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Hyperbolic Ricci solitons on sequential warped product manifolds

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Abstract. We study hyperbolic Ricci solitons on sequential warped products. The necessary conditions are obtained for a hyperbolic Ricci soliton with the structure of a sequential warped product to be an Einstein manifold when we consider the potential field as a Killing or a conformal vector field. Some physical applications are also given.

1. Introduction

A semi-Riemannian manifold (M, g) is said to be a *Ricci soliton* [17], if there exists a smooth vector field $X \in \chi(M)$ satisfying the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

for some constant λ and it is denoted by (M, g, X, λ), where Ric and $\mathcal{L}$ denote the the Ricci tensor and Lie derivative of (M, g), respectively and the vector field X is called the *potential vector field*. Ricci solitons are natural generalizations of Einstein manifolds.

A semi-Riemannian manifold (M, g) is said to be a *hyperbolic Ricci soliton* (see [3] and [10]), if there exists a smooth vector field $X \in \chi(M)$ satisfying the equation

$$\mathrm{Ric} + \lambda\mathcal{L}_X g + (\mathcal{L}_X \circ \mathcal{L}_X)\, g = \mu g \tag{1}$$

for some constants λ and μ and it is denoted by (M, g, X, λ, μ), where Ric and $\mathcal{L}$ denote the the Ricci tensor and Lie derivative of (M, g), respectively. If X vanishes identically, a hyperbolic Ricci soliton is an Einstein manifold. If $\lambda = \frac{1}{2}$ and X is 2-Killing i.e., $(\mathcal{L}_X \circ \mathcal{L}_X)\, g = 0$, (see [23]), then a hyperbolic Ricci soliton is a Ricci soliton.

In [4], O'Neill and Bishop defined the notion of a warped product manifold to construct manifolds with negative curvature. It is known that warped products have important applications in both differential geometry and physics. In general relativity, the main application of them is to model the spacetime. As generalizations of warped product manifolds, doubly, multiply and sequential warped product manifolds

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have been defined and each of them has some different geometric and physical properties. For example see ([7], [26] and [27]). In the recent years many papers have been published in which Ricci solitons on warped product manifolds or generalizations have been studied, for example see ([1], [5], [6], [8], [11], [13], [15], [16], [18], [19], [21] and [22]). Furthermore, for recent studies about sequential warped products see also ([12], [14], [20], [24] and [25]). Moreover, recently, in [3], Azami and Fasihi-Ramandi studied hyperbolic Ricci solitons on warped product manifolds (see also [10]). By a motivation from the above studies, as a generalization of the paper [3], in this paper, we consider hyperbolic Ricci solitons on sequential warped product manifolds.

2. Preliminaries

Let (M_i, g_i) be semi-Riemannian manifolds, $1 \leq i \leq 3$, and $f : M_1 \longrightarrow \mathbb{R}^+$, $h : M_1 \times M_2 \longrightarrow \mathbb{R}^+$ be two smooth functions. The *sequential warped product manifold* M is the triple product manifold $M = (M_1 \times_f M_2) \times_h M_3$ endowed with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ [7]. Here the functions f, h are called the *warping functions*.

Throughout the paper, (M, g) will be considered as a sequential warped product manifold, where $M = M^n = (M_1^{n_1} \times_f M_2^{n_2}) \times_h M_3^{n_3}$ with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. The restriction of the warping function $h : \overline{M} = M_1 \times M_2 \longrightarrow \mathbb{R}$ to $M_1 \times \{0\}$ is $h^1 = h|_{M_1 \times \{0\}}$.

We use the notation ∇, ∇^i; Ric, Ric^i; Hess, Hess^i; $\mathcal{L}$, $\mathcal{L}^i$ for the Levi-Civita connections, Ricci tensors, Hessians and Lie derivatives of M, and M_i, respectively. Hessian of $\overline{M}$ is denoted by $\overline{\text{Hess}}$.

Firstly, we give the following lemmas on sequential warped product manifolds which will be necessary to prove our results:

Lemma 2.1. *[7] Let (M, g) be a sequential warped product and $X_i, Y_i \in \chi(M_i)$ for $1 \leq i \leq 3$. Then*

1. $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1,$
2. $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(ln f) X_2,$
3. $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \nabla^1 f,$
4. $\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(ln h) X_3,$
5. $\nabla_{X_2} X_3 = \nabla_{X_3} X_2 = X_2(ln h) X_3,$
6. $\nabla_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) \nabla h.$

Lemma 2.2. *[7] Let (M, g) be a sequential warped product and $X_i, Y_i \in \chi(M_i)$ for $1 \leq i \leq 3$. Then*

1. $\text{Ric}(X_1, Y_1) = \text{Ric}^1(X_1, Y_1) - \frac{n_2}{f} \text{Hess}^1 f(X_1, Y_1) - \frac{n_3}{h} \overline{\text{Hess}} h(X_1, Y_1),$
2. $\text{Ric}(X_2, Y_2) = \text{Ric}^2(X_2, Y_2) - f^\sharp g_2(X_2, Y_2) - \frac{n_3}{h} \overline{\text{Hess}} h(X_2, Y_2),$
3. $\text{Ric}(X_3, Y_3) = \text{Ric}^3(X_3, Y_3) - h^\sharp g_3(X_3, Y_3),$
4. $\text{Ric}(X_i, X_j) = 0$ *when* $i \neq j$, *where* $f^\sharp = \left(f \Delta^1 f + (n_2 - 1) \left\| \nabla^1 f \right\|^2 \right)$ *and* $h^\sharp = \left(h \Delta h + (n_3 - 1) \|\nabla h\|^2 \right).$

