Filomat 38:3 (2024), 1033–1041
https://doi.org/10.2298/FIL2403033M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On equidistant parabolic Kähler manifolds and geodesic mappings

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Abstract. We study equdistant parabolically Kähler spaces which are generalizations of classical and hyperbolical Kähler spaces. We find the metric form of these spaces in a special system of coordinates. We also find properties of these spaces under geodesic mappings including the projective corresponding metric form.

1. Introduction

In the present paper we consider spaces which, by analogy with Kählerian and hyperbolic Kählerian spaces, we will call parabolic Kählerian spaces $\mathbb{K}_n$. Note that this class of spaces was introduced in [36] when studying spaces over algebras. On the other hand, Sinyukov introduced equidistant spaces [31]. They are pseudo-Riemannian spaces in which concircular vector fields exist, defined by Yano [38]. These vector fields first appear in the work of Brinkmann [4] in the study of conformal mappings of Einstein spaces. As we know [29, 31], equidistant spaces of the basic type always admit geodesic mappings. The problems shown above are studied in detail in [14]. Interesting problems about equidistant spaces, surfaces of revolution and the existence of bifurcations of geodesics on them were studied in [6, 19, 22, 23].

We have constructed metrics of all equidistant basic type parabolic Kähler spaces which admit non-trivial geodesic mappings. This is a continuation of the work [25], and extends the results obtained in [10, 13] for Kähler and hyperbolic Kähler spaces.

Finally, we continue the detailed study of geodesic mappings of parabolically Kähler spaces, which are necessarily equidistant. This property follows from more general results about geodesic mappings in [14] for covariantly constant tensors. These studies are closely connected with works on geodesic and more general mappings of special spaces with structures, for example [1–5, 7, 18, 20, 21, 31–35]. The works cited above are mainly of a local character, global issues of (pseudo-) Riemannian manifolds were devoted to articles [11, 12, 14–16, 24, 39, 41].

2020 *Mathematics Subject Classification*. Primary 53B30; Secondary 53B21.
Keywords. parabolically Kähler spaces, equidistant spaces, geodesic mapping, metric form
Received: 19 July 2023; Accepted: 08 August 2023
Communicated by Mića S. Stanković
Research supported by the project IGA PrF 2023015 Palacky University Olomouc.
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2. Parabolic Kähler Spaces

By analogy with the definition of elliptically and hyperbolically Kähler spaces [13, 36, 41] we introduce the notion of parabolic Kähler spaces:

Definition 2.1. *An even-dimensional Riemannian space will be called a parabolic Kähler space $\mathbb{K}_n$ if along with the metric tensor g there exists a structure F satisfying the following conditions:*

$$a)\ F^2 = 0,\ \ b)\ g(X, FX) = 0,\ \ c)\ \nabla F = 0,\ \ d)\ \operatorname{rank} F = m = n/2, \tag{1}$$

where X is any vector fild and ∇ denotes the Levi-Civita connection in $\mathbb{K}_n$.

Taking into account the studies [36, p. 137] in the adapted coordinate system x in which the structure F has the form

$$F^{a+m}_{b} = \delta^a_b,\ \ F^a_b = F^{a+m}_{b+m} = F^a_{b+m} = 0, \tag{2}$$

where $a, b, c, \dots = 1, 2, \dots, m$; $m = n/2$; δ^h_i is the Kronecker symbol;
components of the metric tensor $g_{ij}(x)$ and the Christoffel symbols $\Gamma^h_{ij}(x)$ satisfy the relations:

$$g_{a\,b+m} + g_{a+m\,b} = 0,\ \ g_{a+m\,b+m} = 0, \tag{3}$$

$$\Gamma^a_{bc} = \Gamma^{a+m}_{b\,c+m},\quad \Gamma^a_{b\,c+m} = \Gamma^a_{b+m\,c+m} = \Gamma^{a+m}_{b+m\,c+m} = 0.$$

For the Christoffel symbols of the 1st kind in the adapted coordinate system the conditions are fulfilled

$$\Gamma_{ab\,c+m} + \Gamma_{a+m\,b\,c} = 0,\quad \Gamma_{a+m\,b+m\,c} = \Gamma_{a\,b+m\,c+m} = \Gamma_{a+a\,b+m\,c+m} = 0. \tag{4}$$

