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# Hadamard functional fractional integrals and derivatives and fractional differential equations

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**Abstract.** This paper introduces a general type of new version of Hadamard fractional integrals and derivatives with respect to another function and studies some of their properties. Further, we prove the existence results for fractional differential equations with this Hadamard type fractional derivative. The results are established by applying the fixed point theorems. Examples are given to illustrate the theory.

## 1. Introduction

Fractional calculus is a generalization of classical calculus and the concepts of differentiation and integration are defined to an arbitrary non-integer order. There are many fractional derivatives and integrals defined by several authors [2, 3, 7–9, 15–17, 19, 21, 26–29]. Fractional differential equations have been received an increasing attention due to their applications in various fields of engineering and science, economics, mechanics, chemistry, physics, viscoelasticity, finance, aerodynamics, electrodynamics of complex medium [10–13, 22, 23, 32].

The $\psi$-fractional operators are different from other classical operators due to the kernel entering in terms of another function. Kilbas et al. [19] investigated the $\psi$-fractional operators as a generalization of Riemann-Liouville operators. Recently Almedia [4] introduced a new type of fractional derivative, which is known as Caputo $\psi$-fractional derivative, with respect to another function, which extended the classical fractional derivative. Further, he studied the important properties such as Taylor's theorem, Fermat's theorem, and semigroup law.

Abdelhedi [1] established the existence and uniqueness results for fractional differential equations involving $\psi$-Hilfer fractional derivative by using the Picard approximation method. Almeida [5] studied the existence and uniqueness results for $\psi$-Caputo fractional derivative by means of the fixed point technique. Sousa and Oliveira [25] introduced a new fractional derivative with respect to another function which is named as $\psi$-Hilfer fractional derivative, and discussed some important results. Srivastava et al. [6, 30, 31] discussed the solutions of fractional differential equations with different Laplacian operators. Motivated by the above discussion, we have introduced a new fractional derivative called Hadamard functional fractional derivative and study some of its properties.

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The rest of this paper is organized as follows. In Section 2, we introduce Hadamard functional fractional integrals and derivatives and discuss a few fundamental properties. In Section 3, we obtain some semigroup laws for the fractional derivative. In Section 4, we discuss the existence results of the fractional differential equations by using the fixed point theorems. Examples are given in Section 5.

## 2. Notions

In this section, we introduce the notion of new fractional differentiation and integration with respect to another function. For some special cases of $\psi$, we obtain the Riemann-Liouville fractional operator [24], the Caputo-Hadamard fractional derivative [14, 18], and the Caputo-Erdelyi-Kober fractional derivative [20].

Let $C[a,b]$ be the Banach space of continuous functions defined on the interval $[a,b]$, and $C^n[a,b]$ be the Banach space of continuously $n^{th}$ differentiable functions defined on $[a,b]$.

**Definition 2.1.** *(Hadamard Functional Fractional Integrals) Let $f \in C[a,b]$ and $\psi \in C^n[a,b]$ be an increasing function with $\psi'(x) \neq 0$, for all $x \in [a,b]$. Then the left-sided Hadamard functional fractional integral of order $\alpha > 0$ is defined by*

$$ {}^{H}I_{a^+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln \psi(x) - \ln \psi(t))^{\alpha-1} f(t) dt, \tag{1} $$

*and the right-sided Hadamard functional fractional integral of order $\alpha$ is defined by*

$$ {}^{H}I_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\psi'(t)}{\psi(t)} (\ln \psi(t) - \ln \psi(x))^{\alpha-1} f(t) dt. $$

**Definition 2.2.** *(Hadamard Functional Fractional Derivatives) Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $f \in C^{n-1}[a,b]$ and $\psi \in C^n[a,b]$ be an increasing function with $\psi'(x) \neq 0$, for all $x \in [a,b]$. The left-sided Hadamard functional fractional derivative of $f$ of order $\alpha$ is defined by*

$$ \begin{aligned} {}^{H}D_{a^+}^{\alpha,\psi} f(x) &= \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n {}^{H}I_{a^+}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln \psi(x) - \ln \psi(t))^{n-\alpha-1} f(t) dt, \end{aligned} \tag{2} $$

*and the right-sided Hadamard functional fractional derivative of $f$ of order $\alpha$ is defined by*

$$ {}^{H}D_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \int_x^b \frac{\psi'(t)}{\psi(t)} (\ln \psi(t) - \ln \psi(x))^{n-\alpha-1} f(t) dt. $$

It is obvious, if $0 < \alpha < 1$, then we have

$$ \begin{aligned} {}^{H}D_{a^+}^{\alpha,\psi} f(x) &= \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right) {}^{H}I_{a^+}^{1-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right) \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln \psi(x) - \ln \psi(t))^{-\alpha} f(t) dt, \end{aligned} $$

and

$$ {}^{H}D_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( -\frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right) \int_x^b \frac{\psi'(t)}{\psi(t)} (\ln \psi(t) - \ln \psi(x))^{-\alpha} f(t) dt. $$

**Remark 2.1.** The following are some particular cases:

- When $\psi(x) = x$, Hadamard functional fractional derivative(integral) becomes Hadamard fractional derivative(integral) [18].
- When $\psi(x) = e^x$, Hadamard functional fractional derivative(integral) becomes Riemann-Liouville fractional derivative(integral) [24].
- When $\psi(x) = x^\rho$, Hadamard functional fractional derivative(integral) becomes generalized fractional derivative(integral) [20].
- When $\psi(x) = e^{g(x)}$, Hadamard functional fractional derivative(integral) becomes fractional derivative(integral) of a function with respect to another function [4].

