18n+6} = \frac{R \prod_{i=0}^{n-1}(1 + (6i + 6)AGR)}{\prod_{i=0}^{n}(1 + (6i + 2)AGR)},$$

$$x_{18n+7} = \frac{P \prod_{i=0}^{n}(1 + (6i + 1)CJT)}{\prod_{i=0}^{n}(1 + (6i + 3)CJT)}, \qquad x_{18n+8} = \frac{M \prod_{i=0}^{n}(1 + (6i + 1)BHS)}{\prod_{i=0}^{n}(1 + (6i + 3)BHS)},$$

$$x_{18n+9} = \frac{L \prod_{i=0}^{n}(1 + (6i + 1)AGR)}{\prod_{i=0}^{n}(1 + (6i + 3)AGR)}, \qquad x_{18n+10} = \frac{J \prod_{i=0}^{n}(1 + (6i + 2)CJT)}{\prod_{i=0}^{n}(1 + (6i + 4)CJT)},$$

$$x_{18n+11} = \frac{H \prod_{i=0}^{n}(1 + (6i + 2)BHS)}{\prod_{i=0}^{n}(1 + (6i + 4)BHS)}, \qquad x_{18n+12} = \frac{G \prod_{i=0}^{n}(1 + (6i + 2)AGR)}{\prod_{i=0}^{n}(1 + (6i + 4)AGR)},$$

$$x_{18n+13} = \frac{F\prod_{i=0}^{n}(1+(6i+3)CJT)}{\prod_{i=0}^{n}(1+(6i+5)CJT)}, \qquad x_{18n+14} = \frac{E\prod_{i=0}^{n}(1+(6i+3)BHS)}{\prod_{i=0}^{n}(1+(6i+5)BHS)},$$
$$x_{18n+15} = \frac{D\prod_{i=0}^{n}(1+(6i+3)AGR)}{\prod_{i=0}^{n}(1+(6i+5)AGR)}, \qquad x_{18n+16} = \frac{C\prod_{i=0}^{n}(1+(6i+4)CJT)}{\prod_{i=0}^{n}(1+(6i+6)CJT)},$$
$$x_{18n+17} = \frac{B\prod_{i=0}^{n}(1+(6i+4)BHS)}{\prod_{i=0}^{n}(1+(6i+6)BHS)}, \qquad x_{18n+18} = \frac{A\prod_{i=0}^{n}(1+(6i+4)AGR)}{\prod_{i=0}^{n}(1+(6i+6)AGR)}.$$

where, $x_0, \ldots, x_{-14}$ defines as in (5).