Hence, on the basis of the definition of the Christoffel symbols of the 1st kind and properties of the metric tensor (3) it is easy to find that in adapted coordinate system the following conditions are fulfilled:

$$\partial_{c+m} g_{a\,b+m} = 0, \tag{5}$$

$$\partial_a g_{b\,c+m} + \partial_b g_{c\,a+m} + \partial_c g_{a\,b+m} = 0, \tag{6}$$

$$\partial_{a+m} g_{b\,c} - \partial_{b+m} g_{a\,c} - \partial_a g_{b\,c+m} + \partial_b g_{a\,c+m} = 0, \tag{7}$$

where $\partial_i = \partial/\partial x^i$.

It is easy to see that the Riemannian space, the metric tensor of which in some coordinate system satisfies conditions (3), (5), (6) and (7), is a parabolic $\mathbb{K}_n$ space whose structure in this coordinate system is defined by formulas (2).

Let us analyze conditions (5), (6), and (7). It follows from (5) that

$$g_{a\,b+m} = g_{a\,b+m}(x^1, x^2, \dots, x^m).$$

As shown in [36], it follows from (6) that the functions $\mathcal{F}(x^1, x^2, \dots, x^m)$ satisfies

$$g_{a\,b+m} = \partial_b \mathcal{F}_a - \partial_a \mathcal{F}_b\,.$$

Then formulas (7) take the following form:

$$\partial_{a+m} g_{b\,c} - \partial_{b+m} g_{a\,c} - \partial_{ac} \mathcal{F}_b + \partial_{bc} \mathcal{F}_a = 0. \tag{8}$$

Let's make equations with respect to the unknown functions $\Phi_a(x)$:

$$\partial_{b+m} \Phi_a = g_{a\,b} - x^{c+m} \partial_{ac} \mathcal{F}_b.$$

These equations are, by virtue of (8), completely integrable, hence they have a solution. Thus,

$$g_{ab} = \partial_{b+m}\Phi_a + x^{c+m}\partial_{ac}\mathcal{F}_b. \tag{9}$$

Since g_{ij} is a symmetric tensor, it follows from (9) that

$$\partial_{b+m}\Phi_a - \partial_{a+m}\Phi_b + x^{c+m}(\partial_{ac}\mathcal{F}_b - \partial_{bc}\mathcal{F}_a) = 0. \tag{10}$$

Similarly, we make equations with respect to the unknown function $\Phi(x)$:

$$\partial_{a+m}\Phi = \Phi_a - \frac{1}{2}\, x^{b+m}x^{c+m}\,\partial_{bc}\mathcal{F}_a\,.$$

These equations, by virtue of (10), are also completely integrable and have a solution. Thus, given the latter and formulas (9), the components have the following representation:

$$g_{ab} = \partial_{a+m\,b+m}\Phi + x^{c+m}(\partial_{ac}\mathcal{F}_b + \partial_{bc}\mathcal{F}_a).$$

Finally, the following is proved

Theorem 2.2. *The metric tensor of the parabolic Kähler space $\mathbb{K}_n$ has the following representation in the adapted coordinate system:*

$$\begin{aligned} &a) \quad g_{a+m\,b+m} = 0;\\ &b) \quad g_{a\,b+m} = \partial_b\mathcal{F}_a - \partial_a\mathcal{F}_b;\\ &c) \quad g_{ab} = \partial_{a+m\,b+m}\Phi + x^{c+m}(\partial_{ac}\mathcal{F}_b + \partial_{bc}\mathcal{F}_a). \end{aligned} \tag{11}$$

where $\mathcal{F}_a = \mathcal{F}_a(x^1, x^2, \ldots, x^m)$ and $\Phi_a = \Phi_a(x^1, x^2, \ldots, x^n)$.

However, the opposite is also true.