**Theorem 2.3.** *Suppose that $f \in C^{n-1}[a,b]$, and $\psi \in C^n[a,b]$. Then, for all $\alpha > 0$, $n-1 < \alpha < n$,*

$$
\begin{aligned}
{}^{H}D^{\alpha,\psi}_{a^+} f(x) &= \frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n (\ln\psi(x) - \ln\psi(a))^{n-\alpha} f(a) \\
&+ \frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x (\ln\psi(x) - \ln\psi(t))^{n-\alpha} f'(t)dt,
\end{aligned}
$$

*and*

$$
\begin{aligned}
{}^{H}D^{\alpha,\psi}_{b^-} f(x) &= \frac{1}{\Gamma(n+1-\alpha)}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n (\ln\psi(b) - \ln\psi(x))^{n-\alpha} f(b) \\
&- \frac{1}{\Gamma(n+1-\alpha)}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b (\ln\psi(t) - \ln\psi(x))^{n-\alpha} f'(t)dt.
\end{aligned}
$$

**Proof:** Applying integration by parts formula to equation (2), by taking $v(t) = f(t)$ and $u'(t) = \frac{\psi'(t)}{\psi(t)}(\ln\psi(x) - \ln\psi(t))^{n-\alpha-1}$, we get

$$
\begin{aligned}
{}^{H}D^{\alpha,\psi}_{a^+} f(x) =& \frac{1}{\Gamma(n-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \Bigg[\left[\frac{-(\ln\psi(x) - \ln\psi(t))^{n-\alpha}}{(n-\alpha)} f(t)\right]_a^x \\
&+ \int_a^x \frac{(\ln\psi(x) - \ln\psi(t))^{n-\alpha}}{(n-\alpha)} f'(t)dt\Bigg] \\
=& \frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n (\ln\psi(x) - \ln\psi(a))^{n-\alpha} f(a) \\
&+ \frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x (\ln\psi(x) - \ln\psi(t))^{n-\alpha} f'(t)dt.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
{}^{H}D^{\alpha,\psi}_{b^-} f(x) =& \frac{1}{\Gamma(n+1-\alpha)}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n (\ln\psi(b) - \ln\psi(x))^{n-\alpha} f(b) \\
&- \frac{1}{\Gamma(n+1-\alpha)}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b (\ln\psi(t) - \ln\psi(x))^{n-\alpha} f'(t)dt.
\end{aligned}
$$

**Remark 2.2.** In accordance with Theorem 2.3, it is clear that

$$\begin{aligned}\lim_{\alpha\to n^+} {}^{H}D^{\alpha,\psi}_{a^+} f(x) = &\lim_{\alpha\to n^+}\Bigg[\frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n (\ln\psi(x)-\ln\psi(a))^{n-\alpha} f(a)\\ &+\frac{1}{\Gamma(n+1-\alpha)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x (\ln\psi(x)-\ln\psi(t))^{n-\alpha}\frac{d}{dt}f(t)dt\Bigg]\\ =&\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(a)+\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n\int_a^x \frac{d}{dt}f(t)dt\\ =&\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(a)+\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(x)-\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(a)\\ =&\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(x).\end{aligned}$$

Similarly we can obtain

$$\lim_{\alpha\to n^-} {}^{H}D^{\alpha,\psi}_{b^-} f(x) = \left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(x).$$

## 3. Main Properties

In this section we prove main properties of Hadamard functional fractional derivatives and integrals such as the composition, semigroup property, etc.

**Theorem 3.1.** *Let $n-1<\alpha,\beta<n$, $f\in C[a,b]$ and $\psi\in C^n[a,b]$. Then the following semigroup properties hold:*

$${}^{H}I^{\alpha,\psi}_{a^+}{}^{H}I^{\beta,\psi}_{a^+} f(x) = {}^{H}I^{\alpha+\beta,\psi}_{a^+} f(x) \ \ and \ \ {}^{H}I^{\alpha,\psi}_{b^-}{}^{H}I^{\beta,\psi}_{b^-} f(x) = {}^{H}I^{\alpha+\beta,\psi}_{b^-} f(x).$$

**Proof:** Using Definition 2.1, we have

$$\begin{aligned}{}^{H}I^{\alpha,\psi}_{a^+}{}^{H}I^{\beta,\psi}_{a^+} f(x) =&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{\psi'(t)}{\psi(t)}(\ln\psi(x)-\ln\psi(t))^{\alpha-1}\\ &\int_a^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\beta-1} f(\tau)d\tau dt.\end{aligned}$$