Proof. The proof of each formula are carried out in similar way. So, we will demonstrate proof using one of the formula. We will employ the mathematical induction method. For $n = 0$ results holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{18n-17} = \frac{CJT\prod_{i=0}^{n-2}(1+(6i+5)CJT)}{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)}, \qquad x_{18n-16} = \frac{BHS\prod_{i=0}^{n-2}(1+(6i+5)BHS)}{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)},$$
$$x_{18n-15} = \frac{AGR\prod_{i=0}^{n-2}(1+(6i+5)AGR)}{DL\prod_{i=0}^{n-1}(1+(6i+1)AGR)}, \qquad x_{18n-14} = \frac{T\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-1}(1+(6i+2)CJT)},$$
$$x_{18n-13} = \frac{S\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-1}(1+(6i+2)BHS)}, \qquad x_{18n-12} = \frac{R\prod_{i=0}^{n-2}(1+(6i+6)AGR)}{\prod_{i=0}^{n-1}(1+(6i+2)AGR)},$$
$$x_{18n-11} = \frac{P\prod_{i=0}^{n-1}(1+(6i+1)CJT)}{\prod_{i=0}^{n-1}(1+(6i+3)CJT)}, \qquad x_{18n-10} = \frac{M\prod_{i=0}^{n-1}(1+(6i+1)BHS)}{\prod_{i=0}^{n-1}(1+(6i+3)BHS)},$$
$$x_{18n-9} = \frac{L\prod_{i=0}^{n-1}(1+(6i+1)AGR)}{\prod_{i=0}^{n-1}(1+(6i+3)AGR)}, \qquad x_{18n-8} = \frac{J\prod_{i=0}^{n-1}(1+(6i+2)CJT)}{\prod_{i=0}^{n-1}(1+(6i+4)CJT)},$$
$$x_{18n-7} = \frac{H\prod_{i=0}^{n-1}(1+(6i+2)BHS)}{\prod_{i=0}^{n-1}(1+(6i+4)BHS)}, \qquad x_{18n-6} = \frac{G\prod_{i=0}^{n-1}-1(1+(6i+2)AGR)}{\prod_{i=0}^{n-1}(1+(6i+4)AGR)},$$
$$x_{18n-5} = \frac{F\prod_{i=0}^{n-1}(1+(6i+3)CJT)}{\prod_{i=0}^{n-1}(1+(6i+5)CJT)}, \qquad x_{18n-4} = \frac{E\prod_{i=0}^{n-1}(1+(6i+3)BHS)}{\prod_{i=0}^{n-1}(1+(6i+5)BHS)},$$
$$x_{18n-3} = \frac{D\prod_{i=0}^{n-1}(1+(6i+3)AGR)}{\prod_{i=0}^{n-1}(1+(6i+5)AGR)}, \qquad x_{18n-2} = \frac{C\prod_{i=0}^{n-1}(1+(6i+4)CJT)}{\prod_{i=0}^{n-1}(1+(6i+6)CJT)},$$
$$x_{18n-1} = \frac{B\prod_{i=0}^{n-1}(1+(6i+4)BHS)}{\prod_{i=0}^{n-1}(1+(6i+6)BHS)}, \qquad x_{18n} = \frac{A\prod_{i=0}^{n-1}(1+(6i+4)AGR)}{\prod_{i=0}^{n-1}(1+(6i+6)AGR)}.$$

Now, using the main (4), one has

$$\begin{aligned}
x_{18n+1} &= \frac{x_{18n-2}x_{18n-8}x_{18n-14}}{(x_{18n-5}x_{18n-11})(1+x_{18n-2}x_{18n-8}x_{18n-14})} \\
&= \frac{\frac{C\prod_{i=0}^{n-1}(1+(6i+4)CJT)}{\prod_{i=0}^{n-1}(1+(6i+6)CJT)}\frac{J\prod_{i=0}^{n-1}(1+(6i+2)CJT)}{\prod_{i=0}^{n-1}(1+(6i+4)CJT)}\frac{T\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-1}(1+(6i+2)CJT)}}{\left(\frac{F\prod_{i=0}^{n-1}(1+(6i+3)CJT)}{\prod_{i=0}^{n-1}(1+(6i+5)CJT)}\frac{P\prod_{i=0}^{n-1}(1+(6i+1)CJT)}{\prod_{i=0}^{n-1}(1+(6i+3)CJT)}\right)\left(1+\frac{C\prod_{i=0}^{n-1}(1+(6i+4)CJT)}{\prod_{i=0}^{n-1}(1+(6i+6)CJT)}\frac{J\prod_{i=0}^{n-1}(1+(6i+2)CJT)}{\prod_{i=0}^{n-1}(1+(6i+4)CJT)}\frac{T\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-1}(1+(6i+2)CJT)}\right)} \\
&= \frac{\frac{CJT\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-1}(1+(6i+6)CJT)}}{\left(\frac{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)}{\prod_{i=0}^{n-1}(1+(6i+5)CJT)}\right)\left(1+\frac{CJT\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-1}(1+(6i+6)CJT)}\right)} = \frac{\frac{CJT\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-2}(1+(6i+6)CJT)(1+(6(n-1)+6)CJT)}}{\left(\frac{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)}{\prod_{i=0}^{n-1}(1+(6i+5)CJT)}\right)\left(1+\frac{CJT\prod_{i=0}^{n-2}(1+(6i+6)CJT)}{\prod_{i=0}^{n-2}(1+(6i+6)CJT)(1+(6(n-1)+6)CJT)}\right)}
\end{aligned}$$

$$= \frac{\frac{CJT}{1+6nCJT}}{\left(\frac{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)}{\prod_{i=0}^{n-1}(1+(6i+5)CJT)}\right)\left(1+\frac{CJT}{1+6nCJT}\right)}$$
$$= \frac{\frac{CJT}{1+6nCJT}\prod_{i=0}^{n-1}(1+(6i+5)CJT)}{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)\left(\frac{1+6nCJT+CJT}{1+6nCJT}\right)}$$
$$= \frac{CJT\prod_{i=0}^{n-1}(1+(6i+5)CJT)}{FP\prod_{i=0}^{n-1}(1+(6i+1)CJT)(1+(6n+1)CJT)}$$
$$= \frac{CJT\prod_{i=0}^{n-1}(1+(6i+5)CJT)}{FP\prod_{i=0}^{n}(1+(6i+1)CJT)}.$$