Lemma 2.3. *[7] Let (M, g) be a sequential warped product manifold. A vector field $X \in \chi(M)$ satisfies the equation*

$$(\mathcal{L}_X g)(Y, Z) = \left(\mathcal{L}^1_{X_1} g_1 \right)(Y_1, Z_1) + f^2 \left(\mathcal{L}^2_{X_2} g_2 \right)(Y_2, Z_2) + h^2 \left(\mathcal{L}^3_{X_3} g_3 \right)(Y_3, Z_3)$$
$$+ 2 f X_1(f) g_2(Y_2, Z_2) + 2h(X_1 + X_2)(h) g_3(Y_3, Z_3)$$

for $Y, Z \in \chi(M)$.

Proposition 2.4. *Let (M, g) be a sequential warped product manifold and $Y_i, Z_i \in \chi(M_i)$ for $1 \leq i \leq 3$. Then*

1. $(\mathcal{L}_X \mathcal{L}_X g)(Y_1, Z_1) = (\mathcal{L}^1_{X_1} \mathcal{L}^1_{X_1} g_1)(Y_1, Z_1),$
2. $(\mathcal{L}_X \mathcal{L}_X g)(Y_2, Z_2) = f^2 (\mathcal{L}^2_{X_2} \mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + 2 X_1(f^2)(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + X_1(X_1(f^2)) g_2(Y_2, Z_2),$

3. $(\mathcal{L}_X\mathcal{L}_X g)(Y_3, Z_3) = h^2(\mathcal{L}^3_{X_3}\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) + 2(X_1(h^2) + X_2(h^2))(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3)$
$+ [(X_1 + X_2)(X_1 + X_2)(h^2)]g_3(Y_3, Z_3)$,
4. $(\mathcal{L}_X\mathcal{L}_X g)(Y_i, Z_j) = 0$, $1 \leq i, j \leq 3$, $i \neq j$ for every vector field $X = X_1 + X_2 + X_3$ on M.

Proof. Let (M, g) be a sequential warped product manifold. Using Lemma 2.1, we have

$$\begin{aligned}
&\mathcal{L}_X Y_1 = \nabla_X Y_1 - \nabla_{Y_1} X \\
&= \nabla_{X_1} Y_1 + \nabla_{X_2} Y_1 + \nabla_{X_3} Y_1 - \nabla_{Y_1} X_1 - \nabla_{Y_1} X_2 - \nabla_{Y_1} X_3 \\
&= \nabla^1_{X_1} Y_1 - \nabla^1_{Y_1} X_1 = \mathcal{L}^1_{X_1} Y_1, \\
&\mathcal{L}_X Y_2 = \nabla_X Y_2 - \nabla_{Y_2} X \\
&= \nabla_{X_1} Y_2 + \nabla_{X_2} Y_2 + \nabla_{X_3} Y_2 - \nabla_{Y_2} X_1 - \nabla_{Y_2} X_2 - \nabla_{Y_2} X_3 \\
&= \nabla^2_{X_2} Y_2 - \nabla^2_{Y_2} X_2 = \mathcal{L}^2_{X_2} Y_2
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_X Y_3 = \nabla_X Y_3 - \nabla_{Y_3} X \\
&= \nabla_{X_1} Y_3 + \nabla_{X_2} Y_3 + \nabla_{X_3} Y_3 - \nabla_{Y_3} X_1 - \nabla_{Y_3} X_2 - \nabla_{Y_3} X_3 \\
&= \nabla^3_{X_3} Y_3 - \nabla^3_{Y_3} X_3 = \mathcal{L}^3_{X_3} Y_3.
\end{aligned}$$