By Fubini's theorem, we interchange the order of integration, and get

$$\begin{aligned}{}^{H}I^{\alpha,\psi}_{a^+}{}^{H}I^{\beta,\psi}_{a^+} f(x) =&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_\tau^x \frac{\psi'(t)}{\psi(t)}(\ln\psi(x)-\ln\psi(t))^{\alpha-1}\\ &\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\beta-1} f(\tau)dtd\tau\\ =&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{\psi'(\tau)}{\psi(\tau)}f(\tau)\int_\tau^x \frac{\psi'(t)}{\psi(t)}(\ln\psi(x)-\ln\psi(t))^{\alpha-1}\\ &(\ln\psi(t)-\ln\psi(\tau))^{\beta-1}dtd\tau.\end{aligned}$$

The substitution $\ln\psi(t) = \ln\psi(\tau) + s(\ln\psi(x) - \ln\psi(\tau))$ yields

$$
\begin{aligned}
{}^{H}I_{a^+}^{\alpha,\psi}\,{}^{H}I_{a^+}^{\beta,\psi} f(x) =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{\psi'(\tau)}{\psi(\tau)} f(\tau) \int_0^1 \Big( (\ln\psi(x) - \ln\psi(\tau))(1-s) \Big)^{\alpha-1} \\
& \Big( s(\ln\psi(x) - \ln\psi(\tau)) \Big)^{\beta-1} (\ln\psi(x) - \ln\psi(\tau)) ds d\tau \\
=& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{\psi'(\tau)}{\psi(\tau)} f(\tau) \int_0^1 (\ln\psi(x) - \ln\psi(\tau))^{\alpha-1} (1-s)^{\alpha-1} s^{\beta-1} \\
& (\ln\psi(x) - \ln\psi(\tau))^{\beta-1+1} ds d\tau \\
=& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{\psi'(\tau)}{\psi(\tau)} f(\tau) (\ln\psi(x) - \ln\psi(\tau))^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds d\tau \\
=& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{\psi'(\tau)}{\psi(\tau)} f(\tau) (\ln\psi(x) - \ln\psi(\tau))^{\alpha+\beta-1} \left( \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \right) d\tau \\
=& \frac{1}{\Gamma(\alpha+\beta)} \int_a^x \frac{\psi'(\tau)}{\psi(\tau)} f(\tau) (\ln\psi(x) - \ln\psi(\tau))^{\alpha+\beta-1} d\tau \\
=& {}^{H}I_{a^+}^{\alpha+\beta,\psi} f(x).
\end{aligned}
$$

Similarly, we have

$$
{}^{H}I_{b^-}^{\alpha,\psi}\,{}^{H}I_{b^-}^{\beta,\psi} f(x) = {}^{H}I_{b^-}^{\alpha+\beta,\psi} f(x).
$$

**Theorem 3.2.** *Given functions $f \in C[a,b]$, $\psi \in C^n[a,b]$ and $n-1 < \alpha < n$ with $n \in \mathbb{N}$, we get*

$$
{}^{H}D_{a^+}^{\alpha,\psi}\,{}^{H}I_{a^+}^{\alpha,\psi} f(x) = f(x) \;\; \text{and} \;\; {}^{H}D_{b^-}^{\alpha,\psi}\,{}^{H}I_{b^-}^{\alpha,\psi} f(x) = f(x).
$$

**Proof:** By Definition (2), and using Leibnitz's formula, we get

$$
\begin{aligned}
{}^{H}D_{a^+}^{\alpha,\psi}\,{}^{H}I_{a^+}^{\alpha,\psi} f(x) =& \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n {}^{H}I_{a^+}^{n-\alpha,\psi}\,{}^{H}I_{a^+}^{\alpha,\psi} f(x) \\
=& \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n {}^{H}I_{a^+}^{n,\psi} f(x) \\
=& \frac{1}{\Gamma(n)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln\psi(x) - \ln\psi(t))^{n-1} f(t) dt \\
=& \frac{1}{\Gamma(n+1)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \Big( (\ln\psi(x) - \ln\psi(a))^n f(a) \\
& + \int_a^x (\ln\psi(x) - \ln\psi(t))^n f'(t) dt \Big) \\
=& \frac{1}{\Gamma(n+1)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^{n-1} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right) \Big( (\ln\psi(x) - \ln\psi(a))^n f(a) \\
& + \int_a^x (\ln\psi(x) - \ln\psi(t))^n f'(t) dt \Big) \\
=& \frac{n}{n\Gamma(n)} \left( \frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^{n-1} \Big( (\ln\psi(x) - \ln\psi(a))^{n-1} f(a) \\
& + \int_a^x (\ln\psi(x) - \ln\psi(t))^{n-1} f'(t) dt \Big)
\end{aligned}
$$

$$=\frac{(n-1)}{(n-1)!}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^{n-2}\Big((\ln\psi(x)-\ln\psi(a))^{n-2}f(a)$$
$$+\int_a^x(\ln\psi(x)-\ln\psi(t))^{n-2}f'(t)dt\Big)$$
$$=\frac{1}{(n-2)!}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^{n-2}\Big((\ln\psi(x)-\ln\psi(a))^{n-2}f(a)$$
$$+\int_a^x(\ln\psi(x)-\ln\psi(t))^{n-2}f'(t)dt\Big)$$
$$\vdots$$
$$=f(a)+\int_a^x f'(t)dt$$
$$=f(x).$$

Similarly, we prove

$$^{H}D_{b^-}^{\alpha,\psi}\,{}^{H}I_{b^-}^{\alpha,\psi}f(x)=f(x).$$

In a similar way one can establish the following result.