Theorem 2.3. *For any functions $\mathcal{F}_a(x^1, x^2, \ldots, x^m) \in C^2$ and $\Phi_a(x^1, x^2, \ldots, x^n) \in C^3$ such that the matrix $\|g_{ij}\|$, constructed using the formulas (11), is not degenerate, then g_{ij} defines the metric tensor of some parabolic $\mathbb{K}_n$ space whose structure has the form (2).*

The validity of Theorem 2.3 follows from the fact that the metric tensor of the form (11) satisfies conditions (3), (5), (6), and (7).

Remark The regularity of g_{ij} in Theorem 2.2 follows from the fact that $\|g_{ij}\|$ is regular in any Riemannian space. In Theorem 2.3 regularity must be required additionally. Since, for example, for $\mathcal{F}_a = 0$, the tensor g_{ij} is obviously non regular.

In [36] it is shown that in some special adapted coordinate system the metric tensor $\mathbb{K}_n$ can be reduced to a simpler form

$$a) \;\; g_{a+m\,b+m} = 0; \qquad b) \;\; g_{a\,b+m} = -g_{b\,a+m} = \text{const}; \qquad c) \;\; g_{ab} = \partial_{a+m\,b+m}\Phi.$$

3. Analytic vector fields on parabolic Kähler spaces

A transformation of coordinates $x'^h = x'^h(x)$ will be called *analytic* if it preserves the canonical form of structure (2). In [36] it is shown that analytic transformations of adapted coordinates are only transformations of the form

$$\begin{aligned} x'^a &= \tilde{x}^a(x^1, x^2, \ldots, x^m);\\ x'^{a+m} &= \tilde{x}^{a+m}(x^1, x^2, \ldots, x^m) + x^{b+m}\partial_b\tilde{x}^a, \end{aligned} \tag{12}$$

where $\tilde{x}^a$ and $\tilde{x}^{a+m}$ are functions of these variables such that $\det\|\partial_a\tilde{x}^b\| \neq 0$.

The vector field ξ is caled *analytic*, if the condition

$$\mathcal{L}_\xi F = 0, \tag{13}$$

where $\mathcal{L}_\xi$ is the Lie derivative in the direction of vector ξ.

Condition (13) can be written in the following form: $\xi^\alpha \nabla_\alpha F_i^h + \nabla_i \xi^\alpha F_\alpha^h - \nabla_\alpha \xi^h F_i^\alpha = 0$. On the basis of (1c), the analytic vector fields in parabolic Kähler space are characterized by the conditions

$$\nabla_i \xi^\alpha F_\alpha^h - \nabla_\alpha \xi^h F_i^\alpha = 0.$$

Considering these conditions in the adapted coordinate system, we obtain the following representation of the components of the analytic vectors

$$\begin{aligned} \xi^a &= \tilde{\xi}^a(x^1, x^2, \dots, x^m); \\ \xi^{a+m} &= \tilde{\xi}^{a+m}(x^1, x^2, \dots, x^m) + x^{b+m}\partial_b \tilde{\xi}^a. \end{aligned} \tag{14}$$

Let us prove the following theorem.

Theorem 3.1. *For a given nonzero analytic vector field x in parabolic Kähler space there exists an adapted coordinate system in which the components of this vector field have the following representation:*

$$a) \quad \xi^h = \delta_1^h \qquad or \qquad b) \quad \xi^h = \delta_{1+m}^h. \tag{15}$$

Proof. Consider a nonzero analytic vector ξ satisfying conditions (13) in the adapted coordinate system of parabolic Kähler space $\mathbb{K}_n$. These conditions in the adapted coordinate system take the form (14). The analytic transformation (12) transforms the components of $\xi^h(x)$ according to the following law

$$\xi'^h = \xi^\alpha \partial_\alpha x'^h \tag{16}$$

and ξ'^h will have a structure similar to (14)

$$\begin{aligned} \xi'^a &= \tilde{\xi}'^a(x^1, x^2, \dots, x^m); \\ \xi'^{a+m} &= \tilde{\xi}'^{a+m}(x^1, x^2, \dots, x^m) + x'^{b+m}\partial'_b \tilde{\xi}^a, \end{aligned}$$

where $\partial'_i = \partial/\partial x'^i$.