Hence, from Lemma 2.3, we have

$$\begin{aligned}
&(\mathcal{L}_X\mathcal{L}_X g)(Y_1, Z_1) = \mathcal{L}_X((\mathcal{L}_X g)(Y_1, Z_1)) - (\mathcal{L}_X g)(\mathcal{L}_X Y_1, Z_1) - (\mathcal{L}_X g)(Y_1, \mathcal{L}_X Z_1) \\
&= \mathcal{L}_X((\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1)) - (\mathcal{L}_X g)(\mathcal{L}^1_{X_1} Y_1, Z_1) - (\mathcal{L}_X g)(Y_1, \mathcal{L}^1_{X_1} Z_1) \\
&= \mathcal{L}^1_{X_1}((\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1)) - (\mathcal{L}^1_{X_1} g_1)(\mathcal{L}^1_{X_1} Y_1, Z_1) - (\mathcal{L}^1_{X_1} g_1)(Y_1, \mathcal{L}^1_{X_1} Z_1) \\
&= (\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1), \\
&(\mathcal{L}_X\mathcal{L}_X g)(Y_2, Z_2) = \mathcal{L}_X((\mathcal{L}_X g)(Y_2, Z_2)) - (\mathcal{L}_X g)(\mathcal{L}_X Y_2, Z_2) - (\mathcal{L}_X g)(Y_2, \mathcal{L}_X Z_2) \\
&= \mathcal{L}_X(f^2(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + X_1(f^2) g_2(Y_2, Z_2)) - (\mathcal{L}_X g)(\mathcal{L}^2_{X_2} Y_2, Z_2) - (\mathcal{L}_X g)(Y_2, \mathcal{L}^2_{X_2} Z_2) \\
&= X_1(f^2)(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + f^2\mathcal{L}^2_{X_2}((\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2)) \\
&+X_1(X_1(f^2)) g_2(Y_2, Z_2) + X_1(f^2)\mathcal{L}^2_{X_2}(g_2(Y_2, Z_2)) \\
&-f^2(\mathcal{L}^2_{X_2} g_2)(\mathcal{L}^2_{X_2} Y_2, Z_2) - X_1(f^2) g_2(\mathcal{L}^2_{X_2} Y_2, Z_2) \\
&-f^2(\mathcal{L}^2_{X_2} g_2)(Y_2, \mathcal{L}^2_{X_2} Z_2) - X_1(f^2) g_2(Y_2, \mathcal{L}^2_{X_2} Z_2) \\
&= f^2(\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + 2X_1(f^2)(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + X_1(X_1(f^2)) g_2(Y_2, Z_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{L}_X\mathcal{L}_X g)(Y_3, Z_3) = \mathcal{L}_X((\mathcal{L}_X g)(Y_3, Z_3)) - (\mathcal{L}_X g)(\mathcal{L}_X Y_3, Z_3) - (\mathcal{L}_X g)(Y_3, \mathcal{L}_X Z_3) \\
&= \mathcal{L}_X(h^2(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) + X_1(h^2) g_3(Y_3, Z_3) + X_2(h^2) g_3(Y_3, Z_3)) \\
&-(\mathcal{L}_X g)(\mathcal{L}^3_{X_3} Y_3, Z_3) - (\mathcal{L}_X g)(Y_3, \mathcal{L}^3_{X_3} Z_3) \\
&= X_1(h^2)(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) + X_2(h^2)(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) \\
&+h^2\mathcal{L}^3_{X_3}((\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3)) + X_1(X_1(h^2)) g_3(Y_3, Z_3) + X_2(X_1(h^2)) g_3(Y_3, Z_3) \\
&+X_1(h^2)\mathcal{L}^3_{X_3}(g_3(Y_3, Z_3)) + X_1(X_2(h^2)) g_3(Y_3, Z_3) + X_2(X_2(h^2)) g_3(Y_3, Z_3)
\end{aligned}$$

$$+X_2(h^2)\mathcal{L}^3_{X_3}(g_3(Y_3,Z_3)) - h^2(\mathcal{L}^3_{X_3}g_3)(\mathcal{L}^3_{X_3}Y_3,Z_3)$$
$$-X_1(h^2)g_3(\mathcal{L}^3_{X_3}Y_3,Z_3) - X_2(h^2)g_3(\mathcal{L}^3_{X_3}Y_3,Z_3)$$
$$-h^2(\mathcal{L}^3_{X_3}g_3)(Y_3,\mathcal{L}^3_{X_3}Z_3) - X_1(h^2)g_3(Y_3,\mathcal{L}^3_{X_3}Z_3) - X_2(h^2)g_3(Y_3,\mathcal{L}^3_{X_3}Z_3)$$
$$= h^2(\mathcal{L}^3_{X_3}\mathcal{L}^3_{X_3}g_3)(Y_3,Z_3) + 2X_1(h^2)(\mathcal{L}^3_{X_3}g_3)(Y_3,Z_3)$$
$$+2X_2(h^2)(\mathcal{L}^3_{X_3}g_3)(Y_3,Z_3) + X_1(X_1(h^2))g_3(Y_3,Z_3) + X_2(X_1(h^2))g_3(Y_3,Z_3)$$
$$+X_1(X_2(h^2))g_3(Y_3,Z_3) + X_2(X_2(h^2))g_3(Y_3,Z_3).$$

Moreover,

$$(\mathcal{L}_X\mathcal{L}_Xg)(Y_1,Z_2) = \mathcal{L}_X(\mathcal{L}_Xg(Y_1,Z_2)) - (\mathcal{L}_Xg)(\mathcal{L}_XY_1,Z_2) - (\mathcal{L}_Xg)(Y_1,\mathcal{L}_XZ_2)$$
$$= -(\mathcal{L}_Xg)(\mathcal{L}^1_{X_1}Y_1,Z_2) - (\mathcal{L}_Xg)(Y_1,\mathcal{L}^2_{X_2}Z_2) = 0$$

and similarly for $1 \leq i,j \leq 3$ and $i \neq j$, we obtain $(\mathcal{L}_X\mathcal{L}_Xg)(Y_i,Z_j) = 0$. □

A vector field V on a Riemannian manifold (M,g) is said to be *conformal*, if there exists a smooth function f on M satisfying the equation $\mathcal{L}_Vg = 2fg$. The function f is called the *potential function* of the conformal vector field V. If $f = 0$, then V is called a *Killing* vector field.