**Theorem 3.3.** *Given a function $f\in C^{n-1}[a,b]$ and $n-1<\alpha<n$ with $n\in\mathbb{N}$, we get*

$$^{H}I_{a^+}^{\alpha,\psi}\,{}^{H}D_{a^+}^{\alpha,\psi}f(x)=f(x)-\sum_{k=0}^{n-1}\left[\left(\frac{\psi(t)}{\psi'(t)}\frac{d}{dt}\right)^k f(t)\right]_{t=a}\left(\frac{(\ln\psi(x)-\ln\psi(a))^k}{k!}\right),$$

*and*

$$^{H}I_{b^-}^{\alpha,\psi}\,{}^{H}D_{b^-}^{\alpha,\psi}f(x)=f(x)-\sum_{k=0}^{n-1}\left[\left(-\frac{\psi(t)}{\psi'(t)}\frac{d}{dt}\right)^k f(t)\right]_{t=b}\left(\frac{(\ln\psi(b)-\ln\psi(x))^k}{k!}\right).$$

The compositions of powers of derivatives and integrals are given by the next result.

**Theorem 3.4.** *If $f\in C^{m(n-1)}[a,b]$ and $\alpha>0$, then for all $k,m\in\mathbb{N}$, we get*

$$\left({}^{H}I_{a^+}^{\alpha,\psi}\right)^k\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(x)=\frac{({}^{H}D_{a^+}^{\alpha,\psi})^m f(c)}{\Gamma(k\alpha+1)}(\ln\psi(x)-\ln\psi(a))^{k\alpha},$$

*and*

$$\left({}^{H}I_{b^-}^{\alpha,\psi}\right)^k\left({}^{H}D_{b^-}^{\alpha,\psi}\right)^m f(x)=\frac{({}^{H}D_{b^-}^{\alpha,\psi})^m f(d)}{\Gamma(k\alpha+1)}(\ln\psi(b)-\ln\psi(x))^{k\alpha},$$

*for some $c\in(a,x)$ and $d\in(x,b)$.*

**Proof:** Applying the semigroup property for fractional integrals, namely Theorem 3.1, we get

$$\left({}^{H}I_{a^+}^{\alpha,\psi}\right)^k f(x)={}^{H}I_{a^+}^{\alpha,\psi}\dots{}^{H}I_{a^+}^{\alpha,\psi}f(x)={}^{H}I_{a^+}^{k\alpha,\psi}f(x).$$

Therefore,

$$\left({}^{H}I_{a^+}^{\alpha,\psi}\right)^k\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(x)={}^{H}I_{a^+}^{k\alpha,\psi}\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(x)$$
$$=\frac{1}{\Gamma(k\alpha)}\int_a^x\frac{\psi'(t)}{\psi(t)}(\ln\psi(x)-\ln\psi(t))^{k\alpha-1}\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(t)dt.$$

For some $c \in (a, x)$ by applying mean value theorem, we obtain

$$\left({}^{H}I_{a^+}^{\alpha,\psi}\right)^k \left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(x) = \frac{\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(c)}{\Gamma(k\alpha)} \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln\psi(x) - \ln\psi(t))^{k\alpha-1} dt$$
$$= \frac{\left({}^{H}D_{a^+}^{\alpha,\psi}\right)^m f(c)}{\Gamma(k\alpha+1)} (\ln\psi(x) - \ln\psi(a))^{k\alpha}.$$

Similarly, we get

$$\left({}^{H}I_{b^-}^{\alpha,\psi}\right)^k \left({}^{H}D_{b^-}^{\alpha,\psi}\right)^m f(x) = \frac{\left({}^{H}D_{b^-}^{\alpha,\psi}\right)^m f(d)}{\Gamma(k\alpha+1)} (\ln\psi(b) - \ln\psi(x))^{k\alpha},$$

for some $d \in (x, b)$.

In addition, we can obtain the next result as a consequence of the above.