Hence, we have

$$x_{18n+1} = \frac{CJT\prod_{i=0}^{n-1}(1+(6i+5)CJT)}{FP\prod_{i=0}^{n}(1+(6i+1)CJT)}.$$

Similarly,

$$x_{18n+2} = \frac{x_{18n-1}x_{18n-7}x_{18n-13}}{(x_{18n-4}x_{18n-10})(1+x_{18n-1}x_{18n-7}x_{18n-13})}$$
$$= \frac{\frac{B\prod_{i=0}^{n-1}(1+(6i+4)BHS)}{\prod_{i=0}^{n-1}(1+(6i+6)BHS)}\frac{H\prod_{i=0}^{n-1}(1+(6i+2)BHS)}{\prod_{i=0}^{n-1}(1+(6i+4)BHS)}\frac{S\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-1}(1+(6i+2)BHS)}}{\left(\frac{E\prod_{i=0}^{n-1}(1+(6i+3)BHS)}{\prod_{i=0}^{n-1}(1+(6i+5)BHS)}\frac{M\prod_{i=0}^{n-1}(1+(6i+1)BHS)}{\prod_{i=0}^{n-1}(1+(6i+3)BHS)}\right)\left(1+\frac{B\prod_{i=0}^{n-1}(1+(6i+4)BHS)}{\prod_{i=0}^{n-1}(1+(6i+6)BHS)}\frac{H\prod_{i=0}^{n-1}(1+(6i+2)BHS)}{\prod_{i=0}^{n-1}(1+(6i+4)BHS)}\frac{S\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-1}(1+(6i+2)BHS)}\right)}$$
$$= \frac{\frac{BHS\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-1}(1+(6i+6)BHS)}}{\left(\frac{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)}{\prod_{i=0}^{n-1}(1+(6i+5)BHS)}\right)\left(1+\frac{BHS\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-1}(1+(6i+6)BHS)}\right)} = \frac{\frac{BHS\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-2}(1+(6i+6)BHS)(1+(6(n-1)+6)BHS)}}{\left(\frac{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)}{\prod_{i=0}^{n-1}(1+(6i+5)BHS)}\right)\left(1+\frac{BHS\prod_{i=0}^{n-2}(1+(6i+6)BHS)}{\prod_{i=0}^{n-2}(1+(6i+6)BHS)(1+(6(n-1)+6)BHS)}\right)}$$
$$= \frac{\frac{BHS}{1+6nBHS}}{\left(\frac{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)}{\prod_{i=0}^{n-1}(1+(6i+5)BHS)}\right)\left(1+\frac{BHS}{1+6nBHS}\right)}$$
$$= \frac{\frac{BHS}{1+6nBHS}\prod_{i=0}^{n-1}(1+(6i+5)BHS)}{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)\left(\frac{1+6nBHS+BHS}{1+6nBHS}\right)}$$
$$= \frac{BHS\prod_{i=0}^{n-1}(1+(6i+5)BHS)}{EM\prod_{i=0}^{n-1}(1+(6i+1)BHS)(1+(6n+1)BHS)}$$
$$= \frac{BHS\prod_{i=0}^{n-1}(1+(6i+5)BHS)}{EM\prod_{i=0}^{n}(1+(6i+1)BHS)}.$$

Then, we have

$$x_{18n+2} = \frac{BHS\prod_{i=0}^{n-1}(1+(6i+5)BHS)}{EM\prod_{i=0}^{n}(1+(6i+1)BHS)}.$$

Other relations can also be obtained in a similar way, and thus the proof is complete. □

Theorem 2.2. *The equation (4) has unique equilibrium point which is the number zero and this equilibrium isn't locally asymptotically stable. Also $\overline{x}$ is non hyperbolic.*

Proof. If we use the definition (1.1), for the equilibriums of (4), we have

$$\overline{x} = \frac{\overline{x}^3}{\overline{x}^2 + \overline{x}^5},$$

then

$$\overline{x}^3 + \overline{x}^6 = \overline{x}^3, \quad \overline{x}^6 = 0.$$

Thus the equilibrium point of equation (4), is $\overline{x} = 0$.