3. Main Results

In this section, we examine the properties of hyperbolic Ricci solitons on sequential warped product manifolds.

Firstly, we have the following theorem:

Theorem 3.1. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2g_2) \oplus h^2g_3$. If (M,g,X,λ,μ) is a hyperbolic Ricci soliton with potential vector field of the form $X = X_1 + X_2 + X_3$, where $X_i \in \chi(M_i)$ for $1 \leq i \leq 3$, then*

(i) $(M_1, g_1, \lambda X_1 - \frac{n_2}{2}\nabla(\ln f) - \frac{n_3}{2}\nabla(\ln h^1), 1, \mu)$ is a hyperbolic Ricci soliton.
(ii) M_2 is an Einstein manifold when X_2 is a Killing vector field and $\overline{\mathrm{Hess}}h = \psi g$.
(iii) M_3 is an Einstein manifold when X_3 is a Killing vector field.

Proof. Let (M,g,X,λ,μ) be a hyperbolic Ricci soliton. Then from (1), we have

$$Ric(Y,Z) + \lambda(\mathcal{L}_Xg)(Y,Z) + (\mathcal{L}_X \circ \mathcal{L}_X)g(Y,Z) = \mu g(Y,Z)$$

for all vector fields $Y, Z \in \chi(M)$.

Let $Y = Y_1$ and $Z = Z_1$. From Lemma 2.2, Lemma 2.3 and Proposition 2.4, we have

$$\mathrm{Ric}^1(Y_1,Z_1) - \frac{n_2}{f}\mathrm{Hess}^1f(Y_1,Z_1) - \frac{n_3}{h}\overline{\mathrm{Hess}}h(Y_1,Z_1) \tag{2}$$
$$+\lambda(\mathcal{L}^1_{X_1}g_1)(Y_1,Z_1) + (\mathcal{L}^1_{X_1} \circ \mathcal{L}^1_{X_1})g_1(Y_1,Z_1) = \mu g_1(Y_1,Z_1).$$

It is noted that

$$\lambda(\mathcal{L}^1_{X_1}g_1)(Y_1,Z_1) - \frac{n_2}{f}\mathrm{Hess}^1f(Y_1,Z_1) - \frac{n_3}{h}\overline{\mathrm{Hess}}h(Y_1,Z_1)$$
$$= \lambda g_1(\nabla^1_{Y_1}X_1,Z_1) - \frac{n_2}{2f}g_1(\nabla^1_{Y_1}\nabla^1f,Z_1) - \frac{n_3}{2h}g_1(\nabla^1_{Y_1}\nabla^1h^1,Z_1)$$

$$+\lambda g_1(Y_1, \nabla^1_{Z_1} X_1) - \frac{n_2}{2f} g_1(Y_1, \nabla^1_{Z_1} \nabla^1 f) - \frac{n_3}{2h} g_1(Y_1, \nabla^1_{Z_1} \nabla^1 h^1)$$

$$= g_1\left(\nabla^1_{Y_1}(\lambda X_1 - \frac{n_2}{2}\nabla^1(\ln f) - \frac{n_3}{2}\nabla^1(\ln h^1)), Z_1\right)$$

$$+g_1\left(Y_1, \nabla^1_{Z_1}(\lambda X_1 - \frac{n_2}{2}\nabla^1(\ln f) - \frac{n_3}{2}\nabla^1(\ln h^1))\right)$$

$$= \left(\mathcal{L}^1_{\lambda X_1 - \frac{n_2}{2}\nabla^1(\ln f) - \frac{n_3}{2}\nabla^1(\ln h^1)} g_1\right)(Y_1, Z_1).$$

So the equation (2) turns into

$$\mathrm{Ric}^1(Y_1, Z_1) + \left(\mathcal{L}^1_{\lambda X_1 - \frac{n_2}{2}\nabla^1(\ln f) - \frac{n_3}{2}\nabla^1(\ln h^1)} g_1\right)(Y_1, Z_1) + (\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1) = \mu g_1(Y_1, Z_1)$$

and therefore, $(M_1, g_1, \lambda X_1 - \frac{n_2}{2}\nabla^1(\ln f) - \frac{n_3}{2}\nabla^1(\ln h^1), 1, \mu)$ is a hyperbolic Ricci soliton.

Now, let $Y = Y_2$ and $Z = Z_2$. Then

$$\mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h}\overline{\mathrm{Hess}} h(Y_2, Z_2) + \lambda f^2(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2)$$

$$+2\lambda X_1(f^2) g_2(Y_2, Z_2) + f^2(\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2) + 2X_1(f^2)(\mathcal{L}^2_{X_2} g_2)(Y_2, Z_2)$$

$$+X_1(X_1(f^2)) g_2(Y_2, Z_2) = \mu f^2 g_2(Y_2, Z_2).$$

Here, if X_2 is a Killing vector field and $\overline{\mathrm{Hess}} h = \psi g$, we get

$$\mathrm{Ric}^2(Y_2, Z_2) = (\mu f^2 + f^\sharp + \frac{n_3}{h}\psi f^2 - 2\lambda X_1(f^2) - X_1(X_1(f^2))) g_2(Y_2, Z_2),$$

which implies that M_2 is an Einstein manifold.