**Theorem 3.5.** *If $f \in C^{m+n-1}[a, b]$, with $m, n \in \mathbb{N}$, and $n - 1 < \alpha < n$, then*

$$\left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m {}^{H}D_{a^+}^{\alpha,\psi} f(x) = {}^{H}D_{a^+}^{\alpha+m,\psi} f(x), \quad \textit{and}$$
$$\left(-\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m {}^{H}D_{b^-}^{\alpha,\psi} f(x) = {}^{H}D_{b^-}^{\alpha+m,\psi} f(x).$$

**Proof:** Firstly, we recall definition of ${}^{H}D_{a^+}^{\alpha,\psi} f(x)$, hence, we have

$$\left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m {}^{H}D_{a^+}^{\alpha,\psi} f(x) = \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^n$$
$$\frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln\psi(x) - \ln\psi(t))^{n-\alpha-1} f(t) dt$$
$$= \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^{m+n} \frac{1}{\Gamma((n+m) - (\alpha+m))}$$
$$\int_a^x \frac{\psi'(t)}{\psi(t)} (\ln\psi(x) - \ln\psi(t))^{(n+m)-(\alpha+m)-1} f(t) dt$$
$$= {}^{H}D_{a^+}^{\alpha+m,\psi} f(x).$$

Similarly, we get

$$\left(-\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m {}^{H}D_{b^-}^{\alpha,\psi} f(x) = {}^{H}D_{b^-}^{\alpha+m,\psi} f(x).$$

**Remark 3.1.** *Observe that*

$${}^{H}D_{a^+}^{\alpha,\psi} \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m f(x) \neq {}^{H}D_{a^+}^{\alpha+m,\psi} f(x),$$

*and*

$${}^{H}D_{b^-}^{\alpha,\psi} \left(-\frac{\psi(x)}{\psi'(x)} \frac{d}{dx}\right)^m f(x) \neq {}^{H}D_{b^-}^{\alpha+m,\psi} f(x).$$

Let us close this section by considering special functions and obtaining their fractional integrals and derivatives.

**Lemma 3.6.** *Consider the functions*

$$f(x) = (\ln \psi(x) - \ln \psi(a))^{\beta-1}, \quad and \quad g(x) = (\ln \psi(b) - \ln \psi(x))^{\beta-1},$$

*where $\beta > 0$, then, for $\alpha > 0$,*

$${}^{H}I_{a^+}^{\alpha,\psi} f(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln \psi(x) - \ln \psi(a))^{\beta+\alpha-1},$$

*and*

$${}^{H}I_{b^-}^{\alpha,\psi} g(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln \psi(b) - \ln \psi(x))^{\beta+\alpha-1}.$$

**Proof:** By Definition 2.1, we obtain

$$\begin{aligned}{}^{H}I_{a^+}^{\alpha,\psi} f(x) =& \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\psi'(t)}{\psi(t)}(\ln \psi(x) - \ln \psi(t))^{\alpha-1} f(t)dt\\ =&\frac{(\ln \psi(x) - \ln \psi(a))^{\alpha-1}}{\Gamma(\alpha)}\int_a^x \frac{\psi'(t)}{\psi(t)}\left(1 - \frac{(\ln \psi(t) - \ln \psi(a))}{(\ln \psi(x) - \ln \psi(a))}\right)^{\alpha-1}\\ &(\ln \psi(t) - \ln \psi(a))^{\beta-1}dt.\end{aligned}$$

With the change of variable $u = \frac{(\ln \psi(t) - \ln \psi(a))}{(\ln \psi(x) - \ln \psi(a))}$, and with the help of beta function, we obtain

$$\begin{aligned}{}^{H}I_{a^+}^{\alpha,\psi} f(x) =&\frac{(\ln \psi(x) - \ln \psi(a))^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}(\ln \psi(x) - \ln \psi(a))^{\beta}du\\ =&\frac{(\ln \psi(x) - \ln \psi(a))^{\beta+\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du\\ =&\frac{(\ln \psi(x) - \ln \psi(a))^{\beta+\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\beta+\alpha)}\\ =&\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln \psi(x) - \ln \psi(a))^{\beta+\alpha-1}.\end{aligned}$$

Similarly we can get

$${}^{H}I_{b^-}^{\alpha,\psi} g(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln \psi(b) - \ln \psi(x))^{\beta+\alpha-1}.$$

The derivative can be obtained similarly and hence we omit its proof.

**Lemma 3.7.** *If the functions*

$$f(x) = (\ln \psi(x) - \ln \psi(a))^{\beta-1}, \quad and \quad g(x) = (\ln \psi(b) - \ln \psi(x))^{\beta-1},$$

*then, for $\alpha, \beta > 0$,*

$${}^{H}D_{a^+}^{\alpha,\psi} f(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+n)}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n(\ln \psi(x) - \ln \psi(a))^{\beta-\alpha+n-1},$$

*and*

$${}^{H}D_{b^-}^{\alpha,\psi} g(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+n)}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n(\ln \psi(b) - \ln \psi(x))^{\beta-\alpha+n-1}.$$

## 4. Existence Results

In this section we prove existence theorems for Hadamard functional fractional differential equations by using fixed point theorems. Let $E = C(J, \mathbb{R})$ be the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with sup norm.

Consider the following fractional differential equation

$$\begin{aligned} {}^{H}D^{\alpha,\psi}_{a^+}u(t) =& f(t, u(t)), \ t \in J = [a, b], \\ u(a) =& u_a, \end{aligned} \tag{3}$$

where ${}^{H}D^{\alpha,\psi}_{a^+}$ is the Hadamard functional fractional derivative of order $0 < \alpha < 1$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\psi \in C^1(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. Applying Theorem 3.3, the equation (3) is equivalent to the following integral equation

$$u(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} f(\tau, u(\tau)) d\tau.$$

Now, in our first result, we use the Banach fixed point theorem to prove the existence and uniqueness of a solution to problem (3).