Let $f : (0,\infty)^5 \to (0,\infty)$ be the function defined by

$$f(\alpha,\beta,\varrho,\eta,\zeta) = \frac{\alpha\varrho\zeta}{\beta\eta(1+\alpha\varrho\zeta)}.$$

Therefore it follows that,

$$f_\alpha(\alpha,\beta,\varrho,\eta,\zeta) = \frac{\zeta\varrho}{\eta\beta(1+\alpha\varrho\zeta)^2}, \qquad f_\beta(\alpha,\beta,\varrho,\eta,\zeta) = -\frac{\alpha\varrho\zeta}{\eta\beta^2(1+\alpha\varrho\zeta)}, \qquad f_\varrho(\alpha,\beta,\varrho,\eta,\zeta) = \frac{\alpha\zeta}{\eta\beta(1+\alpha\varrho\zeta)^2},$$
$$f_\eta(\alpha,\beta,\varrho,\eta,\zeta) = -\frac{\alpha\varrho\zeta}{\eta^2\beta(1+\alpha\varrho\zeta)}, \qquad f_\zeta(\alpha,\beta,\varrho,\eta,\zeta) = \frac{\alpha\varrho}{\eta\beta(1+\alpha\varrho\zeta)^2}.$$

$$\begin{aligned} f_\alpha(\alpha,\beta,\varrho,\eta,\zeta) &= \frac{\zeta\varrho}{\eta\beta(1+\alpha\varrho\zeta)^2} \\ f_\alpha(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{\overline{x}\overline{x}}{\overline{x}\overline{x}(1+\overline{x}\overline{x}\overline{x})^2} \\ &= \frac{\overline{x}^2}{\overline{x}^2(1+\overline{x}^3)^2} \\ &= \frac{1}{(1+\overline{x}^3)^2} \end{aligned}$$

Equilibrium point $\overline{x} = 0$ was found. Substituting this above, we get

$$f_\alpha(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1.$$

Similarly way we can obtain

$$f_\beta(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1, \qquad f_\varrho(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1, \qquad f_\eta(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1, \qquad f_\zeta(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1.$$

It has been used from *e*) of the definition (1.2). The proof now follows by using Theorem 2.1. □

To illustrate the results of this section, we now consider numerical examples which represent different types of solutions to equation (4).

Example 2.3. Suppose that,

$$x_{-14} = 1.06, \quad x_{-13} = 1.07, \quad x_{-12} = 1.08, \quad x_{-11} = 1.09, \quad x_{-10} = 1.1, \quad x_{-9} = 1.11, \quad x_{-8} = 1.12,$$
$$x_{-7} = 1.13, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 1.16, \quad x_{-3} = 1.17, \quad x_{-2} = 1.18, \quad x_{-1} = 1.19,$$
$$x_0 = 1.2.$$

According to the above initial conditions, figure 1 was obtained. Figure 1 shows the dynamic behavior of (4).

Example 2.4. Assume that,

$$x_{-14} = 1.6, \quad x_{-13} = 3, \quad x_{-12} = 2, \quad x_{-11} = 1.09, \quad x_{-10} = 1.1, \quad x_{-9} = 1.11, \quad x_{-8} = 1.12,$$
$$x_{-7} = 1.9, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 1.16, \quad x_{-3} = 1.17, \quad x_{-2} = 1.18, \quad x_{-1} = 1.19,$$
$$x_0 = 1.2.$$

According to the above initial conditions, figure 2 was obtained. Figure 2 shows the dynamic behavior of (4).