Finally, let $Y = Y_3$ and $Z = Z_3$. Then

$$\mathrm{Ric}^3(Y_3, Z_3) - h^\sharp g_3(Y_3, Z_3) + \lambda h^2(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) + 2\lambda(X_1 + X_2)(h^2) g_3(Y_3, Z_3)$$

$$+h^2(\mathcal{L}^3_{X_3}\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3) + 2(X_1(h^2) + X_2(h^2))(\mathcal{L}^3_{X_3} g_3)(Y_3, Z_3)$$

$$+((X_1 + X_2)(X_1 + X_2)(h^2)) g_3(Y_3, Z_3) = \mu h^2 g_3(Y_3, Z_3),$$

which means that (M_3, g_3) is an Einstein manifold when X_3 is a Killing vector field.

This completes the proof of the theorem. □

In the following theorem, we provide some conditions for a sequential warped product $M = (M_1 \times_f M_2) \times_h M_3$ to be an Einstein manifold.

Theorem 3.2. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and (M, g, X, λ, μ) a hyperbolic Ricci soliton. Then M is an Einstein manifold, if the following conditions hold:*

(i) *X_i are conformal vector fields on M_i with factor ρ_i, $1 \le i \le 3$,*

(ii) $\mu f^2 - \lambda X_1(f^2) - X_1(X_1(f^2)) - \rho_2(\lambda f^2 + 2X_1(f^2)) - f^2(X_2(\rho_2) + \rho_2^2)$
$= f^2(\mu - \lambda\rho_1 - X_1(\rho_1) - \rho_1^2)$
and

(iii) $\mu h^2 - \lambda X_1(h^2) - \lambda X_2(h^2) - X_1(X_1(h^2)) - X_2(X_1(h^2)) - X_1(X_2(h^2))$
$-X_2(X_2(h^2)) - \rho_3(\lambda h^2 + 2X_1(h^2) + 2X_2(h^2)) - h^2(X_3(\rho_3) + \rho_3^2)$
$= h^2(\mu - \lambda\rho_1 - X_1(\rho_1) - \rho_1^2).$

Proof. Since X_i are conformal vector fields on M_i with factor ρ_i, $1 \le i \le 3$, we have $\mathcal{L}^1_{X_1} g_1 = \rho_1 g_1$, $\mathcal{L}^2_{X_2} g_2 = \rho_2 g_2$ and $\mathcal{L}^3_{X_3} g_3 = \rho_3 g_3$. So we get

$$\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1} g_1 = \mathcal{L}^1_{X_1}(\rho_1 g_1) = (X_1(\rho_1) + \rho_1^2) g_1,$$

$$\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2} g_2 = (X_2(\rho_2) + \rho_2^2) g_2$$

and

$$\mathcal{L}^3_{X_3}\mathcal{L}^3_{X_3} g_3 = (X_3(\rho_3) + \rho_3^2) g_3.$$

Now, since (M, g, X, λ, μ) is a hyperbolic Ricci soliton, from (1), we have

$$\mathrm{Ric}(Y_1, Z_1) + \lambda(\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1) + (\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1) = \mu g_1(Y_1, Z_1).$$

Hence we get

$$\mathrm{Ric}(Y_1, Z_1) = (\mu - \lambda\rho_1 - X_1(\rho_1) - \rho_1^2) g_1(Y_1, Z_1).$$

Similarly, we can write

$$\mathrm{Ric}(Y_2, Z_2) = (\mu f^2 - \lambda X_1(f^2) - X_1(X_1(f^2)) - \rho_2(\lambda f^2 + 2X_1(f^2)) - f^2(X_2(\rho_2) + \rho_2^2)) g_2(Y_2, Z_2)$$

and

$$\mathrm{Ric}(Y_3, Z_3) = [\mu h^2 - \lambda X_1(h^2) - \lambda X_2(h^2) - X_1(X_1(h^2)) - X_2(X_1(h^2)) - X_1(X_2(h^2))$$
$$-X_2(X_2(h^2)) - \rho_3(\lambda h^2 + 2X_1(h^2) + 2X_2(h^2)) - h^2(X_3(\rho_3) + \rho_3^2)] g_3(Y_3, Z_3).$$

Therefore $\mathrm{Ric}(Y, Z) = (\mu - \lambda\rho_1 - X_1(\rho_1) - \rho_1^2) g(Y, Z)$, which completes the proof. □

Using Lemma 2.3 and Proposition 2.4, we can state the following theorem:

Theorem 3.3. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and (M, g, X, λ, μ) a hyperbolic Ricci soliton. Then (M, g, X, λ, μ) is Einstein if one of the following conditions hold:*