Formulas (16) for $h = a$ will take the form $\tilde{\xi}'^h = \tilde{\xi}^\alpha \partial_\alpha \tilde{x}'^h$.

It is possible to find solutions $\tilde{x}'^a(x^1, x^2, \dots, x^m)$ satisfying this condition, at $\tilde{\xi}'^a = e_1\, \delta_1^a$; $e_1 = 0, 1$, with $\det \|\partial_b \tilde{x}^a\| \neq 0$. Then the transformation $x'^a = \tilde{x}^a$ and $x'^{a+m} = x^{b+m}\partial_b \tilde{x}^a$ generates a nondegenerate analytic transformation of coordinates $x'^h = x'^h(x)$.

Therefore, we can assume that the adapted coordinate system x is given such that $\tilde{\xi}^a = e_1 \delta_1^a$, $e_1 = 0, 1$. Then the conditions (14) are simplified

$$\xi^a = e_1 \delta_1^a \qquad \text{and} \qquad \xi^{a+m} = \tilde{\xi}^{a+m}.$$

Formulas (16), considering (12) and (14), look like

$$\xi'^a = e_1\, \partial_1 \tilde{x}^a, \qquad\qquad \xi'^{a+m} = \tilde{\xi}^{b+m}\partial_b \tilde{x}^{a+m} + e_1\, \partial_1 \tilde{x}^{a+m}. \tag{17}$$

If $e = 1$, the coordinate transformation (12) at $\tilde{x}^a = x^a$ and $\tilde{x}^{a+m} = -\int \tilde{\xi}^a\, dx^1$ is a nondegenerate analytic transformation, which (17) leads to the form $\xi'^h = \delta_1^h$, i.e. (15a).

If $e_1 = 0$, we can find $\tilde{x}^a(x^1, x^2, \dots, x^m)$, that satisfy (17) for $\xi'^{a+m} = \delta_{1+m}^{a+m}$. These functions give rise to a regular analytic transformation leading vector ξ to the form (15b). The theorem is proved. □

4. Equidistant parabolic Kähler spaces

Since the structure of F is covariantly constant and $F \neq \text{const} \cdot Id$, it follows from [8] that parabolic Kähler spaces $\mathbb{K}_n$ admitting geodesic mappings are equidistant.

A space is called *equidistant* if there exists a *concircular* vector field ξ which satisfies the conditions

$$\nabla \xi = \varrho \cdot Id, \tag{18}$$

where ϱ is a function [14, 31, 38]. If $\varrho \neq 0$, then it is called equidistant of the basic type. Equidistant spaces of basic type always admit non-trivial geodesic mappings [14, 31].

Studying the integrability conditions (18), it is easy to see that in a parabolic Kähler space $\mathbb{K}_n$: $\varrho = \text{const}$. After normalizing the vector ξ, we can assume that either $\varrho = 0$ or $\varrho = 1$.

Consider an equidistant parabolic Kähler space $\mathbb{K}_n$ of the basic type. Then, without loss of generality, (18) will as $\nabla \xi = Id$, in the coordinates

$$\nabla_i \xi^h = \delta_i^h. \tag{19}$$

Obviously, the concircular vector field ξ is analytic. Thus, we can choose an adapted coordinate system in which

$$\xi^h = e_1\, \delta_1^h + e_2\, \delta_{1+m}^h; \qquad e_1, e_2 = 0, 1; \qquad e_1 + e_2 = 1.$$

In this coordinate system the conditions (19) will be written as follows;

$$e_1\, \Gamma_{1i}^h + e_2\, \Gamma_{1+m\, i}^h = \delta_i^h;$$

omitting the index h using the metric tensor, we obtain

$$e_1\, \Gamma_{1ih} + e_2\, \Gamma_{1+m\, ih} = g_{ih}. \tag{20}$$

We put $i = a$, $j = b + m$ in (20), given (4), we find $e_1\, \Gamma_{1a\, b+m} = g_{a\, b+m}$. Since $|g_{a\, b+m}| \neq 0$ then $e_1 = 1$ and hence $e_2 = 0$. Then (20) is simplified:

$$\Gamma_{1ih} = g_{ih}.$$

The latter conditions can be written as follows:

$$\partial_1 g_{a\, b+m} + \partial_a g_{1\, b+m} - \partial_{b+m} g_{1\, a} = 2\, g_{a\, b+m}; \qquad \partial_1 g_{a\, b} + \partial_a g_{1\, b} - \partial_b g_{1\, a} = 2\, g_{a\, b}\, . \tag{21}$$