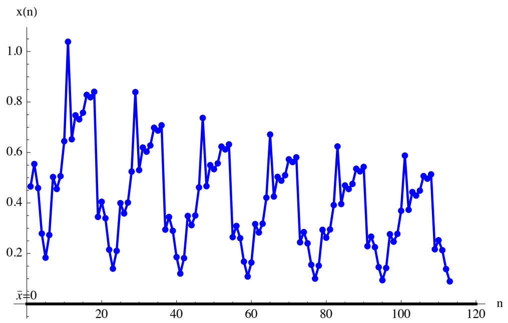

Figure 1: Example 2.3

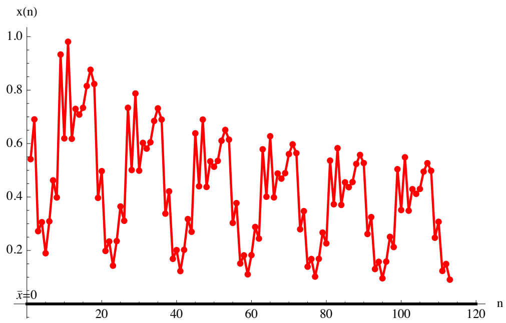

Figure 2: Example 2.4

3. Second case

In this part, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11} - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}. \tag{6}$$

where, $x_0, \ldots, x_{-14}$ defines as in (5).

Theorem 3.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of equation (6). Then,*

$$x_{18n+1} = \frac{-CJT\prod_{i=0}^{n-1}(-1+(6i+5)CJT)}{FP\prod_{i=0}^{n}(-1+(6i+1)CJT)}, \quad x_{18n+2} = \frac{-BHS\prod_{i=0}^{n-1}(-1+(6i+5)BHS)}{EM\prod_{i=0}^{n}(-1+(6i+1)BHS)},$$

$$x_{18n+3} = \frac{-AGR\prod_{i=0}^{n-1}(-1+(6i+5)AGR)}{DL\prod_{i=0}^{n}(-1+(6i+1)AGR)}, \quad x_{18n+4} = \frac{-T\prod_{i=0}^{n-1}(-1+(6i+6)CJT)}{\prod_{i=0}^{n}(-1+(6i+2)CJT)},$$

$$x_{18n+5} = \frac{-S\prod_{i=0}^{n-1}(-1+(6i+6)BHS)}{\prod_{i=0}^{n}(-1+(6i+2)BHS)}, \quad x_{18n+6} = \frac{-R\prod_{i=0}^{n-1}(-1+(6i+6)AGR)}{\prod_{i=0}^{n}(-1+(6i+2)AGR)},$$

$$x_{18n+7} = \frac{P\prod_{i=0}^{n}(-1+(6i+1)CJT)}{\prod_{i=0}^{n}(-1+6i+3)CJT)}, \quad x_{18n+8} = \frac{M\prod_{i=0}^{n}(-1+(6i+1)BHS)}{\prod_{i=0}^{n}(-1+(6i+3)BHS)},$$

$$x_{18n+9} = \frac{L\prod_{i=0}^{n}(-1+(6i+1)AGR)}{\prod_{i=0}^{n}(-1+(6i+3)AGR)}, \quad x_{18n+10} = \frac{J\prod_{i=0}^{n}(-1+(6i+2)CJT)}{\prod_{i=0}^{n}(-1+(6i+4)CJT)},$$

$$x_{18n+11} = \frac{H\prod_{i=0}^{n}(-1+(6i+2)BHS)}{\prod_{i=0}^{n}(-1+(6i+4)BHS)}, \quad x_{18n+12} = \frac{G\prod_{i=0}^{n}(-1+(6i+2)AGR)}{\prod_{i=0}^{n}(-1+(6i+4)AGR)},$$

$$x_{18n+13} = \frac{F\prod_{i=0}^{n}(-1+(6i+3)CJT)}{\prod_{i=0}^{n}(1+(6i+5)CJT)}, \quad x_{18n+14} = \frac{E\prod_{i=0}^{n}(-1+(6i+3)BHS)}{\prod_{i=0}^{n}(-1+(6i+5)BHS)},$$

$$x_{18n+15} = \frac{D\prod_{i=0}^{n}(-1+(6i+3)AGR)}{\prod_{i=0}^{n}(-1+(6i+5)AGR)}, \quad x_{18n+16} = \frac{C\prod_{i=0}^{n}(-1+(6i+4)CJT)}{\prod_{i=0}^{n}(-1+(6i+6)CJT)},$$

$$x_{18n+17} = \frac{B\prod_{i=0}^{n}(-1+(6i+4)BHS)}{\prod_{i=0}^{n}(-1+(6i+6)BHS)}, \quad x_{18n+18} = \frac{A\prod_{i=0}^{n}(-1+(6i+4)AGR)}{\prod_{i=0}^{n}(-1+(6i+6)AGR)},$$

holds.