(i) *$X = X_1$ is a Killing vector field on M_1 and*
$$\begin{cases} \lambda X_1(f^2) + X_1(X_1(f^2)) = 0 \\ \lambda X_1(h^2) + X_1(X_1(h^2)) = 0, \end{cases}$$
(ii) *$X = X_2$ is a Killing vector field on M_2 and $\lambda X_2(h^2) + X_2(X_2(h^2)) = 0$,*
(iii) *$X = X_3$ is a Killing vector field on M_3,*
(iv) *X_i is a Killing vector field on M_i, $i = 1, 2, 3$ and*
$$\begin{cases} \lambda X_1(f^2) + X_1(X_1(f^2)) = 0, \\ \lambda X_1(h^2) + \lambda X_2(h^2) + X_1(X_1(h^2)) + X_2(X_1(h^2)) + X_1(X_2(h^2)) + X_2(X_2(h^2)) = 0. \end{cases}$$

Proof. If X_i is a Killing vector field on M_i, $1 \le i \le 3$, then $\mathcal{L}^i_{X_i} g_i = 0$ and $\mathcal{L}^i_{X_i}\mathcal{L}^i_{X_i} g_i = 0$.
Assume that $X = X_1$ and X_1 is a Killing vector field on M_1. Then we have

$$(\mathcal{L}_{X_1} g)(Y, Z) = (\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1) + X_1(f^2) g_2(Y_2, Z_2) + X_1(h^2) g_3(Y_3, Z_3)$$
$$= X_1(f^2) g_2(Y_2, Z_2) + X_1(h^2) g_3(Y_3, Z_3)$$

and

$$(\mathcal{L}_{X_1}\mathcal{L}_{X_1} g)(Y, Z) = (\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1)$$

$$+X_1(X_1(f^2))g_2(Y_2,Z_2)+X_1(X_1(h^2))g_3(Y_3,Z_3)$$
$$=X_1(X_1(f^2))g_2(Y_2,Z_2)+X_1(X_1(h^2))g_3(Y_3,Z_3).$$

From the hypothesis, the hyperbolic Ricci soliton equation (1) turns into

$$\mu g(Y,Z)=\mathrm{Ric}(Y,Z)+\lambda(\mathcal{L}_{X_1}g)(Y,Z)+(\mathcal{L}_{X_1}\mathcal{L}_{X_1}g)(Y,Z)$$
$$=\mathrm{Ric}(Y,Z)+(\lambda X_1(f^2)+X_1(X_1(f^2)))g_2(Y_2,Z_2)+(\lambda X_1(h^2)+X_1(X_1(h^2)))g_3(Y_3,Z_3)=\mathrm{Ric}(Y,Z),$$

which shows that M is an Einstein manifold.

Using the same pattern, it can be shown that M is Einstein for the remaining cases. □

A vector field X on a Riemannian manifold (M,g) is called *Ricci bi-conformal* [9], if it satisfies $(\mathcal{L}_X g)(Y,Z)=\alpha g(Y,Z)+\beta\mathrm{Ric}(Y,Z)$ and $(\mathcal{L}_X\mathrm{Ric})(Y,Z)=\alpha\mathrm{Ric}(Y,Z)+\beta g(Y,Z)$ for arbitrary non-zero smooth functions α and β.

In [3], Azami and Fasihi-Ramandi proved that on the warped product manifolds, the hyperbolic Ricci soliton with Ricci bi-conformal potential vector field is Einstein. Inspiring from this point of view, by a similar proof of Theorem 2.13 given in [3], we obtain a similar result for the sequential warped product manifolds as follows:

Theorem 3.4. *Let $M=(M_1\times_f M_2)\times_h M_3$ be a sequential warped product equipped with the metric $g=(g_1\oplus f^2g_2)\oplus h^2g_3$. If (M,g,X,λ,μ) is a hyperbolic Ricci soliton with Ricci bi-conformal potential vector field X, then M is an Einstein manifold or*

$$1+\lambda\beta+2\alpha\beta+X(\beta)=0,\quad \lambda\alpha+X(\alpha)+\alpha^2+\beta^2-\mu=0.$$

4. Hyperbolic Ricci-Solitons on Sequential Warped Product Space-Times

In this section, we consider hyperbolic Ricci-solitons admitting sequential standard static space-times and sequential generalized Robertson-Walker space-times.

Let (M_i,g_i) be semi-Riemannian manifolds, $1\le i\le 2$, and $f:M_1\longrightarrow\mathbb{R}^+$, $h:M_1\times M_2\longrightarrow\mathbb{R}^+$ two smooth functions. The (n_1+n_2+1)- dimensional *sequential standard static space-time* $\overline{M}$ is the triple product manifold $\overline{M}=(M_1\times_f M_2)\times_h I$ endowed with the metric tensor $\overline{g}=(g_1\oplus f^2g_2)\oplus h^2(-dt^2)$. Here I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I [7].

By using of Lemma 2.3 and Proposition 2.4, it is easy to state the following Corollary:

Corollary 4.1. *Let $(\overline{M}=(M_1\times_f M_2)\times_h I,\overline{g})$ be a sequential standard static space-time and $Y_i,Z_i\in\chi(M_i)$ for $1\le i\le 2$. Then*

1. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_1,Z_1)=(\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1}g_1)(Y_1,Z_1)$,
2. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_2,Z_2)=f^2(\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2}g_2)(Y_2,Z_2)+2X_1(f^2)(\mathcal{L}^2_{X_2}g_2)(Y_2,Z_2)+X_1(X_1(f^2))g_2(Y_2,Z_2)$,
3. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(\partial_t,\partial_t)=-X_1(X_1(h^2))-X_2(X_1(h^2))-X_1(X_2(h^2))-X_2(X_2(h^2))$,
4. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_i,Z_j)=0$, $1\le i,j\le 2$, $i\neq j$,
5. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(\partial_t,Y_j)=0$, $1\le i,j\le 2$

for every vector field $\overline{X}=X_1+X_2+\partial_t$ on $\overline{M}$.