**Theorem 4.1.** *If there exists a positive constant L such that*

$$|f(t, u) - f(t, v)| \leq L|u - v|,$$

*for all $t \in J$ and $u, v \in \mathbb{R}$ with*

$$\frac{L}{\Gamma(\alpha + 1)} (\ln \psi(b) - \ln \psi(a))^{\alpha} < 1,$$

*then the problem* (3) *has a unique solution on J.*

**Proof:** Define the operator $F : E \to E$ by

$$Fu(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} f(\tau, u(\tau)) d\tau.$$

Let $u, v \in E$, then we have

$$\begin{aligned} |Fu(t) - Fv(t)| \leq& \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau \\ \leq& \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} L|u(\tau) - v(\tau)| d\tau. \end{aligned}$$

Taking the supremum, we obtain

$$\begin{aligned} \|Fu - Fv\| \leq& \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} L\|u - v\| d\tau \\ \leq& \frac{L\|u - v\|}{\Gamma(\alpha)} \int_a^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} d\tau \\ \leq& \frac{L}{\Gamma(\alpha + 1)} (\ln \psi(b) - \ln \psi(a))^{\alpha} \|u - v\|. \end{aligned}$$

Thus $F$ is a contraction mapping and so by the Banach fixed point theorem, $F$ has a unique fixed point $u \in E$ such that $Fu(t) = u(t)$. This fixed point is the solution of the problem (3).

Next, we apply the Schauder fixed point theorem to prove the existence of a solution to problem (3).

**Theorem 4.2.** *If $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $M > 0$ such that*

$$|f(t,u)| \leq M,$$

*for all $t \in J$ and $u \in \mathbb{R}$, then the problem* (3) *has at least one solution on $J$.*

**Proof:** Choose $r \geq \frac{M}{\Gamma(\alpha+1)}(\ln\psi(b) - \ln\psi(a))^{\alpha}$ and $S = \{u \in E : \|u - u_a\| \leq r\}$. Clearly $S$ is a bounded closed convex subset of $E$. Define the operator $F : S \to S$ by

$$Fu(t) = u_a + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} f(\tau, u(\tau))d\tau.$$

Since $f$ is a continuous function, clearly $F$ is also continuous. First we prove that $F$ maps $S$ into itself.

Let $u \in S$, then for $t \in J$, we get

$$\begin{aligned}|Fu(t) - u_a| \leq& \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}|f(\tau, u(\tau))|d\tau\\ \leq& \frac{M}{\Gamma(\alpha)}\int_a^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}d\tau\\ \leq& \frac{M}{\Gamma(\alpha+1)}(\ln\psi(b) - \ln\psi(a))^{\alpha} \leq r.\end{aligned}$$

Therefore $F$ is bounded and maps $S$ into itself. Next we show that $F$ maps $S$ into a precompact subset of $S$. Let us prove that $F(S) = \{Fu : u \in S\}$ is an equicontinuous family of functions. For $a < t_1 < t_2$, we have

$$\begin{aligned}|Fu(t_2) - Fu(t_1)| \leq& \frac{1}{\Gamma(\alpha)}\int_a^{t_2} \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1}|f(\tau, u(\tau))|d\tau\\ &- \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t_1) - \ln\psi(\tau))^{\alpha-1}|f(\tau, u(\tau))|d\tau\\ \leq& \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \frac{\psi'(\tau)}{\psi(\tau)}\Big((\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} - (\ln\psi(t_1) - \ln\psi(\tau))^{\alpha-1}\Big)\\ &|f(\tau, u(\tau))|d\tau + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1}|f(\tau, u(\tau))|d\tau\\ \leq& \frac{M}{\Gamma(\alpha+1)}\Big[(\ln\psi(t_2) - \ln\psi(a))^{\alpha} - (\ln\psi(t_1) - \ln\psi(a))^{\alpha}\\ &- (\ln\psi(t_2) - \ln\psi(t_1))^{\alpha} + (\ln\psi(t_2) - \ln\psi(t_1))^{\alpha}\Big],\end{aligned}$$

which tends to zero as $t_2 \to t_1$. Thus by Arzela-Ascoli's theorem, $S$ is precompact. Hence by the Schauder fixed point theorem, $F$ has a fixed point in $S$ and any fixed point of $F$ is a solution of the problem (3).

**Remark 4.1.** The above results can be easily extended to the following fractional integrodifferential equation

$$\begin{aligned}{}^{H}D_{a^+}^{\alpha,\psi}u(t) =& f\Big(t, u(t), \int_a^t h(t, s, u(s))ds\Big),\ t \in J = [a, b], \qquad (4)\\ u(a) =& u_a,\end{aligned}$$

where ${}^{H}D_{a^+}^{\alpha,\psi}$, $0 < \alpha < 1$, $\psi$ are as before and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Applying Theorem 3.3, the equation (4) is equivalent to the following integral equation

$$u(t) = u_a + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1} f\Big(s, u(s), \int_a^s h(s, \tau, u(\tau))d\tau\Big)ds.$$

Using similar arguments as in the Theorems 4.1 and 4.2 we can establish existence results for the above integral equations.