Proof. The proof is similar to the proof of Theorem 2.1 and therefore it will be omitted. □

Theorem 3.2. *The equation (6) has a unique equilibrium point $\overline{x} = 0$, which is not locally asymptotically stable.*

Proof. The proof is similar to the proof Theorem 2.2 and there it will be omitted. □

For confirming the outcomes of this section, we take into consideration mathematical instances which stand for various kind of solutions to (6).

Example 3.3. Suppose that,

$$x_{-14} = 1.6, \quad x_{-13} = 3, \quad x_{-12} = 2, \quad x_{-11} = 1.09, \quad x_{-10} = 3.1, \quad x_{-9} = 2.3, \quad x_{-8} = 1.12,$$
$$x_{-7} = 1.9, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 1.16, \quad x_{-3} = 1.17, \quad x_{-2} = 2.7, \quad x_{-1} = 1.3,$$
$$x_0 = 3.5.$$

According to the above initial conditions, figure 3 was obtained. Figure 3 shows the dynamic behavior of (6).

Example 3.4. We consider,

$$x_{-14} = 1.6, \quad x_{-13} = 3, \quad x_{-12} = 6, \quad x_{-11} = 1.09, \quad x_{-10} = 3.1, \quad x_{-9} = 2.3, \quad x_{-8} = 1.12,$$
$$x_{-7} = 5.5, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 6.16, \quad x_{-3} = 1.17, \quad x_{-2} = 2.7, \quad x_{-1} = 1.3,$$
$$x_0 = 3.5.$$

According to the above initial conditions, figure 4 was obtained. Figure 4 shows the dynamic behavior of (6).

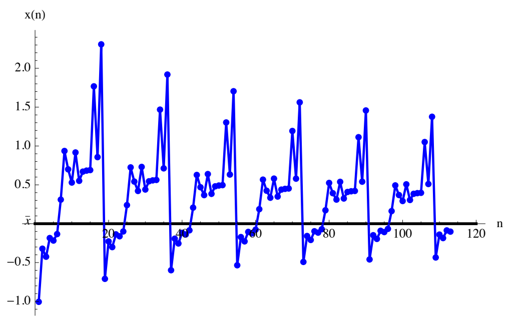

Figure 3: Example 3.3

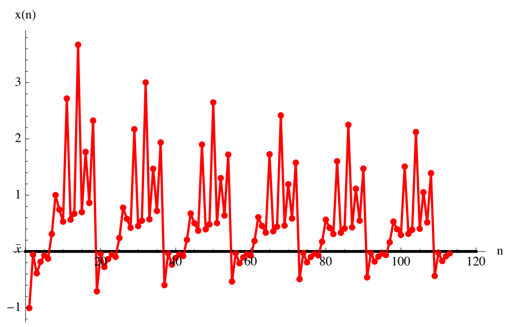

Figure 4: Example 3.4

4. Third case

In this case, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{-x_{n-5}x_{n-11} + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \tag{7}$$

where, $x_0, \ldots, x_{-14}$ defines as in (5).

Theorem 4.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of equation (7). Then every solution of equation (7) for, $n = 0, 1, 2, \ldots$*

$$x_{18n+1} = \frac{CJT}{FP(-1+CJT)}, x_{18n+2} = \frac{BHS}{EM(-1+BHS)}, x_{18n+3} = \frac{AGR}{DL(-1+AGR)},$$
$$x_{18n+4} = T, x_{18n+5} = S, x_{18n+6} = R, x_{18n+7} = P, x_{18n+8} = M, x_{18n+9} = L,$$
$$x_{18n+10} = J, x_{18n+11} = H, x_{18n+12} = G, x_{18n+13} = F, x_{18n+14} = E, x_{18n+15} = D,$$
$$x_{18n+16} = C, x_{18n+17} = B, x_{18n+18} = A,$$

solutions have 18 periods.