Now we consider a hyperbolic Ricci soliton with the structure of the sequential standard static space-times. By using Theorem 3.1, the following result can be given:

Theorem 4.2. *Let $\overline{M}=(M_1\times_f M_2)\times_h I$ be a sequential standard static space-time equipped with the metric $\overline{g}=(g_1\oplus f^2g_2)\oplus h^2(-dt^2)$. If $(\overline{M},\overline{g},\overline{X},\overline{\lambda},\overline{\mu})$ is a hyperbolic Ricci soliton with $\overline{X}=X_1+X_2+\partial_t$, where $X_i\in\chi(M_i)$ for $1\le i\le 2$ and $\partial_t\in\chi(I)$, then* *(i)* $(M_1, g_1, \lambda X_1 - \frac{n_2}{2}\nabla(\ln f) - \frac{1}{2}\nabla(\ln h^1), 1, \overline{\mu})$ *is a hyperbolic Ricci soliton.*

(ii) M_2 *is an Einstein manifold when* X_2 *a Killing vector field and* $\overline{\mathrm{Hess}}h = \psi\overline{g}$.

(iii) $\overline{\mu}h^2 = \lambda(X_1 + X_2)(h^2) + X_1(X_1(h^2)) + X_2(X_1(h^2)) + X_1(X_2(h^2)) + X_2(X_2(h^2)) - h\Delta h.$

Proof. Let $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\mu})$ be a hyperbolic Ricci soliton with the structure of the sequential warped product. Then from (1), for $\overline{Y}, \overline{Z} \in \chi(\overline{M})$, the equation

$$\overline{\mathrm{Ric}}(\overline{Y}, \overline{Z}) + \lambda(\mathcal{L}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) + (\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) = \overline{\mu}\overline{g}(Y, Z) \tag{3}$$

is satisfied. In the equation (3), using Lemma 2.2, Lemma 2.3 and Corollary 4.1 for vector fields $\overline{Y} = Y_1+Y_2+\partial_t$ and $\overline{Z} = Z_1 + Z_2 + \partial_t$, we get

$$\mathrm{Ric}^1(Y_1, Z_1) - \frac{n_2}{f}\mathrm{Hess}^1 f(Y_1, Z_1) - \frac{1}{h}\overline{\mathrm{Hess}}h(Y_1, Z_1) \tag{4}$$

$$+\lambda(\mathcal{L}^1_{X_1}g_1)(Y_1, Z_1) + (\mathcal{L}^1_{X_1}\mathcal{L}^1_{X_1}g_1)(Y_1, Z_1) = \overline{\mu}g_1(Y_1, Z_1),$$

$$\mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{1}{h}\overline{\mathrm{Hess}}h(Y_2, Z_2) + \lambda f^2\mathcal{L}^2_{X_2}g_2(Y_2, Z_2) \tag{5}$$

$$+\lambda X_1(f^2)g_2(Y_2, Z_2) + (\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2}g_2)(Y_2, Z_2) + 2X_1(f^2)(\mathcal{L}^2_{X_2}g_2)(Y_2, Z_2)$$

$$+X_1(X_1(f^2))g_2(Y_2, Z_2) = \overline{\mu}f^2g_2(Y_2, Z_2)$$

and

$$h\Delta h - \lambda(X_1 + X_2)(h) - X_1(X_1(h^2)) - X_2(X_1(h^2)) - X_1(X_2(h^2)) - X_2(X_2(h^2)) = -\overline{\mu}h^2,$$

which imply *(iii)*.

In the equation (4), by following the same pattern as in the Theorem 3.1, we arrive that $(M_1, g_1, \lambda X_1 - \frac{n_2}{2}\nabla(\ln f) - \frac{1}{2}\nabla(\ln h^1), 1, \overline{\mu})$ is a hyperbolic Ricci soliton. Moreover, in the equation (5), if X_2 is a Killing vector field and $\overline{\mathrm{Hess}}h = \psi\overline{g}$, we obtain that M_2 is an Einstein manifold, which completes the proof. □

Now we consider a hyperbolic Ricci soliton with the structure of the sequential generalized Robertson-Walker space-times. Firstly, we give the definition of the sequential generalized Robertson-Walker space-time.

Let (M_i, g_i) be semi-Riemannian manifolds, $2 \leq i \leq 3$, and $f : I \longrightarrow \mathbb{R}^+$, $h : I \times M_2 \longrightarrow \mathbb{R}^+$ two smooth functions. The $(n_2 + n_3 + 1)$-dimensional *sequential generalized Robertson-Walker space-time* $\overline{M}$ is the triple product manifold $\overline{M} = I \times_f M_2 \times_h M_3$ endowed with the metric tensor $\overline{g} = (-dt^2 \oplus f^2g_2) \oplus h^2g_3$. Here I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I [7].