By putting $a, b = 1$ in (21), we find $\partial_1 g_{11} = 2\, g_{11}$, $\partial_{1+m} g_{11} = 0$. By integrating the latter, we get

$$g_{11} = 4\, \exp(2x^1)\, \mathcal{G}, \tag{22}$$

where $\mathcal{G}$ is a function independent of x^1 and x^{1+m}.

Given (21) $b = 1$ and $a > 1$, given (22), we obtain

$$g_{1\, a+m} = 2\, \exp(2x^1)\, \partial_{a+m} \mathcal{G}, \quad \text{and} \quad g_{1\, a} = 2\, \exp(2x^1)\, \partial_a \mathcal{G}. \tag{23}$$

For $a, b > 1$ conditions (21) by virtue of (22) and (23) will take the form $\partial_1 g_{a\, b+m} = 2\, g_{a\, b+m}$ and $\partial_1 g_{a\, b} = 2\, g_{a\, b}$. From this it is easy to obtain that

$$g_{ab} = \exp(2x^1)\, \tilde{g}_{ab}, \qquad g_{a\, b+b} = \exp(2x^1)\, \tilde{g}_{a\, b+m}. \tag{24}$$

where $\tilde{g}_{ab}$ and $\tilde{g}_{ab+m}$ $(a, b > 1)$ are functions independent of x^1.

The metric tensor of the parabolic Kähler space $\mathbb{K}_n$ must satisfy conditions (5), (6) and (7) in the adapted coordinate system. Putting in (5) $b = 1$ and $a, c > 1$, we obtain $\partial_{a+m\, b+m}\, \mathcal{G} = 0$. It follows that

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_a\, x^{a+m} \qquad (a > 1), \tag{25}$$

where $\mathcal{G}_0, \mathcal{G}_a$ are functions of variables $x^2, x^3, \dots, x^m$.

Assuming in (6) $c = 1$ and $a, c > 1$, given (22), (23), (24), and (25), we obtain

$$\tilde{g}_{a\,b+m} = -\partial_a \mathcal{G}_b + \partial_b \mathcal{G}_a. \tag{26}$$

Then by substituting $a = 1$ and $b, c > 1$ in (7) we make sure that $\tilde{g}_{bc}$ do not depend on x^{1+m}.

By substituting in (7) $a, b, c > 1$, we obtain the conditions

$$\partial_{a+m}\tilde{g}_{bc} - \partial_{b+m}\tilde{g}_{ac} - \partial_{ac}\mathcal{G}_b + \partial_{bc}\mathcal{G}_a = 0.$$

Solving them, it is easy to see that the general solution has the following form:

$$\tilde{g}_{ab} = \partial_{a+m\,b+m}\,\mathcal{H} + x^{c+m}\,(\partial_{ac}\mathcal{G}_b + \partial_{bc}\mathcal{G}_a), \tag{27}$$

where $a, b, c = 2, \dots, m$, $\mathcal{H}$ is a function that does not depend on x^1 and x^{1+m}.

Given (22), (23), (24), (25), (26), and (27), we obtain the following theorem.

Theorem 4.1. *In any equidistant basic type parabolic Kähler space $\mathbb{K}_n$ there exists a coordinate system in which the structure F is expressed by formulas (2) and the nonzero components of the metric tensor g by formulas*

$$\begin{array}{lll} a) & g_{11} & = 4\,\exp(2x^1)\,\mathcal{G}, \\ b) & g_{1\,a+m} & = 2\,\exp(2x^1)\,\partial_{a+m}\mathcal{G}, \\ c) & g_{1\,a} & = 2\,\exp(2x^1)\,\partial_a\mathcal{G}. \\ d) & g_{ab} & = \exp(2x^1)\,(\partial_{a+m\,b+m}\,\mathcal{H} + x^{c+m}\,(\partial_{ac}\mathcal{G}_b + \partial_{bc}\mathcal{G}_a)), \\ e) & g_{a\,b+m} & = \exp(2x^1)\,(-\partial_a\mathcal{G}_b + \partial_b\mathcal{G}_a), \end{array} \tag{28}$$