Proof. The proof of each formula are carried out in similar way. So, we will demonstrate proof using one of the formula. We will employ the mathematical induction method. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
&x_{18n-17} = \frac{CJT}{FP(-1+CJT)}, x_{18n-16} = \frac{BHS}{EM(-1+BHS)}, x_{18n-15} = \frac{AGR}{DL(-1+AGR)},\\
&x_{18n-14} = T, x_{18n-13} = S, x_{18n-12} = R, x_{18n-11} = P, x_{18n-10} = M, x_{18n-9} = L,\\
&x_{18n-8} = J, x_{18n-7} = H, x_{18n-6} = G, x_{18n-5} = F, x_{18n-4} = E, x_{18n-3} = D,\\
&x_{18n-2} = C, x_{18n-1} = B, x_{18n} = A.
\end{aligned}$$

Now, it follows from (7) that

$$\begin{aligned}
x_{18n+1} &= \frac{x_{18n-2}x_{18n-8}x_{18n-14}}{-x_{18n-5}x_{18n-11} + x_{18n-2}x_{18n-5}x_{18n-8}x_{18n-11}x_{18n-14}} = \frac{CJT}{FP(-1+CJT)},\\
x_{18n+2} &= \frac{x_{18n-1}x_{18n-7}x_{18n-13}}{-x_{18n-4}x_{18n-10} + x_{18n-1}x_{18n-4}x_{18n-7}x_{18n-10}x_{18n-13}} = \frac{BHS}{EM(-1+BHS)}
\end{aligned}$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed. □

Theorem 4.2. *The equation (7) has three equilibrium points which are* $0, \pm\sqrt[3]{2}$, *and these equilibrium points are not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 2.2 and hence is omitted. □

Example 4.3. Suppose that,

$$\begin{aligned}
&x_{-14} = 1.6, \quad x_{-13} = 2.5, \quad x_{-12} = 2, \quad x_{-11} = 1.09, \quad x_{-10} = 3.1, \quad x_{-9} = 2.8, \quad x_{-8} = 1.12,\\
&x_{-7} = 3.1, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 3.16, \quad x_{-3} = 1.17, \quad x_{-2} = 2.7, \quad x_{-1} = 1.3,\\
&x_0 = 3.2.
\end{aligned}$$

According to the above initial conditions, figure 5 was obtained. Figure 5 shows the dynamic behavior of (7).

Example 4.4. Assume that,

$$\begin{aligned}
&x_{-14} = 3.6, \quad x_{-13} = 2.5, \quad x_{-12} = 2, \quad x_{-11} = 1.09, \quad x_{-10} = 3.1, \quad x_{-9} = 2.8, \quad x_{-8} = 1.12,\\
&x_{-7} = 3.1, \quad x_{-6} = 1.14, \quad x_{-5} = 1.15, \quad x_{-4} = 3.16, \quad x_{-3} = 2.17, \quad x_{-2} = 1.75, \quad x_{-1} = 1.32,\\
&x_0 = 3.2.
\end{aligned}$$

According to the above initial conditions, figure 6 was obtained. Figure 6 shows the dynamic behavior of (7).

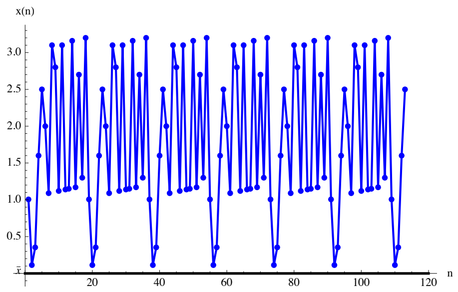

Figure 5: Example 4.3

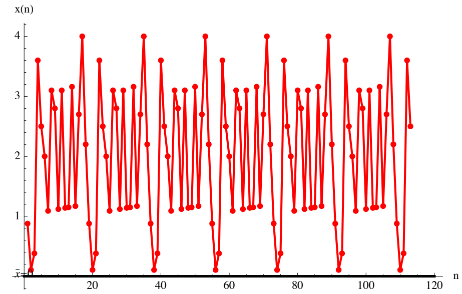

Figure 6: Example 4.4

5. Fourth case

In this section, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{-x_{n-5}x_{n-11} - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \tag{8}$$

where, $x_0, \ldots, x_{-14}$ defines as in (5).