## 5. Examples

Some examples are introduced in this section. We begin by some graphical shapes of the Hadamard functional fractional integrals with particular cases.

**Example 5.1.** Consider the Hadamard functional fractional integrals introduced in Lemma 3.6. In this special cases, when $\beta = k+1$, we get

$$^{H}I_{a^+}^{\alpha,\psi} f(x) = \frac{k!}{\Gamma(k+1+\alpha)} (\ln\psi(x) - \ln\psi(a))^{k+\alpha}.$$

Put $k = 2$ and $a = 0$, we get

$$^{H}I_{0^+}^{\alpha,\psi} f(x) = \frac{2!}{\Gamma(3+\alpha)} (\ln\psi(x) - \ln\psi(0))^{2+\alpha}.$$

If $\alpha = 1$, we get

$$^{H}I_{0^+}^{1,\psi} f(x) = \frac{1}{3} (\ln\psi(x) - \ln\psi(0))^{3}.$$

In Figure 1, 2, 3 and 4, we show some graphs of $^{H}I_{0^+}^{\alpha,\psi} f(x)$, for different values of $\alpha$ and different kernels $\psi$.

For Figure.1, $\psi(x) = x+1$ and $f(x) = \frac{2}{\Gamma(\alpha+3)}[\ln(x+1)]^{\alpha+2}$.

For Figure.2, $\psi(x) = \sqrt{(}x+1)$ and $f(x) = \frac{2}{\Gamma(\alpha+3)}[\frac{1}{2}\ln(x+1)]^{\alpha+2}$.

For Figure.3, $\psi(x) = e^x$ and $f(x) = \frac{2}{\Gamma(\alpha+3)} x^{\alpha+2}$.

For Figure.4, $\psi(x) = (x+1)^2$ and $f(x) = \frac{2}{\Gamma(\alpha+3)}[2\ln(x+1)]^{\alpha+2}$.

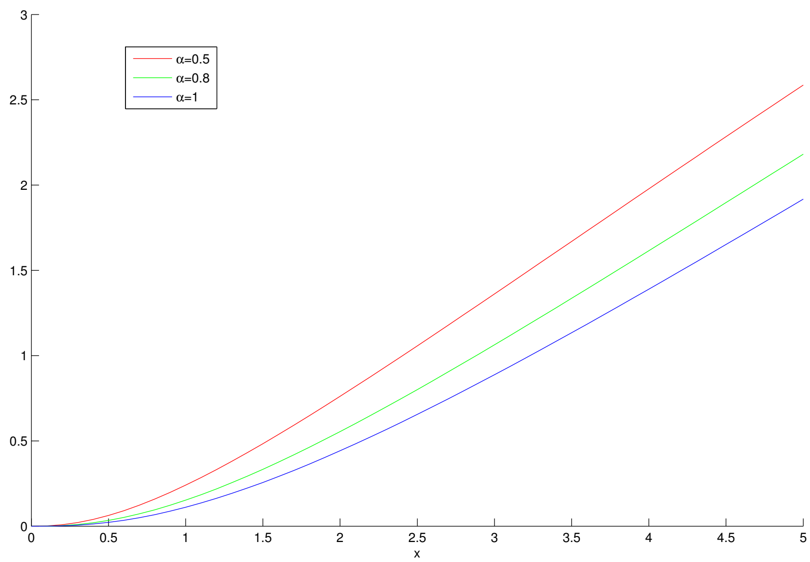


Figure 1: Graph of $^{H}I_{0^+}^{\alpha,\psi} f(x)$, for the kernel $\psi(x) = x+1$ and for different values of $\alpha = 0.5, 0.8, 1$.

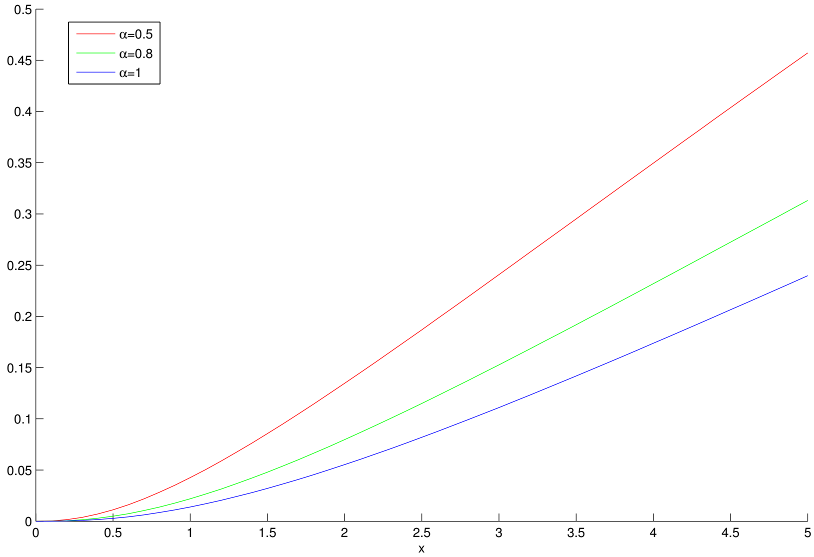


Figure 2: Graph of ${}^{H}I_{0^+}^{\alpha,\psi} f(x)$, for the kernel $\psi(x) = \sqrt{x+1}$ and for different values of $\alpha = 0.5, 0.8, 1$.