where $a, b, c = 2, \dots, m$, $m = n/2$, $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_c x^{c+m}$, $\mathcal{G}_0$ and $\mathcal{G}_c$ are functions that depend of $x^2, \dots, x^m$, $\mathcal{H}$ is a function that does not depend on x^1 and x^{1+m}.

On the other hand, it is true

Theorem 4.2. *Riemannian space, the metric tensor g of which has structure (31), for arbitrary functions $\mathcal{G}_0 \in C^3, \mathcal{G}_a \in C^2$ and $\mathcal{H} \in C^3$ of the above variables, provided that $\det g \neq 0$, is an equidistant basic type parabolic Kähler space $\mathbb{K}_n$ whose structure is defined by formulas (2).*

The proof consists in checking if relations (3), (5), (6), (7) and (28) are fulfilled.

Remark The regularity of g in Theorem 4.1 follows from the fact that g is regular in any Riemannian space. In Theorem 4.2, regularity must be required additionally, since e.g., for $\mathcal{G}_0 = \mathcal{G}_a = 0$ the tensor g is obviously non regular.

Coordinate transformation:

$$x'^1 = x^1, \quad x'^{1+m} = x^{1+m}, \quad x'^a = \tilde{x}^a, \quad x'^{a+m} = \tilde{x}^{a+m} + x^{b+m}\partial_b\tilde{x}^a, \tag{29}$$

where $\tilde{x}^a, \tilde{x}^{a+m}$ are depends of $x^2, \dots, x^m$ such that $\det\|\partial_b\tilde{x}^{a+m}\| \neq 0$, $a, b = 2, \dots, m$, $m = n/2$.

This transformation is analytic, i.e., it preserves the adapted coordinate systems, and moreover, it preserves the canonicity of analytic vector fields (15).

It can be seen that with a suitable coordinate transformation (29) it is possible to simplify the components of the metric tensor (28) so that

$$\partial_b\mathcal{G}_a - \partial_a\mathcal{G} = C_{ab}, \tag{30}$$

where C_{ab} are constants some that $C_{ab} + C_{ba} = 0$ for $a, b = 2, \dots, m$.

From (29) it follows that there is a function $h(x^2, \dots, x^m)$ such that

$$\mathcal{G}_a = \partial_a h + 1/2\, C_{ab}\, x^b, \tag{31}$$

for $a, b = 2, \dots, m$.

Thus, in any equidistant basic-type parabolic Kähler space $\mathbb{K}_n$ there exists a more special coordinate system in which the structure F is expressed by formulae (2) and the nonzero components of the metric tensor g are expressed by formulae (28a-c) and

$$g_{ab} = \exp(2x^1)\, \partial_{a+m\, b+m}\mathcal{H}, \qquad g_{a\, b+m} = \exp(2x^1)\, C_{ab}\,,$$

for $a, b = 2, \dots, m$; $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_a\, x^{a+m}$; $\mathcal{G}_a = \partial_a h + 1/2\, C_{ab} x^b$, C_{ab} are constants some that $C_{ab} + C_{ba} = 0$, and $\mathcal{G}_0, h$ are functions of $x^2, \dots, x^m$ and $\mathcal{H}$ is a function independ of x^1, x^{1+m}.

5. Geodesic mappings of parabolically Kähler spaces

The mapping of Riemannian space $\mathbb{V}_n$ onto Riemannian space $\overline{\mathbb{V}}_n$ is called *geodesic* if any geodesic on $\mathbb{V}_n$ is mapped into a geodesic on $\overline{\mathbb{V}}_n$. We consider general questions and the case when the structures of the corresponding spaces are preserved under geodesic mapping. We prove

Theorem 5.1. *m-parabolic Kähler spaces $\mathbb{K}_n$ do not admit non-trivial geodesic mappings to each other, provided the structural affinor is preserved.*

This result is derived from more general reasoning and is similar to the previously proved properties of Kähler spaces with respect to geodesic mappings obtained by Yano, Nagano [40, 42], and W. Westlake [37].