By using of Lemma 2.3 and Proposition 2.4, it is easy to state the following Corollary:

Corollary 4.3. *Let* $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g})$ *be a sequential standard static space-time and* $Y_i, Z_i \in \chi(M_i)$ *for* $2 \leq i \leq 3$. *Then*

1. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(\partial_t, \partial_t) = 0,$
2. $$(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_2, Z_2) = f^2(\mathcal{L}^2_{X_2}\mathcal{L}^2_{X_2}g_2)(Y_2, Z_2) + 2f\dot{f}(\mathcal{L}^2_{X_2}g_2)(Y_2, Z_2) + \partial_t(\partial_t(f^2))g_2(Y_2, Z_2),$$
3. $$(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_3, Z_3) = h^2(\mathcal{L}^3_{X_3}\mathcal{L}^3_{X_3}g_3)(Y_3, Z_3) + 2(2f\dot{f} + X_2(h^2))(\mathcal{L}^3_{X_3}g_3)(Y_3, Z_3)$$
$$+\partial_t(\partial_t(h^2))g_3(Y_3, Z_3) + X_2(\partial_t(h^2))g_3(Y_3, Z_3) + \partial_t(X_2(h^2))g_3(Y_3, Z_3) + X_2(X_2(h^2))g_3(Y_3, Z_3),$$

4. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(Y_i, Z_j) = 0$, $2 \leq i, j \leq 3$, $i \neq j$,
5. $(\mathcal{L}_{\overline{X}}\mathcal{L}_{\overline{X}}g)(\partial_t, Y_j) = 0$, $2 \leq i, j \leq 3$

for every vector field $\overline{X} = \partial_t + X_1 + X_2$ *on* $\overline{M}$.

We give the following theorem as an application of Theorem 3.1.

Theorem 4.4. *Let* $\overline{M} = (I \times_f M_2) \times_h M_3$ *be a sequential generalized Robertson-Walker space-time. Assume that* $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\mu})$ *is a hyperbolic Ricci soliton with* $\overline{X} = \partial_t + X_2 + X_3$ *on* $\overline{M}$, *where* $X_i \in \chi(M_i)$ *for* $2 \leq i \leq 3$ *and* $\partial_t \in \chi(I)$. *Then*

(i) $-\dfrac{n_2}{f}\ddot{f} - \dfrac{n_3}{h}\dfrac{\partial^2 h}{\partial t^2} = \overline{\mu}$,

(ii) (M_2, g_2) *is Einstein when* $\overline{\mathrm{Hess}h} = \psi\overline{g}$ *and* X_2 *is Killing on* M_2.

(iii) (M_3, g_3) *is Einstein when* X_3 *is Killing on* M_3.

Proof. Assume that $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\mu})$ is a hyperbolic Ricci soliton with the structure of the generalized Robertson-Walker space-time $\overline{M} = (I \times_f M_2) \times_h M_3$. By Lemma 2.2, Lemma 2.3 and Corollary 4.3, the proof is clear. □

Now, we give the following result for gradient hyperbolic Ricci soliton with the structure of the generalized Robertson-Walker space-time.

Theorem 4.5. *Let* $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g})$ *be a sequential generalized Robertson-Walker space-time and* $(\overline{M}, \overline{g}, \nabla u, \overline{\lambda}, \overline{\mu})$ *a hyperbolic Ricci soliton, where*

$$u = \int_a^t f(r)dr \ \text{ for some constant } a \in I,$$

then $\overline{M}$ *is an Einstein manifold with factor* $(\overline{\mu} - 2\lambda\dot{f} - 2f\ddot{f} - 4\dot{f}^2)\overline{g}$.

Proof. Suppose that $X = \nabla u$. Then $X = f(t)\partial_t$.
Let $\{\partial_t, \partial_1, \partial_2, \ldots, \partial_{n_2}, \partial_{n_2+1}, \ldots, \partial_{n_2+n_3}\}$ be an orthonormal basis for $\chi(\overline{M})$. Using the proof of Theorem 4.7 in [2], we have

$$\begin{aligned}
(\mathcal{L}_X\mathcal{L}_X\overline{g})(Y, Z) &= \mathcal{L}_X((\mathcal{L}_X\overline{g})(Y, Z)) - (\mathcal{L}_X\overline{g})(\mathcal{L}_X Y, Z) - (\mathcal{L}_X\overline{g})(Y, \mathcal{L}_X Z) \\
&= 2\mathcal{L}_X(\dot{f}\overline{g}(Y, Z)) - 2\dot{f}\overline{g}(\mathcal{L}_X Y, Z) - 2\dot{f}\overline{g}(Y, \mathcal{L}_X Z) \\
&= 2(X(\dot{f}))\overline{g}(Y, Z) + 2\dot{f}(\mathcal{L}_X\overline{g})(Y, Z) = (2f\ddot{f} + 4\dot{f}^2)\overline{g}(Y, Z)
\end{aligned}$$

Therefore, $\overline{\mathrm{Ric}} = (\overline{\mu} - \lambda\dot{f} - 2f\ddot{f} - 4\dot{f}^2)\overline{g}$ is satisfied. This completes the proof. □

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