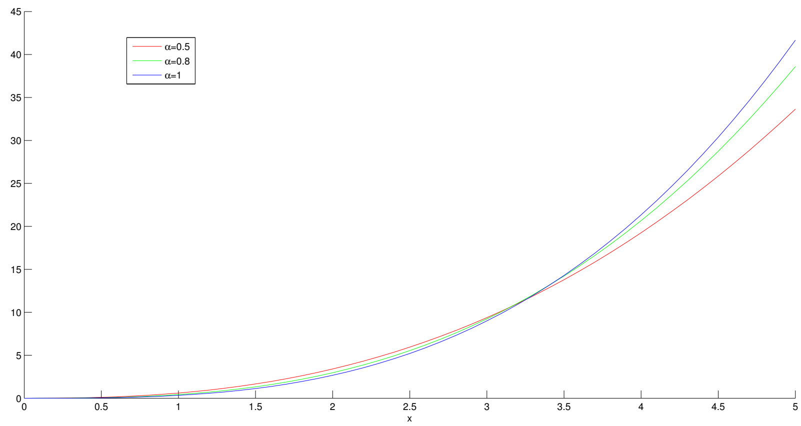


Figure 3: Graph of ${}^{H}I_{0^+}^{\alpha,\psi} f(x)$, for the kernel $\psi(x) = e^x$ and for different values of $\alpha = 0.5, 0.8, 1$.

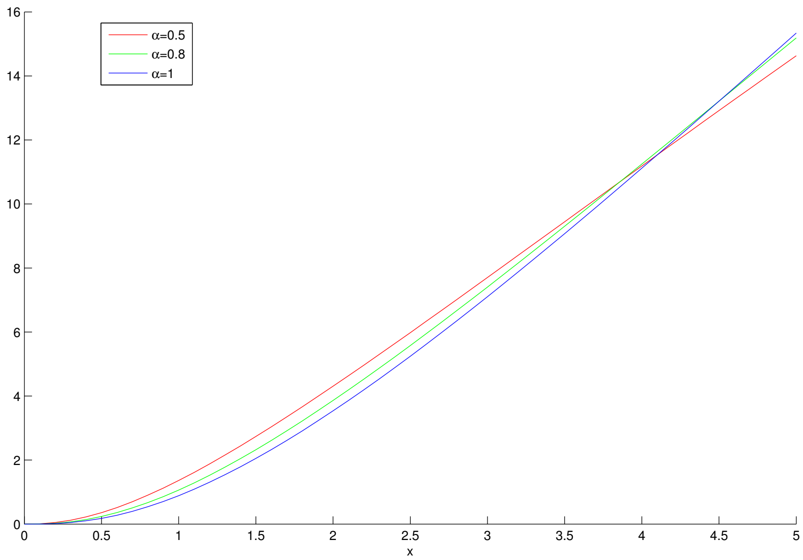


Figure 4: Graph of ${}^{H}I^{\alpha,\psi}_{0^+}f(x)$, for the kernel $\psi(x) = (x+1)^2$ and for different values of $\alpha = 0.5, 0.8, 1$.

**Example 5.2.** Consider the fractional differential equation

$$\begin{aligned} {}^{H}D^{\alpha,\psi}_{a^+}u(t) =& \frac{u(t)}{5(1+t)} + \frac{e^{-t}}{4},\ t \in J, \\ u(0) =& 1, \end{aligned} \tag{5}$$

where $J = [0,1]$, $X = \mathbb{R}$, $\psi(t) = 2^t$ and $\alpha = \frac{1}{2}$. Here the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(t,u) = \frac{u(t)}{5(1+t)} + \frac{e^{-t}}{4},$$

and is continuous. Further, $|f(t,u) - f(t,v)| \le \frac{1}{5(1+t)}|u - v|$. By a simple calculation, we see that

$$\begin{aligned} \frac{L}{\Gamma(\alpha+1)}(\ln\psi(b) - \ln\psi(a))^{\alpha} =& \frac{L}{\Gamma(\frac{1}{2}+1)}(\ln 2 - \ln 1)^{\frac{1}{2}} \\ & < 0.1878 < 1. \end{aligned}$$

Therefore, by Theorem 4.1, the problem (5) has a unique solution on $J$.

## 6. Conclusion

This paper introduced a new type of Hadamard fractional integrals and derivatives with respect to another function and studied several of their properties. Existence of solutions for fractional differential equations with this new Hadamard fractional derivative are established by utilizing the Banach and Schauder fixed point theorems. Theories developed in this work are illustrated with examples. These results provide a framework for further research and applications of fractional calculus in various fields.

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