Proof. Let m-parabolic Kähler spaces $\mathbb{K}_n$ with structure F admit geodesic mapping f onto m-parabolic Kähler spaces $\overline{\mathbb{K}}_n$ with structure $\bar{F}$ and structures pereserves, i.e. $\bar{F} = F$. Then in common coordinates x respective mapping f satifies the Levi-Civita equations $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \psi_j + \delta^h_j \psi_i$, where ψ_i are components of linear form ψ, Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of connections $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$, respectively. The assumption that structures are preserved implies $\bar{\nabla} F = \nabla F$, in coordinates $\partial_j F^h_i + F^\alpha_i \bar{\Gamma}^h_{\alpha j} - F^h_\alpha \bar{\Gamma}^\alpha_{ij} = \partial_j F^h_i + F^\alpha_i \Gamma^h_{\alpha j} - F^h_\alpha \Gamma^\alpha_{ij}$.

After substitution the Levi-Civita equation we obtain

$$F^\alpha_i \psi_\alpha\, \delta^h_j - \psi_i F^h_j = 0.$$

By analysing this expression, we can see that $\psi_i = 0$. Thus, the geodesic mapping is trivial. The theorem is therefore proved by contradiction. □

Finaly, we show that m-parabolic Kähler spaces $\mathbb{K}_n$ admit geodesic mappings if there exists a convergent vector field in them. This result follows from the more general results of Mikeš [8] in his study of the Sinyukov equations of geodesic mappings. These equations have the form [30, 31], see [14]:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where a_{ij} are components of a regular symmetric tensor and λ_i are components of gradient vector, which are connected to Levi-Civita equations by conditions $a_{ij} = \exp(-\psi)\, \bar{g}^{\alpha\beta} g_{\alpha i}\, g_{\beta j}$ and $\lambda_i = -\exp(-\psi)\, \bar{g}^{\alpha\beta} g_{\alpha i}\, \psi_\beta$, here $\bar{g}^{ij}$ are components of matrix, which is inverse of metric tensor of (pseudo-) Riemannian space $\overline{\mathbb{V}}_n$ which geodesically corresponds to the (pseudo-) Riemannian space $\mathbb{V}_n$. If $\lambda_i = 0$, then the mapping is trivial.

Since $\nabla F = 0$, it follows from [8], see [14], that $\nabla_j \lambda_i = \mu\, g_{ij}$, $\mu =$ const. Thus the vector λ_i is concircular, and even convergent in the sense of P.A. Shirokov [28].

If there exists a non-constant convergent vector field in $\mathbb{K}_n$, then $\mathbb{K}_n$ can be related to a coordinate system y such that

$$ds^2 = e\,(dy^1)^2 + (y^1)^2\, d\tilde{s}^2,$$

where $e = \pm 1$, $d\tilde{s}^2$ is a metric of an m-parabolic Sasaki space.

This space geodesically corresponds to a two-parametric family of Riemannian spaces $\overline{\mathbb{V}}_n$ with metric

$$d\bar{s}^2 = \frac{\alpha e}{(1+\beta (y^1)^2)^2}\,(dy^1)^2 + \frac{\alpha}{1+\beta (y^1)^2}\,(y^1)^2\, d\tilde{s}^2,$$

where α, β are constants such that $\alpha \neq 0$ and $1 + \beta (y^1)^2 \neq 0$.

The mapping will be nontrivial at $\beta \neq 0$ and at spaces $\overline{\mathbb{V}}_n$ need not to be parabolically Kähler spaces. On the other hand, these spaces are parabolically almost Hermitian with structure F. We can show that the formulas for the structure F are valid $F^2 = 0$ and $\bar{g}(X, FX) = 0$, for any tangent vector field.

The above solution corresponds to the following solution of the Sinyukov equation when we put

$$a_{ij} = \alpha\, g_{ij} + \beta\, \lambda_i\, \lambda_j.$$

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