Theorem 5.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of (8). Then every solution of (8) for, $n = 0, 1, 2, \ldots$*

$$x_{18n+1} = \frac{-CJT}{FP(1+CJT)}, x_{18n+2} = \frac{-BHS}{EM(1+BHS)}, x_{18n+3} = \frac{-AGR}{DL(1+AGR)},$$
$$x_{18n+4} = T, x_{18n+5} = S, x_{18n+6} = R, x_{18n+7} = P, x_{18n+8} = M, x_{18n+9} = L,$$
$$x_{18n+10} = J, x_{18n+11} = H, x_{18n+12} = G, x_{18n+13} = F, x_{18n+14} = E, x_{18n+15} = D,$$
$$x_{18n+16} = C, x_{18n+17} = B, x_{18n+18} = A,$$

have 18 periods.

Proof. The proof is the same as the proof of Theorem 4.1 and hence is omitted. □

Theorem 5.2. *The equation (8) has three equilibrium point which are $0, \pm\sqrt[3]{-2}$ and this equilibrium points is not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 2.2 and hence is omitted. □

Example 5.3. Assume that,

$$x_{-14} = 3, \quad x_{-13} = 3.2, \quad x_{-12} = 3.4, \quad x_{-11} = 1.6, \quad x_{-10} = 3.7, \quad x_{-9} = 2.8, \quad x_{-8} = 2.5, \quad x_{-7} = 2.7,$$
$$x_{-6} = 1.3, \quad x_{-5} = 1.19, \quad x_{-4} = 1.2, \quad x_{-3} = 1.45, \quad x_{-2} = 1.56, \quad x_{-1} = 1.6, \quad x_0 = 1.5.$$

According to the above initial conditions, figure 7 was obtained. Figure 7 shows the dynamic behavior of (8).

Example 5.4. Suppose that,

$$x_{-14} = 3, \quad x_{-13} = 3.2, \quad x_{-12} = 3.4, \quad x_{-11} = 1.6, \quad x_{-10} = 3.7, \quad x_{-9} = 2.8, \quad x_{-8} = 2.5, \quad x_{-7} = 2.7,$$
$$x_{-6} = 1.3, \quad x_{-5} = 1.19, \quad x_{-4} = 1.2, \quad x_{-3} = 2.7, \quad x_{-2} = 4.7, \quad x_{-1} = 4.1, \quad x_0 = 1.9.$$

According to the above initial conditions, figure 8 was obtained. Figure 8 shows the dynamic behavior of (8).

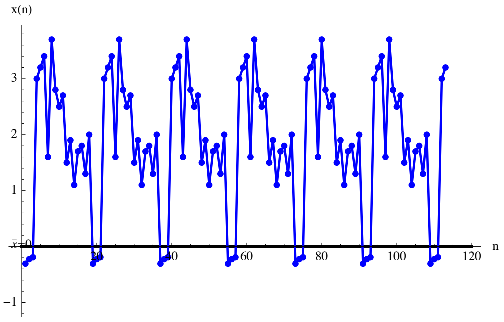

Figure 7: Example 5.3

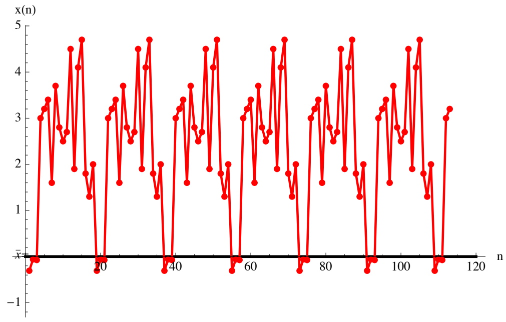

Figure 8: Example 5.4

6. Conclusion

We study the behavior of the difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{\pm x_{n-5}x_{n-11} \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}},$$

where the initials are positive real numbers. Local stability is discussed. Moreover, we get the solution of some special cases. Finally, some numerical examples are given.

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